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Regularity results for a singular elliptic equation involving variable exponents *

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ABSTRACT: In this paper, we investigate the existence and regularity of nonnegative weak solutions for specific class of nonlinear singular anisotropic elliptic problems with degenerate coercivity involving variable exponents. We show that certain lower-order term contribute to the regularization of solutions, as well as the regularization induced by a singular nonlinearity term. Our methodology employs an approximation technique that integrates anisotropic variable exponent Sobolev spaces, truncation methods, compactness arguments, and Schauder's fixed-point theorem. These findings extend some previous results established under constant exponents.

Key Words: Nonlinear elliptic equations, Singular elliptic equations, Degenerate coercivity, Measure data, Variable exponents, Regularity of weak solutions.

Contents

1. Introduction

Let us consider the nonlinear anisotropic elliptic problem with variable exponents and degenerate coercivity, involving a singular nonlinearity term of the form

$$
\begin{aligned}
-\sum_{i=1}^{N} D_i\big(b_i(x,u)a_i(x,Du)\big) + F(x,u) &= H(u)f + \mu \text{ in } \Omega,\\
u > 0 \quad &\text{in } \Omega,\\
u = 0 \quad &\text{on } \partial\Omega.
\end{aligned}
\tag{1.1}
$$

where Ω is a bounded open domain in $\mathbb{R}^N$ $(N \geq 2)$ with Lipschitz boundary $\partial\Omega$, μ is a non-negative bounded Radon measure on Ω, and f is a non-negative function belonging to $L^1(\Omega)$ (or $L^{m(\cdot)}(\Omega)$) with $m(\cdot) > 1$ being small.

We assume that the nonlinearity term $H : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous and possibly singular function, such that

$$
\begin{aligned}
&H(0) \neq 0, \quad \lim_{u\to+\infty} H(u) := H(+\infty) < +\infty,\\
&\text{There exists } M > 0, \text{ for all } u \in (0,+\infty), \text{ such that } : H(u) \leq M,\\
&\text{There exists } \widehat{M} > 0, \text{ such that } : H(u) \leq \frac{\widehat{M}}{u^{\gamma(x)}}, \text{ for all } u \in (0,+\infty), \text{ for all } x \in \overline{\Omega},.
\end{aligned}
\tag{1.2}
$$

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where $\gamma : \overline{\Omega} \longrightarrow (0,1)$ is continuous functions satisfies:

$$0 < \gamma^- := \min_{x\in\overline{\Omega}} \gamma(x) \leq \gamma^+ := \max_{x\in\overline{\Omega}} \gamma(x) < 1, \quad |\nabla\gamma| \in L^\infty(\Omega), \text{ for all } x \in \overline{\Omega}. \tag{1.3}$$

Suppose that $a_i : \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}$, $b_i : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$, $i = 1, \dots, N$, $F : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions and satisfying
For a.e.$x \in \Omega$, for all $u \in \mathbb{R}$, for all $\xi, \xi' \in \mathbb{R}^N$, and for all $i = 1, \dots, N$ the following:

$$a_i(x,\xi).\xi_i \geq \alpha|\xi_i|^{p_i(x)}, \tag{1.4}$$

$$|a_i(x,\xi)| \leq \beta\Big(1 + \sum_{j=1}^{N} |\xi_j|^{p_j(x)}\Big)^{1-\frac{1}{p_i(x)}}, \tag{1.5}$$

$$\Big(a_i(x,\xi) - a_i(x,\xi')\Big) \cdot (\xi_i - \xi_i') > 0, \;\; \xi_i \neq \xi_i', \tag{1.6}$$

$$\frac{C_2}{(1+|u|)^{\sigma_i(x)}} \leq b_i(x,u) \leq C_1, \tag{1.7}$$

$$\sup_{|u|\leq k} |F(x,u)| = g_k(x) \in L^1(\Omega), \quad \forall k > 0, \tag{1.8}$$

$$F(x,u)\,\mathrm{sign}(u) \geq |u|^{r(x)}, \tag{1.9}$$

where $\alpha, \beta, C_1,$ and C_2 are strictly nonnegative real numbers.

Here, the variable exponents $p_i : \overline{\Omega} \longrightarrow (1,\infty)$, $\sigma_i : \overline{\Omega} \longrightarrow [0,\infty)$, $i = 1, \dots, N$, $r : \overline{\Omega} \longrightarrow (0,\infty)$ and $m : \overline{\Omega} \longrightarrow (1,\infty)$ are continuous functions for all $x \in \overline{\Omega}$ such that

$$1 < \overline{p}(x) < N, \quad \text{where} \quad \frac{1}{\overline{p}(x)} = \frac{1}{N}\sum_{i=1}^{N} \frac{1}{p_i(x)},. \tag{1.10}$$

$$\max\Big\{1; \frac{N}{1 + N(\overline{p}(x)-1) + (N-1)(\gamma(x) - \sigma_+(x))}\Big\} < m(x) < \widehat{m}_1(x), \quad |\nabla m| \in L^\infty(\Omega), \tag{1.11}$$

where

$$\widehat{m}_1(x) = \frac{N\overline{p}(x)}{N\overline{p}(x) - (N - \overline{p}(x))(\sigma_+(x) + 1 - \gamma(x))} = \Big(\frac{\overline{p}^\star(x)}{\sigma_+(x) + 1 - \gamma(x)}\Big)',$$

$$\sigma_+(x) = \max_{1\leq i\leq N} \sigma_i(x), \quad |\nabla\sigma_+| \in L^\infty(\Omega), \quad |\nabla\gamma| \in L^\infty(\Omega), \quad \overline{p}^\star(x) = \frac{N\overline{p}(x)}{N - \overline{p}(x)},$$

As a prototype example, we consider the model problem

$$\begin{gathered} -\sum_{i=1}^{N} D^i\left(\frac{|D^i u|^{p_i(x)-2}D^i u}{(\ln(e+|u|))^{\sigma_i(\cdot)}}\right) + u|u|^{r(\cdot)-1} = \frac{f}{u^{\gamma(\cdot)}} + \mu \text{ in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.12}$$

where γ as in (1.3), $\sigma_i \in C(\overline{\Omega})$, $\sigma_i(\cdot) \geq 0$ for all $i = 1, \dots, N$, $r \in C(\overline{\Omega})$, $r(\cdot) \geq 0$, while f is a nonnegative function in $L^1(\Omega)$, and $\mu \geq 0$ is a nonnegative bounded Radon measure.

Boundary value problems involving anisotropic operators have garnered considerable attention and have arisen in several areas of science in recent years. Notably, important applications of this complex issue in biology, physics, and image processing can be found in [3,10,11,34]. Another point worth mentioning is that the study of both isotropic and anisotropic problems involving singular nonlinearities is still developing slowly due to the very limited number of results available on this topic. For further information, we refer to the seminal works in [7,9,12,15,23,25]. Additionally, for existence results related to problem

(1.1), as well as some model equations with either isotropic or anisotropic principal parts, nonlinear terms of lower order with variable nonlinearity, involving singular nonlinearity terms, and under different regularity assumptions on the data using various methods, see, e.g., [22,32,36] and further references therein.

In the present paper, we aim to extend results recently addressed in [6,33] to the anisotropic case, focusing on singular elliptic problems with degenerate coercivity and variable exponents. Compared to [14] and [16], the key feature of this work is the combination of an anisotropic degenerate operator with lower-order terms involving variable exponents, singular nonlinearity term characterized by a nonnegative function which has some Lebesgue regularity, and a nonnegative bounded Radon measure. Inspired by [1,20,27], we prove the existence and regularity of nonnegative weak solutions to problem (1.1), noting that classical methods cannot be directly applied in this setting. More precisely, we show that the presence of certain lower-order terms has a regularizing effect on the solutions, as does the singular nonlinearity term. Our regularity results depend on the summability of f, as well as on the values of $p_i(\cdot)$, $\sigma_i(\cdot)$, $r(\cdot)$, and $m(\cdot)$, which are novel in the literature and have not been previously investigated in either the isotropic or anisotropic case, as demonstrated in [8,17,21,37].

The proof of the existence and regularity results under assumptions (1.2)-(1.11) is essentially based on deriving uniform estimates for suitable approximate solutions $(u_n)_n$ and passing to the limit as $n \to +\infty$. This strategy addresses two main challenges associated with this approach. Given the lower-order term in the model problem (1.12), characterized by a singular nonlinearity term, namely $\frac{f}{u^{\gamma(\cdot)}}$, where f is a nonnegative function in $L^1(\Omega)$ (or $L^{m(\cdot)}(\Omega)$), with $m(\cdot)$ and $\gamma(\cdot)$ as defined in (1.3)-(1.11), and μ is a nonnegative bounded Radon measure on Ω, the first challenge is to obtain suitable uniform estimates on $(u_n)_n$ and the partial derivatives $(D^i u_n)_n$ for all $i = 1, \dots, N$, which are independent of n. The second challenge involves passing to the limit in the degenerate nonlinear anisotropic vector field with variable exponents

$$Au = -\sum_{i=1}^{N} D^i \left(\frac{|D^i u|^{p_i(x)-2} D^i u}{(\ln(e+|u|))^{\sigma_i(\cdot)}} \right)$$

The nonlinear term $u|u|^{r(\cdot)-1}$, where $p_i(\cdot)$ satisfies (1.10), $\sigma_i \in C(\overline{\Omega})$, $\sigma_i(\cdot) \geq 0$ for all $i = 1, \dots, N$, and $r \in C(\overline{\Omega})$ with $r(\cdot) \geq 0$, also presents challenges. To address these, we truncate the singular term $\frac{f}{u^{\gamma(\cdot)}}$ to make it non-singular at the origin and study the behavior of approximate solutions $(u_n)_n$. We employ Schauder's fixed-point theorem to ensure the existence of $(u_n)_n$ and apply the anisotropic Sobolev inequality, along with certain truncation techniques as in [2,28,30,31], to establish uniform estimates. We believe that our gradient estimates (see Lemma 4.5) are new compared to those in [24]. Specifically, if $p_i(\cdot) > 1 + \frac{\sigma_i(\cdot)+1+\gamma(\cdot)}{r(\cdot)}$, then the regularity provided in Theorem 3.3 is an improvement over that in [24, Theorem 1]; see Remark 3.1. To prove Theorem 3.3, a key result (Lemma 4.5) involving an $L^{q_i(\cdot)}$ estimate for the solution to problem (1.1) is established. It is important to note that the results of Theorems 3.3-3.4 are distinct from those in [33, Theorem 3.2] (and also in [32, Theorem 3.1]) due to the degenerate coercivity of the anisotropic operator involving a singular nonlinearity term. According to Remark 3.1, the results of Theorem 3.3 align with the regularity results of [6, Theorem 1.1] in the case of constant exponents. Additionally, in the nonsingular case where $\gamma(\cdot) \longrightarrow 0$ and $\mu = 0$, similar results to those in Theorems 3.1-3.2-3.5 can be found in [5,29] under different appropriate assumptions on the data.

This paper is organized as follows: In Section 2, we recall some basic properties of anisotropic spaces with variable exponents and present an anisotropic Sobolev inequality. The main results are established in Section 3. In Section 4, we provide some uniform estimates for the solutions of the approximate problems. Finally, the proofs are presented in Section 5.

2. Mathematical background and auxiliary results

We first recall some definitions, facts, and basic properties of anisotropic spaces with variable exponents. For further details on Lebesgue-Sobolev spaces with variable exponents, we refer to [4,13,18] and the references therein. In this paper we denote

$$C_+^0(\overline{\Omega}) = \{p \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} p(x) > 0\},$$

$$C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} p(x) > 1\},$$

For any $p \in C_+^0(\overline{\Omega})$, we denote

$$p^+ := \max_{x\in\overline{\Omega}} p(x) \quad \text{and} \quad p^- := \min_{x\in\overline{\Omega}} p(x).$$

We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $u : \Omega \to \mathbb{R}$ for which the convex modular

$$\rho_{p(\cdot)}(u) = \int_\Omega |u|^{p(x)}\, dx,$$

is finite.

The space $L^{p(\cdot)}(\Omega)$ equipped with the norm

$$\|u\|_{p(\cdot)} := \|u\|_{L^{p(\cdot)}(\Omega)} = \inf\big\{k > 0 : \rho_{p(\cdot)}\big(\frac{u}{k}\big) \le 1\big\},$$

which called the Luxemburg norm. The space $(L^{p(\cdot)}(\Omega), \|u\|_{p(\cdot)})$ is a separable Banach space. Moreover, if $1 < p^- \le p^+ < +\infty$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

For all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, the Hölder type inequality

$$\left|\int_\Omega uv\, dx\right| \le \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big)\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)} \le 2\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)}, \tag{2.1}$$

holds.

We define also the Banach space

$$W^{1,p(\cdot)}(\Omega) = \big\{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\big\},$$

endowed with the norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

The space $(W^{1,p(\cdot)}(\Omega), \|u\|_{1,p(\cdot)})$ is a Banach space. while

$$W_0^{1,p(\cdot)}(\Omega) = \big\{u \in W^{1,p(\cdot)}(\Omega) : u = 0 \text{ on } \partial\Omega\big\},$$

is Sobolev space with zero boundary values endowed with the norm $\|\cdot\|_{1,p(\cdot)}$. The space $W_0^{1,p(\cdot)}(\Omega)$ is separable and reflexive provided that $1 < p^- \le p^+ < +\infty$. For $u \in W_0^{1,p(\cdot)}(\Omega)$ with $p \in C_+(\overline{\Omega})$, the Poincaré inequality holds

$$\|u\|_{p(\cdot)} \le C\|\nabla u\|_{p(\cdot)}, \tag{2.2}$$

for some $C > 0$ which depends on Ω and $p(\cdot)$. Therefore, $\|\nabla u\|_{p(\cdot)}$ and $\|u\|_{1,p(\cdot)}$ are equivalent norms.

The following results will be use later.

Proposition 2.1 ([13]) *If $u_n, u \in L^{p(\cdot)}(\Omega)$ and $p^+ < +\infty$, then the following properties hold:*

- $\|u\|_{p(\cdot)} < 1$ *(resp. $= 1, > 1$)* $\iff \rho_{p(\cdot)}(u) < 1$ *(resp. $= 1, > 1$),*
- $\min\big(\rho_{p(\cdot)}(u)^{\frac{1}{p^+}}, \rho_{p(\cdot)}(u)^{\frac{1}{p^-}}\big) \le \|u\|_{p(\cdot)} \le \max\big(\rho_{p(\cdot)}(u)^{\frac{1}{p^+}}, \rho_{p(\cdot)}(u)^{\frac{1}{p^-}}\big),$
- $\min\big(\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\big) \le \rho_{p(\cdot)}(u) \le \max\Big(\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\Big),$

- $\|u\|_{p(\cdot)} \leq \rho_{p(\cdot)}(u) + 1$,
- $\|u_n - u\|_{p(\cdot)} \to 0 \Longleftrightarrow \rho_{p(\cdot)}(u_n - u) \to 0$.

Remark 2.1 As in [19], the following inequality

$$\int_\Omega |u|^{p(x)}\,dx \leq C \int_\Omega |Du|^{p(x)}\,dx,$$

in general does not hold. So, thanks to Proposition 2.1 and (2.2), we get the following inequality which will be used later

$$\min\{\|Du\|_{p(\cdot)}^{p^-}; \|Du\|_{p(\cdot)}^{p^+}\} \leq \int_\Omega |u(x)|^{p(x)}\,dx \leq \max\{\|Du\|_{p(\cdot)}^{p^-}; \|Du\|_{p(\cdot)}^{p^+}\}. \tag{2.3}$$

Next, we recall the definition of Marcinkiewicz space which called also 'weak Lebesgue space'.

Definition 2.1 **([26])** For $0 < q < +\infty$, the set

$$\mathcal{M}^q(\Omega) := \big\{u : \Omega \to \mathbb{R} \text{ measurable } : [u]_q = \sup_{k>0} k \operatorname{meas}\{x \in \Omega : |u(x)| > k\}^{1/q} < \infty\big\},$$

is called a Marcinkiewicz space.

Note that if $|\Omega| < \infty$ and $0 < \varepsilon < q-1$, then it is easy to show that $L^q(\Omega) \subset \mathcal{M}^q(\Omega) \subset L^{q-\varepsilon}(\Omega)$.

In addition, another important aspect worth mentioning is the anisotropic Sobolev space with variable exponents, which helps us study our problem (1.1).

Firstly, let $p_i(\cdot) : \overline{\Omega} \to [1, +\infty)$, $i = 1, \ldots, N$ be continuous functions, we set $\overrightarrow{p}(\cdot) = (p_1(\cdot), \ldots, p_N(\cdot))$ and $p_+(x) = \max_{1 \leq i \leq N} p_i(x)$, for all $x \in \overline{\Omega}$.

The anisotropic variable exponent Sobolev space $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is defined as

$$W^{1,\overrightarrow{p}(\cdot)}(\Omega) = \big\{u \in L^{p_+(\cdot)}(\Omega) : D_i u \in L^{p_i(\cdot)}(\Omega),\ i = 1, \ldots, N\big\},$$

which is Banach space with respect to the norm

$$\|u\|_{W^{1,\overrightarrow{p}(\cdot)}(\Omega)} = \|u\|_{p_+(\cdot)} + \sum_{i=1}^N \|D_i u\|_{p_i(\cdot)}.$$

We denote by $W_0^{1,\overrightarrow{p}\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$, and we define

$$\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) = W^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap W_0^{1,1}(\Omega).$$

If Ω is a bounded open set with Lipschitz boundary $\partial\Omega$, then

$$\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) = \big\{u \in W^{1,\overrightarrow{p}(\cdot)}(\Omega) : u|_{\partial\Omega} = 0\big\}.$$

It is well-known that in the constant exponent case, that is, when $\overrightarrow{p}(\cdot) = \overrightarrow{p} \in [1, +\infty)^N$, $W_0^{1,\overrightarrow{p}}(\Omega) = \mathring{W}^{1,\overrightarrow{p}}(\Omega)$. However in the variable exponent case, in general $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \subset \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and the smooth functions are in general not dense in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$, but if for each $i = 1, \ldots, N$, p_i is log-Hölder continuous, that is, there exists a nonnegative constant $\underline{M}$ such that

$$|p_i(x) - p_i(y)| \leq \frac{\underline{M}}{-\ln|x-y|}, \quad \forall x, y \in \Omega,\ |x-y| < 1.$$

Then $C_0^\infty(\Omega)$ is dense in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$, thus $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) = \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$. We set for all $x \in \overline{\Omega}$

$$\overline{p}(x) = \frac{N}{\sum_{i=1}^N \frac{1}{p_i(x)}},$$

we define

$$\overline{p}^{\star}(x)=\begin{cases}\frac{N\overline{p}(x)}{N-\overline{p}(x)}, & \text{for } \overline{p}(x)<N,\\ +\infty, & \text{for } \overline{p}(x)\geq N.\end{cases}$$

and denote

$$p_+^+=\max_{1\leq i\leq N}p_i^+(x) \quad \text{and} \quad p_-^-=\min_{1\leq i\leq N}p_i^-(x).$$

For $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ we have the following compact embedding results.

Lemma 2.1 ([18]) *Let $\Omega\subset\mathbb{R}^N$ be a bounded domain, and $\overrightarrow{p}(\cdot)\in(C_+(\overline{\Omega}))^N$. If $q\in C_+(\overline{\Omega})$ and for all $x\in\overline{\Omega}$, $q(x)<\max(p_+(x),\overline{p}^{\star}(x))$. Then*

$$\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)\hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega),$$

Lemma 2.2 ([18]) *Let $\Omega\subset\mathbb{R}^N$ be a bounded domain, and $\overrightarrow{p}(\cdot)\in(C_+(\overline{\Omega}))^N$. Suppose that*

$$\forall x\in\overline{\Omega},\ p_+(x)<\overline{p}^{\star}(x). \tag{2.4}$$

Then the following Poincaré-type inequality holds

$$\|u\|_{L^{p_+(\cdot)}(\Omega)}\leq C\sum_{i=1}^{N}\|D_iu\|_{L^{p_i(\cdot)}(\Omega)}, \quad \forall u\in\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega), \tag{2.5}$$

where C is a positive constant independent of u. Thus $\sum_{i=1}^{N}\|D_iu\|_{L^{p_i(\cdot)}(\Omega)}$ is an equivalent norm on $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

Remark 2.2 The assumption (2.4) play a significant role in our investigation which provide us with, for all $u\in\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$, we have

$$\frac{1}{N^{p_-^--1}}\|u\|^{p_-^-}_{\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)}-N\leq\sum_{i=1}^{N}\int_\Omega|D^iu|^{p_i(x)}\,dx\leq N+\|u\|^{p_+^+}_{\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)} \tag{2.6}$$

The following embedding results for the anisotropic constant exponent Sobolev space are well-known [35,39].

Lemma 2.3 *Let $q_i\geq1,\ i=1,\dots,N$, we pose $\overrightarrow{q}=(q_1,\dots,q_N)$. Suppose $u\in W_0^{1,\overrightarrow{q}}(\Omega)$, and set*

$$\frac{1}{\overline{q}}=\frac{1}{N}\sum_{i=1}^{N}\frac{1}{q_i}, \quad s=\begin{cases}\overline{q}^{\star}=\frac{N\overline{q}}{N-\overline{q}} & \text{if } \overline{q}<N,\\ \text{any number in } [1,+\infty) & \text{if } \overline{q}\geq N.\end{cases}$$

Then, there exists a constant C depending on $N,p_1,\dots,p_N$ if $\overline{q}<N$ and also on s and $|\Omega|$ if $\overline{q}\geq N$, such that

$$\|u\|_{L^s(\Omega)}\leq C\prod_{i=1}^{N}\|D^iu\|^{1/N}_{L^{q_i}(\Omega)}. \tag{2.7}$$

Lemma 2.4 *Let Q be a cube of $\mathbb{R}^N$ with faces parallel to the coordinate planes and $q_i\geq1,\ i=1,\dots,N$. Suppose $u\in W^{1,\overrightarrow{q}}(Q)$, and set*

$$s=\overline{q}^{\star} \quad \text{if } \overline{q}<N,$$
$$s\in[1,+\infty) \quad \text{if } \overline{q}\geq N.$$

Then, there exists a constant C depending on $N,q_1,\dots,q_N$ if $\overline{q}<N$ and also on r and $|Q|$ if $\overline{s}\geq N$, such that

$$\|u\|_{L^s(Q)}\leq C\prod_{i=1}^{N}\left(\|u\|_{L^{q_i}(Q)}+\|D_iu\|_{L^{q_i}(Q)}\right)^{1/N}. \tag{2.8}$$

For the notion of nonnegative weak solutions to problem (1.1), we will use through this paper, the truncation function T_k at height k ($k > 0$) and the associated function which denoted by

$$T_k(t) = \max\Big\{ -k, \min\{k,t\}\Big\}, \quad \varphi_k(s) = T_{k+1}(s) - T_k(s), \tag{2.9}$$

It is obvious that T_k and φ_k are Lipschitz functions satisfying $|T_k(s)| \leq k$ and $|\varphi_k(s)| \leq 1$.
We denote

$$\mathcal{T}_0^{1,\overrightarrow{p}(\cdot)}(\Omega) = \Big\{u : \Omega \to \mathbb{R} \text{ measurable }; T_k(u) \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega), \text{ for every } k > 0\Big\}.$$

Finally, we define the weak gradient of a measurable function $u \in \mathcal{T}_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. The proof due to the fact that $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) \subset \mathring{W}^{1,p_-^-}(\Omega)$.

Definition 2.2 *Suppose that $u \in \mathcal{T}_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. For $i = 1, \cdots, N$, there exists a unique measurable function $\nu_i : \Omega \longmapsto \mathbb{R}$, such that*

$$D^i T_k(u) = \nu_i \cdot \chi_{\{|u|<k\}}, \text{ almost everywhere in } \Omega, \text{ for every } k > 0, \tag{2.10}$$

where χ_A denotes the characteristic function of a measurable set A. The functions ν_i are called the weak partial derivatives of u and are still denoted $D^i u$. Moreover, if u belongs to $W_0^{1,1}(\Omega)$, then ν_i coincides with the standard distributional derivative of u, that is $\nu_i = D^i u$.

Remark 2.3 A function u such that $T_k(u) \in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$ for any $k > 0$, does not necessarily belong to $W_0^{1,1}(\Omega)$. However, according to the above definition, it is possible to define its weak gradient, still denoted by $D^i u$, as the unique function v_i which satisfies (2.10).

Throughout this paper, C will denote a nonnegative constant that depends only on the data and may vary from line to line.

3. Main results and approximation of problem (1.1)

Our aim is to prove the existence of nonnegative weak solutions to problem (1.1). The following is the notion of solutions we will consider

Definition 3.1 Let (μ_n) be a sequence of measures in the set of nonnegative bounded Radon measure. We say (μ_n) weakly converges to μ, if for any continuous function $\varphi \in C_c^\infty(\Omega)$

$$\int_\Omega \varphi \, d\mu_n \to \int_\Omega \varphi \, d\mu \quad \text{as} \quad n \to +\infty.$$

Definition 3.2 If $0 < \gamma^+ < 1$ and μ is a nonnegative bounded Radon measure on Ω. Then, we say A function $u \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is a nonnegative weak solution to problem (1.1) if the following conditions are satisfied:

1. u is strictly positive on each compact set of Ω, i.e., for every $\omega \subset\subset \Omega$, there exists $C_\omega > 0$ such that: $u(\cdot) \geq C_\omega > 0$ in ω.
2. $u \in W_0^{1,1}(\Omega)$, $F(x, Du) \in L^1(\Omega)$, $a_i(x, Du) \in L^1(\Omega), \forall i = 1, \ldots, N, \quad H(u)f \in L^1_{loc}(\Omega)$.
3. $$\begin{aligned} &\sum_{i=1}^N \int_\Omega b_i(x,u) a_i(x,Du) \cdot D^i \phi \, dx + \int_\Omega F(x,u) \cdot \phi \, dx \\ &= \int_\Omega H(u) f \cdot \phi \, dx + \int_\Omega \phi \, d\mu. \end{aligned} \tag{3.1}$$

for every $\phi \in C_0^\infty(\Omega)$.

The main results of the paper are the following theorems:

Theorem 3.1 *Under the assumptions (1.2)-(1.10). If μ is a nonnegative bounded Radon measure on Ω, and $f \in L^{m(\cdot)}(\Omega)$ is a nonnegative function with $m(\cdot)$ as in (1.11), the variable exponents $p_i : \overline{\Omega} \to (1,+\infty)$, $\sigma_i : \overline{\Omega} \to [0,+\infty)$ $i = 1,\cdots,N$, and $r : \overline{\Omega} \to (0,+\infty)$ are continuous functions such that (2.4) holds and for all $x \in \overline{\Omega}$*

$$\frac{1+\sigma_+(x)+\gamma(x)}{m(x)-1} \leq r(x), \quad \sigma_+(x) := \max_{1\leq i\leq N} \sigma_i(x), \tag{3.2}$$

where

$$|\nabla r| \in L^\infty(\Omega), \quad |\nabla \sigma_+| \in L^\infty(\Omega), \quad |\nabla \gamma| \in L^\infty(\Omega),$$

Then, problem (1.1) has at least one solution $u \in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega) \cap L^{r(\cdot)m(\cdot)}(\Omega)$ satisfying (3.1).

Theorem 3.2 *Under the assumptions (1.2)-(1.10). If μ is a nonnegative bounded Radon measure on Ω, and $f \in L^{m(\cdot)}(\Omega)$ is a nonnegative function with $m(\cdot)$ as in (1.11), the variable exponents $p_i : \overline{\Omega} \to (1,+\infty)$, $\sigma_i : \overline{\Omega} \to [0,+\infty)$ $i = 1,\cdots,N$, and $r : \overline{\Omega} \to (0,+\infty)$ are continuous functions such that (2.4) holds, for all $x \in \overline{\Omega}$ and for all $i = 1,\ldots,N$*

$$\frac{1}{m(x)}\Big(1+\frac{\sigma_i(x)+1+\gamma(x)}{r(x)}\Big) < p_i(x) < m'(x), \tag{3.3}$$

$$\max\left(\frac{\sigma_i(x)+1+\gamma(x)}{p_i(x)m(x)-1};(\sigma_i(x)+1+\gamma(x))(p_i(x)-1)\right) < r(x) < \frac{\sigma_+(x)+1+\gamma(x)}{m(x)-1} \tag{3.4}$$

where

$$\sigma_+(x) := \max_{1\leq i\leq N} \sigma_i(x), \quad |\nabla r| \in L^\infty(\Omega), \quad |\nabla \sigma_+| \in L^\infty(\Omega), \quad |\nabla \gamma| \in L^\infty(\Omega).$$

Then, problem (1.1) has at least one solution $u \in \mathring{W}^{1,\vec{q}(\cdot)}(\Omega)$ satisfying (3.1) where $q_i(\cdot)$ are continuous functions on $\overline{\Omega}$ satisfying

$$1 < q_i(x) = \frac{p_i(x)m(x)r(x)}{r(x)+\sigma_i(x)+1+\gamma(x)}, \quad \forall x \in \overline{\Omega},\ i = 1,\ldots,N. \tag{3.5}$$

Moreover, $|u|^{m(x)r(x)+\gamma(x)} \in L^1(\Omega)$.

Theorem 3.3 *Under the assumptions (1.2)-(1.10). If μ is a nonnegative bounded Radon measure on Ω, and $f \in L^1(\Omega)$ is a nonnegative function, the variable exponents $p_i : \overline{\Omega} \to (1,+\infty)$, $\sigma_i : \overline{\Omega} \to [0,+\infty)$ $i = 1,\cdots,N$, and $r : \overline{\Omega} \to (0,+\infty)$ are continuous functions such that (2.4) holds, for all $x \in \overline{\Omega}$ and for all $i = 1,\ldots,N$*

$$1+\frac{\sigma_i(x)+1+\gamma(x)}{r(x)} < p_i(x) < 1+\frac{r(x)}{\sigma_i(x)+1+\gamma(x)}, \tag{3.6}$$

$$\max\left(\frac{\sigma_i(x)+1+\gamma(x)}{p_i(x)-1};(\sigma_i(x)+1+\gamma(x))(p_i(x)-1)\right) < r(x), \tag{3.7}$$

where

$$|\nabla r| \in L^\infty(\Omega), \quad |\nabla \sigma_+| \in L^\infty(\Omega), \quad |\nabla \gamma| \in L^\infty(\Omega),$$

Then, problem (1.1) has at least one solution $u \in \mathring{W}^{1,\vec{q}(\cdot)}(\Omega) \cap L^{r(\cdot)}(\Omega)$ satisfying (3.1) where $q_i(\cdot)$ are continuous functions on $\overline{\Omega}$ satisfying

$$1 < q_i(x) < \frac{p_i(x)r(x)}{r(x)+\sigma_i(x)+1+\gamma(x)}, \quad \forall x \in \overline{\Omega},\ i = 1,\ldots,N, \tag{3.8}$$

Theorem 3.4 *Under the assumptions (1.2)-(1.10). If μ is a nonnegative bounded Radon measure on Ω, and $f \in L^1(\Omega)$ is a nonnegative function, the variable exponents $p_i : \overline{\Omega} \to (1,+\infty)$, $\sigma_i : \overline{\Omega} \to [0,+\infty)$ $i = 1,\cdots,N$, and $r : \overline{\Omega} \to (0,+\infty)$ are continuous functions such that, for all $x \in \overline{\Omega}$*

$$p_i(x) \leq r(x), \tag{3.9}$$

$$\frac{\overline{p}(x)\Big(N-1-\sigma_+^+ + \gamma^-\Big)}{N\Big(\overline{p}(x)-1-\sigma_+^+ + \gamma^-\Big)} < p_i(x) < \frac{\overline{p}(x)\Big(N-1-\sigma_+^+ + \gamma^-\Big)}{\Big(\sigma_+^+ + 1 - \gamma^-\Big)(N-\overline{p}(x))}, \tag{3.10}$$

where

$$\gamma^- := \min_{x\in\overline{\Omega}} \gamma(x), \quad \textit{and} \quad \sigma_+^+ := \max_{1\leq i\leq N} \max_{x\in\overline{\Omega}} \sigma_i(x),$$

$$0 \leq \sigma_+(x) < \overline{p}(x) - 1 + \gamma(x), \quad |\nabla\sigma_+| \in L^\infty(\Omega), \quad |\nabla\gamma| \in L^\infty(\Omega). \tag{3.11}$$

Then, problem (1.1) has at least one solution $u \in \mathring{W}^{1,\vec{q}(\cdot)}(\Omega)$ satisfying (3.1) where $q_i(\cdot)$ are continuous functions on $\overline{\Omega}$ satisfying

$$1 \leq q_i(x) < \frac{N\Big(\overline{p}(x)-1-\sigma_+^+ + \gamma^-\Big)p_i(x)}{\overline{p}(x)\Big(N-1-\sigma_+^+ + \gamma^-\Big)}, \quad \forall x \in \overline{\Omega},\ i = 1,\ldots,N. \tag{3.12}$$

Theorem 3.5 *Under the assumptions (1.2)-(1.10). If μ is a nonnegative bounded Radon measure on Ω, and $f \in L^{m(\cdot)}(\Omega)$ is a nonnegative function with $m(\cdot) = m$ as in (1.11), the variable exponents $p_i : \overline{\Omega} \to (1,+\infty)$, $\sigma_i : \overline{\Omega} \to [0,+\infty)$, $i = 1,\cdots,N$, and $r : \overline{\Omega} \to (0,+\infty)$ are continuous functions such that, for all $x \in \overline{\Omega}$, for all $i = 1,\cdots,N$*

$$p_i(x) \leq r(x), \tag{3.13}$$

$$\frac{\overline{p}(x)\Big(N - m(\sigma_+^+ + 1 - \gamma^-)\Big)}{Nm\Big(\overline{p}(x)-1-\sigma_+^+ + \gamma^-\Big)} < p_i(x) < \frac{\overline{p}(x)\Big(N - m(\sigma_+^+ + 1 - \gamma^-)\Big)}{m(\sigma_+^+ + 1 - \gamma^-)(N-\overline{p}(x)) - N\overline{p}(x)(m-1)}, \tag{3.14}$$

where

$$\gamma^- := \min_{x\in\overline{\Omega}} \gamma(x), \quad \textit{and} \quad \sigma_+^+ := \max_{1\leq i\leq N} \max_{x\in\overline{\Omega}} \sigma_i(x),$$

$$0 \leq \sigma_+(x) < \max\left(\overline{p}(x) - 1 + \gamma(x); \frac{N}{m} - 1 + \gamma(x)\right), \quad |\nabla\sigma_+| \in L^\infty(\Omega), \quad |\nabla\gamma| \in L^\infty(\Omega). \tag{3.15}$$

Then, problem (1.1) has at least one solution $u \in \mathring{W}^{1,\vec{q}(\cdot)}(\Omega)$ satisfying (3.1) where $q_i(\cdot)$ are continuous functions on $\overline{\Omega}$ satisfying

$$1 \leq q_i(x) < \frac{Nm\Big(\overline{p}(x)-1-\sigma_+^+ + \gamma^-\Big)p_i(x)}{\overline{p}(x)\Big(N - m(\sigma_+^+ + 1 - \gamma^-)\Big)}, \quad \forall x \in \overline{\Omega},\ i = 1,\ldots,N. \tag{3.16}$$

Remark 3.1

If $\overline{p}(\cdot) = p_i(\cdot) = p = 2$, $\sigma_i(\cdot) = 0$, $\gamma^- \to 0$, and $F(x,u) = 0$, then the results of Theorem 3.4 coincide with the regularity results of [33, Theorem 2.6].

If $\overline{p}(\cdot) = p_i(\cdot) = p$, $\sigma_i(\cdot) = 0$, $\gamma^- \to 0$, and $F(x,u) = 0$, then results of Theorem 3.4 coincide with regularity results of [33, Theorem 3.2].

If $\overline{p}(\cdot) = p_i(\cdot) = p$, $\mu = 0$, $\sigma_i(\cdot) = 0$, $F(x,u) = |u|^{r-1}u$, and $\gamma(\cdot) = \theta$ with take the vector field $-\sum_{i=1}^N \partial_{x_i}\Big(a(\cdot)|\partial_{x_i}|^{p_i-2}\partial_{x_i}u\Big)$, then results of Theorem 3.3 has been obtained in [6, Theorem 1.1].

If $\overline{p}(\cdot) = p_i(\cdot) = p(\cdot)$, $\mu = 0$, $\sigma_i(\cdot) = 0$, and $F(x,u) = b(x)|u|^{r(\cdot)-1}u$,then results of Theorem 3.3 coincide with regularity results of [24, Theorem 1].

If $\gamma(\cdot) \equiv 0$ and $\mu = 0$, then the results of Theorems 3.1-3.2-3.3-3.4 has been obtained in [5].

Remark 3.2 Let $f \in L^1(\Omega)$. Assume that for all $x \in \overline{\Omega}$, $\overline{p}(x) < N$, and

$$r(x) > \frac{(1+\sigma_i(x)-\gamma(x))N(\overline{p}(x)-1-\sigma_+^+ + \gamma^-)}{(1+\sigma_+^+ - \gamma^+)(N-\overline{p}(x))}$$

for all $i = 1, \ldots, N$. Then, assumption (3.10) implies (3.7) and

$$\frac{p_i(x)r(x)}{r(x)+\sigma_i(x)+1+\gamma(x)} > \frac{N(\overline{p}(x)-1-\sigma_+^+ + \gamma^-)p_i(x)}{\overline{p}(x)(N-1-\sigma_+^+ + \gamma^-)}, \quad \forall x \in \overline{\Omega},\ i = 1, \ldots, N.$$

So Theorem 3.3 improves Theorems 3.4-3.5 (and [5, Theorem 3.2]).

Remark 3.3 In Theorem 3.5. If m tends to be 1 and γ^- tends to be 0, then $\frac{Nm\left(\overline{p}(x)-1-\sigma_+^+ + \gamma^-\right)p_i(x)}{\overline{p}(x)\left(N-m(\sigma_+^+ + 1-\gamma^-)\right)}$ tends to be $\frac{N\left(\overline{p}(x)-1-\sigma_+^+\right)p_i(x)}{\overline{p}(x)\left(N-1-\sigma_+^+\right)}$ which is bound on q_i obtained in [5].

Remark 3.4 In Theorem 3.4, it is clear that the assumptions (1.10) and (3.10) imply that (2.4) holds since we have

$$\frac{\overline{p}(x)(N-1-\sigma_+^+ + \gamma^-)}{(\sigma_+^+ + 1-\gamma^-)(N-\overline{p}(x))} \le \overline{p}^\star(x),\ \forall x \in \overline{\Omega}.$$

Remark 3.5 Observe that the assumptions (1.11), (3.2), and (2.4) guarantee that

$$r(x) > (\sigma_+(x)+1+\gamma(x))(p_i(x)-1), \quad \forall x \in \overline{\Omega},\ i = 1, \ldots, N.$$

Remark 3.6 In Theorem 3.2, the assumptions (1.11) and (2.4) imply that the assumption (3.4) is not empty since we have

$$\frac{1}{m(x)-1} > p_i(x) - 1, \quad \forall x \in \overline{\Omega},\ i = 1, \ldots, N. \tag{3.17}$$

In order to prove our results, we will truncate the singular term $H(u)$ so that it becomes non-singular at the origin and study the behavior of a sequence u_n of solutions to the approximate problems.

Let $\{f_n\}_{n\in\mathbb{N}}$, $\{\mu_n\}_{n\in\mathbb{N}}$ be sequences of functions satisfying

$$\begin{cases} f_n := T_n(f), \quad \forall n \ge 1, \\ 0 \le \{f_n\}_{n\in\mathbb{N}} \in C_c^\infty(\Omega), \quad \|f_n\|_{L^1(\Omega)} \le \|f\|_{L^1(\Omega)}, \\ f_n \longrightarrow f \text{ strongly in } L^1(\Omega), \quad \text{as } n \to +\infty, \\ 0 \le \{\mu_n\}_{n\in\mathbb{N}} \in C_c^\infty(\Omega), \quad \|\mu_n\|_{L^1(\Omega)} \le C, \\ \mu_n \longrightarrow \mu \text{ as per Definition 3.1}, \quad \text{as } n \to +\infty. \end{cases} \tag{3.18}$$

Moreover, suppose that

$$H(0) = \lim_{t\to 0} H(t), \quad H_n(t) := \begin{cases} T_n(H(t)), & \text{for } t > 1, \quad \forall n \ge 1, \\ \min(H(0); n), & \text{otherwise}, \quad \forall n \ge 1, \end{cases} \tag{3.19}$$

Let us consider the following scheme of approximation

$$\begin{gathered} -\sum_{i=1}^N D^i\Big(b_i(x, T_n(u_n))a_i(x, Du_n)\Big) + F(x, u_n) = H_n(u_n)f_n + \mu_n \quad \text{in } \Omega, \\ u_n > 0 \quad \text{in } \Omega, \\ u_n = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.20}$$

First, we need to show the existence of a weak solution to the problem 3.20. The proof relies on the Schauder fixed-point theorem.

Theorem 3.6 *Suppose that μ is a nonnegative bounded Radon measure on Ω, and $f \in L^1(\Omega)$ is a nonnegative function with (1.2) holds. Assume that γ as in (1.3), $r : \overline{\Omega} \to (0, +\infty)$, $p_i : \overline{\Omega} \to (1, +\infty)$, and $\sigma_i : \overline{\Omega} \to [0, +\infty)$ $i = 1, \ldots, N$ are continuous functions such that (2.4) holds. Then, there exists at least one nonnegative weak solution $u_n \in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$ to problem 3.20 in the sense that*

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega b_i(x, T_n(u_n)) a_i(x, Du_n) D^i\phi \, dx + \int_\Omega F(x, u_n)\phi \, dx \\ &= \int_\Omega H_n(u_n) f_n \phi \, dx + \int_\Omega \mu_n \phi \, dx, \end{aligned} \tag{3.21}$$

for every $\phi \in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

In addition, there exists a constant $C > 0$ depending on $\|f\|_{L^1(\Omega)}$, $\|\mu\|_{L^1(\Omega)}$ but not on u_n, such that

$$\|F(\cdot, u_n)\|_{L^1(\Omega)} \le C, \tag{3.22}$$

and

$$\int_\Omega |u_n|^{r(x)} \, dx \le C. \tag{3.23}$$

Moreover, the sequence $(u_n)_n$ is increasing with respect to n, $u_n > 0$ in Ω, and for every $\omega \subset\subset \Omega$, there exists $C_\omega > 0$ (independent of n) satisfied

$$u_n(x) \ge C_\omega > 0, \text{ for every } x \in \omega, \text{ for every } n \in \mathbf{N}^\star. \tag{3.24}$$

In particular, there exists the pointwise limit u of the sequence u_n with u that satisfies (3.24). Furthermore, for all γ as in (1.3) and for all $\phi \in C_0^\infty(\Omega)$, we have

$$\int_\Omega H_n(u_n) f_n \phi \, dx \to \int_\Omega H(u) f \phi \, dx, \quad \text{as} \quad n \to +\infty. \tag{3.25}$$

Proof: Let $n \in \mathbb{N}$ be fixed, $v \in L^{\overline{p}^\star(\cdot)}(\Omega)$ and consider the following non singular problem

$$\begin{aligned} -\sum_{i=1}^{N} D^i\Big(b_i(x, T_n(w)) a_i(x, Dw)\Big) + T_k(F(x, w)) &= H_n(|v|) f_n + \mu_n \quad \text{in } \Omega, \\ w > 0 \quad &\text{in } \Omega, \\ w = 0 \quad &\text{on } \partial\Omega. \end{aligned} \tag{3.26}$$

It follows from [18] that problem 3.26 admits at least one solution $u_n \in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$. Furthermore, since the datum $H_n(v) f_n + \mu_n$ is bounded, we have that $w \in L^\infty(\Omega)$ and there exists a positive constant ν, independents of v and w (but possibly depending in n), such that $\|w\|_{L^\infty(\Omega)} \le \nu$.

Our aim is to prove the existence of fixed point of the map

$$\widetilde{G} : L^{\overline{p}^\star(\cdot)}(\Omega) \longrightarrow L^{\overline{p}^\star(\cdot)}(\Omega)$$

where $\widetilde{G}(v) = w \in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$, and w the weak solution of (3.26).

we know that $\widetilde{G}$ is compact if and only of it is continuous and it maps every bounded subset of $L^{\overline{p}^\star(\cdot)}(\Omega)$ into a relatively compact set.

we will show that $\widetilde{G}$ maps the ball $B_c(0) \subset L^{\overline{p}^\star(\cdot)}(\Omega)$ of radius C into itself, the constant C is independent of v.

Thanks to the regularity of the datum $H_n(v) f_n + \mu_n$, allows us to take w as test function in the weak formulation of (3.26) which gives

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega b_i(x, T_n(w)) a_i(x, D^i w) \times D^i w \, dx + \int_\Omega T_k(F(x, w)) w \, dx \\ &= \int_\Omega H_n(|v|) f_n \cdot w \, dx + \int_\Omega \mu_n w \, dx, \end{aligned}$$

By using (1.4), (1.7), (1.8), (1.9), (3.18), (3.19), and using that $T_k(F(x,w)) \geq 0$, we find

$$\frac{\alpha C_2}{(n^2 + C(n))(1+n)^{\sigma_+^+}} \sum_{i=1}^{N} \int_\Omega |D^i w|^{p_i(x)}\, dx \leq \int_\Omega |w|\, dx.$$

Using Young's inequality for all $\eta > 0$, we obtain

$$\begin{aligned} \sum_{i=1}^{N} \int_\Omega |D^i w|^{p_i(x)}\, dx &\leq \eta \int_\Omega |w|^{p_-^-}\, dx + C_3 \\ &\leq \eta C_4 \int_\Omega |D^i w|^{p_-^-}\, dx + C_3 \\ &\leq \eta C_4 \sum_{i=1}^{N} \int_\Omega |D^i w|^{p_i(x)}\, dx + C_5, \end{aligned}$$

where C_3, C_4, and C_5 are nonnegative constants. Now, we can choose $\eta = 1/(2C_4)$, which gives

$$\sum_{i=1}^{N} \int_\Omega |D^i w|^{p_i(x)}\, dx \leq C(n),$$

with $C(n)$ is independent from v.

Thanks to (2.4), (2.6) and Lemma 2.1, we have

$$\|w\|_{L^{\overline{p}^\star(\cdot)}(\Omega)} \leq C\|w\|_{\mathring{W}^{1,\vec{p}(\cdot)}(\Omega)} \leq C(n), \tag{3.27}$$

for some constant C_n independent on v.

In particular, we have that the ball $B := B_{C_n}(0)$ of $L^{\overline{p}^\star(\cdot)}(\Omega)$ of large enough radius C_n is invariant for the map $\widetilde{G}$. Moreover, from the compact Sobolev embedding, we deduce that $\widetilde{G}$ is continuous and compact on $L^{\overline{p}^\star(\cdot)}(\Omega)$.

Indeed, first we prove that the map $\widetilde{G}$ is continuous in B. Let us choose a sequence v_k that converges strongly to v in $L^{\overline{p}^\star(\cdot)}(\Omega)$, then the dominated convergence theorem gives

$$H_n(|v_k|)f_n + \mu_n \to H_n(|v|)f_n + \mu_n \quad \text{in} \quad L^{\overline{p}^\star(\cdot)}(\Omega),$$

then we need to prove that $\widetilde{G}(v_k)$ converge to $\widetilde{G}(v)$ in $L^{\overline{p}^\star(\cdot)}(\Omega)$. By compactness we already know that the sequence $w_k = \widetilde{G}(v_k)$ converge to some function w in $L^{\overline{p}^\star(\cdot)}(\Omega)$. So we only need to prove that $w = \widetilde{G}(v)$.
Since the sequence w_k is bounded in $\mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$ and by uniqueness, we deduce the desired. Second we need to check that the set $\widetilde{G}(B)$ is relatively compact, Let v_k be a bounded sequence in B. and let $w_k = \widetilde{G}(v_k)$. Analogously to (3.27), for any $v_k \in L^{\overline{p}^\star(\cdot)}(\Omega)$, we get

$$\|w_k\|_{L^{\overline{p}^\star(\cdot)}(\Omega)} = \|\widetilde{G}(v_k)\|_{L^{\overline{p}^\star(\cdot)}(\Omega)} \leq C_n,$$

for some constant C_n independent on v_k.

so that $w_k = \widetilde{G}(v_k)$ is relatively compact in $L^{\overline{p}^\star(\cdot)}(\Omega)$.

Thus, we can use Schauder's fixed point theorem to prove the existence of $u_n \in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$ solving the problem:

$$\begin{gathered} -\sum_{i=1}^{N} D^i\Big(b_i(x, T_n(u_n))a_i(x, Du_n)\Big) + F(x, u_n) = H_n(|u_n|)f_n + \mu_n \quad \text{in } \Omega, \\ u_n > 0 \quad \text{in } \Omega, \\ u_n = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.28}$$

Using as a test function $u_n^- = \min\{u_n, 0\}$, one has $u_n \geq 0$, (a classical regularity results as in [38]).

For the estimation (3.22), since $H_n(u_n)f_n + \mu_n \in L^\infty(\Omega)$. Then, the results in [18] provide us with the existence of a function $u \in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega) \cap L^{r(\cdot)}(\Omega)$, satisfying the weak formulation

$$\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega b_i(x, T_n(u_n))a_i(x, Du_n)D^i\varphi\, dx + \int_\Omega F(x, u_n)\varphi\, dx \\
&= \int_\Omega H_n(u_n)f_n\varphi\, dx + \int_\Omega \mu_n\varphi\, dx,
\end{aligned} \tag{3.29}$$

for every $\varphi \in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

Let $\nu > 0$, we define the function $\varrho_\nu(\cdot) : \mathbb{R} \longmapsto \mathbb{R}$ given by

$$\varrho_\nu(s) = \begin{cases} sign(s) & \text{if } |s| \geq \nu, \\ \frac{s}{\nu} & \text{if } |s| < \nu. \end{cases}$$

Taking $\varrho_\nu(u_n)$ in (3.29), by (1.2), (1.4),(1.7), and (1.9), we obtain

$$\int_{\{|u_n|\geq\nu\}} |F(\cdot, u_n)|\, dx \leq CM + C, \tag{3.30}$$

from this and (1.8), we get the result (3.22).

To prove (3.23), we choose $\phi = \frac{T_k(u_n)}{k}$ as a test function in (3.21), we obtain

$$\begin{aligned}
\int_\Omega |u_n|^{r(x)} \frac{T_k(u_n)}{k}\, dx &\leq \int_\Omega H_n(u_n)f_n \frac{T_k(u_n)}{k}\, dx + \int_\Omega \mu_n \frac{T_k(u_n)}{k}\, dx \\
&\leq \int_\Omega T_n(\frac{\widehat{M}}{u_n})f_n \frac{T_k(u_n)}{k}\, dx + \int_\Omega \mu_n \frac{T_k(u_n)}{k}\, dx \\
&\leq M\|f\|_{L^1(\Omega)} + C,
\end{aligned}$$

The assumption (3.18) and Fatou's lemma implies that estimate (3.23) holds as $k \to 0$.

Now, we will closely follow the proof of [Lemma 4.2, [16]] and of [Lemma 2, [24]] hence we will omit the details, giving only a sketch of the passages.

By (1.6), (1.3) and the fact that $0 \leq f_n \leq f_{n+1}$, we can prove that the sequence u_n is increasing with respect to n. knowing that, for $n \in \mathbf{N}^\star$ fixed, $u_n \in L^\infty(\Omega)$. So, in particular for $n = 1$, we have that

$$\begin{aligned}
-\sum_{i=1}^{N} D^i\Big(b_i(x, T_1(u_1))a_i(x, Du_1)\Big) + F(x, u_1) &= H_1(u_1)f_1 + \mu_1 \\
&\geq \frac{f_1}{(\|u_1\|_{L^\infty(\Omega)} + 1)^{\gamma(x)}} + \mu_1 \\
&\geq 0.
\end{aligned}$$

Since $\frac{f_1}{(\|u_1\|_{L^\infty(\Omega)}+1)^{\gamma(x)}} + \mu_1$ is not identically zero, we apply the strong maximum principle (As in [40]), which ensures that, for all $\omega \subset\subset \Omega$, there exists $C_\omega > 0$ (independent of n) such that

$$u_1(x) \geq C_\omega \text{ in } \omega.$$

Thus, (3.24) holds, because $u_n \geq u_1$ for all $n \in \mathbf{N}^\star$. Since u_n is increasing in n, we can define u as the pointwise limit of u_n. It follows that $u \geq u_n$ and by (3.24) we get, for every $\omega \subset\subset \Omega$, there exists $C_\omega > 0$ (independent of n) satisfied

$$u(x) \geq C_\omega > 0, \text{ for every } x \in \omega, \text{ for every } n \in \mathbf{N}^\star.$$

Observe that for all γ as in (1.3) and for all $\phi \in C_0^\infty(\Omega)$, if $\omega = \Big\{x \in \Omega : |\phi| > 0\Big\}$, we get

$$|H_n(u_n) f_n \phi| \leq \frac{\widehat{M} f \|\phi\|_{L^\infty(\Omega)}}{\min\left\{C_\omega^{\gamma^-}; C_\omega^{\gamma^+}\right\}} \in L^1(\Omega).$$

If $u = +\infty$ then $H(u) f \phi = 0$ and that, for $n \to +\infty$, we have

$$H_n(u_n) f_n \phi \to H(u) f \phi \quad \text{a.e. in} \quad \Omega.$$

Therefore, by Lebesgue theorem, it follows that (3.25) holds.
This concludes the proof. □

4. Uniform estimates

This section is devoted to obtaining some estimates for the sequence $(u_n)_n$ that do not depend on n. As we showed in the previous section, u_n belongs to $L^\infty(\Omega)$ for each fixed $n \in \mathbb{N}$. Consequently, this allows us to assume that u_n is a nonnegative solution of (3.20).

Lemma 4.1 *Let $p_i(\cdot)$, $\sigma_i(\cdot)$, $r(\cdot)$, $\gamma(\cdot)$ and $m(\cdot)$ be restricted as in Theorem 3.1. Then, there exists a constant $C > 0$ such that*

$$\sum_{i=1}^N \int_\Omega |D^i u_n|^{p_i(x)}\, dx + \int_\Omega |u_n|^{r(x)+1+\sigma_+(x)+\gamma(x)}\, dx \leq C. \tag{4.1}$$

Proof: Inserting $\varphi(x, u_n) = \big((1 + |u_n|)^{1+\sigma_+(x)+\gamma(x)} - 1\big)\, \mathrm{sign}(u_n)$ into (3.20), gives

$$\begin{aligned}&\sum_{i=1}^N \int_\Omega b_i(x, T_n(u_n)) a_i(x, Du_n) D^i \varphi\, dx + \int_\Omega F(x, u_n) \varphi\, dx \\ &= \int_\Omega H_n(u_n) f_n \varphi\, dx + \int_\Omega \mu_n \varphi\, dx.\end{aligned}$$

From (1.2), (1.4), (1.5), (1.7), (1.9), and the fact that for a.e. $x \in \Omega$, for all $i = 1, \dots, N$,

$$\begin{aligned}D^i \varphi(x, u_n) = {} & D^i \sigma_+(x) (1 + |u_n|)^{1+\sigma_+(x)+\gamma(x)} \ln(1 + |u_n|)\, \mathrm{sign}(u_n) \\ & + (1 + \sigma_+(x) + \gamma(x))(1 + |u_n|)^{\sigma_+(x)+\gamma(x)} D^i u_n \\ & + D^i \gamma(x) (1 + |u_n|)^{1+\sigma_+(x)+\gamma(x)} \ln(1 + |u_n|)\, \mathrm{sign}(u_n),\end{aligned}$$

we obtain

$$\begin{aligned}&\alpha C_2 (1 + \sigma_+^- + \gamma^-) \sum_{i=1}^N \int_\Omega (1 + |u_n|)^{\gamma(x)} |D_i u_n|^{p_i(x)}\, dx + \int_\Omega |u_n|^{r(x)} \left((1 + |u_n|)^{1+\sigma_+(x)+\gamma(x)} - 1\right) dx \\ &\leq M \int_\Omega |f| \left((1 + |u_n|)^{1+\sigma_+(x)+\gamma(x)} - 1\right) dx + \int_\Omega |\mu_n| \left((1 + |u_n|)^{1+\sigma_+(x)+\gamma(x)} - 1\right) dx \\ &\quad + C\beta \sum_{i=1}^N \int_\Omega (1 + |u_n|)^{1+\sigma_+(x)+\gamma(x)} \ln(1 + |u_n|) \cdot \left(1 + \sum_{j=1}^N |D_j u_n|^{p_j(x)}\right)^{1-\frac{1}{p_i(x)}} dx,\end{aligned}$$

where $\sigma_+^- := \max\limits_{1 \leq i \leq N} \min\limits_{x \in \overline{\Omega}} \sigma_i(x)$ and $\gamma^- := \min\limits_{x \in \overline{\Omega}} \gamma(x)$.

Using that $|u_n|^{r(x)} \geq \min\{1, 2^{1-r^+}\}(1+|u_n|)^{r(x)} - 1$, Proposition 2.1, and Young's inequality and since $m'(\cdot)(1+\sigma_+(x)+\gamma(x)) \leq r(x)+1+\sigma_+(x)+\gamma(x)$, we have

$$\begin{aligned} &\sum_{i=1}^{N}\int_\Omega (1+|u_n|)^{\gamma(x)}|D_i u_n|^{p_i(x)} + \frac{1}{2}\int_\Omega (1+|u_n|)^{r(x)+1+\sigma_+(x)+\gamma(x)}\,dx \\ &\leq C + C\max\left(\|f\|^{m^+}_{L^{m(\cdot)}(\Omega)}, \|f\|^{m^-}_{L^{m(\cdot)}(\Omega)}\right) \\ &\quad + C\beta\sum_{i=1}^{N}\int_\Omega (1+|u_n|)^{1+\sigma_+(x)+\gamma(x)}\ln(1+|u_n|)\cdot\left(1+\sum_{j=1}^{N}|D_j u_n|^{p_j(x)}\right)^{1-\frac{1}{p_i(x)}}\,dx, \end{aligned} \tag{4.2}$$

We can estimate the last term in (4.2) by applying Young's inequality,

$$\begin{aligned} &(1+|u_n|)^{1+\sigma_+(x)+\gamma(x)}\ln(1+|u_n|)\times\left(1+\sum_{j=1}^{N}|D_j u_n|^{p_j(x)}\right)^{1-\frac{1}{p_i(x)}} \\ &= (1+|u_n|)^{\sigma_+(x)+1+\frac{\gamma(x)}{p_i(x)}}\ln(1+|u_n|) \\ &\times\left(1+\sum_{j=1}^{N}|D_j u_n|^{p_j(x)}\right)^{1-\frac{1}{p_i(x)}}\quad(1+|u_n|)^{\frac{\gamma(x)(p_i(x)-1)}{p_i(x)}} \\ &\leq C(1+|u_n|)^{\gamma(x)+p_i(x)(\sigma_+(x)+1)}(\ln(1+|u_n|))^{p_i(x)} + \frac{1}{4C\beta}(1+|u_n|)^{\gamma(x)} \\ &+ \frac{1}{4C\beta}\sum_{i=1}^{N}(1+|u_n|)^{\gamma(x)}|D_i u_n|^{p_i(x)}, \end{aligned} \tag{4.3}$$

We combine (4.2) and (4.3) to obtain

$$\begin{aligned} &\sum_{i=1}^{N}\int_\Omega (1+|u_n|)^{\gamma(x)}|D_i u_n|^{p_i(x)}\,dx + \int_\Omega (1+|u_n|)^{r(x)+1+\sigma_+(x)+\gamma(x)}\,dx \\ &\leq C + C\sum_{i=1}^{N}\int_\Omega (1+|u_n|)^{\gamma(x)+p_i(x)(\sigma_+(x)+1)}\ln(1+|u_n|)^{p_i(x)}\,dx = I. \end{aligned} \tag{4.4}$$

Thanks to Remark 3.5, we have $r(x) > (\sigma_+(x)+1+\gamma(x))(p_i(x)-1)$, so

$$(\sigma_+(x)+1+\gamma(x))(p_i(x)-1) - r(x) \leq ((\sigma_+(x)+1+\gamma(x))(p_i(x)-1) - r(x))^+ = e_i < \frac{e_i}{2} < 0,$$

and $(1+|u_n|)^{(\sigma_+(x)+1+\gamma(x))(p_i(x)-1)-r(x)-\frac{e_i}{2}}\ln(1+|u_n|)^{p_i(x)}$ is bounded for all $x \in \overline{\Omega}$. We write

$$\begin{aligned} &(1+|u_n|)^{\gamma(x)+p_i(x)(\sigma_+(x)+1)}\ln(1+|u_n|)^{p_i(x)} \\ &= (1+|u_n|)^{r(x)+1+\sigma_+(x)+\gamma(x)+\frac{e_i}{2}}(1+|u_n|)^{(\sigma_+(x)+1+\gamma(x))(p_i(x)-1)-r(x)-\frac{e_i}{2}}\ln(1+|u_n|)^{p_i(x)}. \end{aligned}$$

By another application of Young's inequality, we obtain

$$I \leq \frac{1}{4}\int_\Omega (1+|u_n|)^{r(x)+1+\sigma_+(x)+\gamma(x)}\,dx + C. \tag{4.5}$$

Using (4.4) and (4.5), we obtain (4.1). □

Lemma 4.2 *Suppose that the assumptions of Theorem 3.2 are satisfied. Then, there exists a constant $C > 0$ such that*

$$\sum_{i=1}^{N}\int_\Omega \frac{|D^i u_n|^{p_i(x)}}{(1+|u_n|)^{\sigma_i(x)+1-\gamma(x)-(m(x)-1)r(x)}}\,dx + \int_\Omega |u_n|^{\gamma(x)+m(x)r(x)}\,dx \leq C. \tag{4.6}$$

Proof: We take $\phi(x,u_n)=\big((1+|u_n|)^{\gamma(x)+(m(x)-1)r(x)}-1\big)\operatorname{sign}(u_n)$ as a test function in (3.20), by (1.2), (1.4), (1.5), (1.7), (1.9), and the fact that for a.e. $x\in\Omega$ and for all $i=1,\dots,N$

$$\begin{aligned} D_i\phi(x,u_n) &= D_i\gamma(x)(1+|u_n|)^{\gamma(x)+(m(x)-1)r(x)}\operatorname{sign}(u_n)\\ &\quad + D_i m(x)(1+|u_n|)^{\gamma(x)+(m(x)-1)r(x)}\operatorname{sign}(u_n) r(x)\ln(1+|u_n|)\\ &\quad + (m(x)-1)(1+|u_n|)^{\gamma(x)+(m(x)-1)r(x)}\operatorname{sign}(u_n) D_i r(x)\ln(1+|u_n|)\\ &\quad + \frac{(m(x)-1)r(x)D_iu_n}{(1+|u_n|)^{1-\gamma(x)-(m(x)-1)r(x))}}, \end{aligned}$$

we obtain

$$\begin{aligned} &\alpha r^-(m^- -1)C_2\sum_{i=1}^N\int_\Omega \frac{|D_iu_n|^{p_i(x)}}{(1+|u_n|)^{\sigma_i(x)+1-\gamma(x)-(m(x)-1)r(x)}}\,dx\\ &+\int_\Omega |u_n|^{r(x)}\Big((1+|u_n|)^{\gamma(x)+(m(x)-1)r(x)}-1\Big)\,dx\\ &\le M\int_\Omega |f|\Big((1+|u_n|)^{\gamma(x)+(m(x)-1)r(x)}-1\Big)\,dx+\int_\Omega|\mu_n|\Big((1+|u_n|)^{\gamma(x)+(m(x)-1)r(x)}-1\Big)\,dx\\ &\quad + C_{15}\beta\sum_{i=1}^N\int_\Omega(1+|u_n|)^{\gamma(x)+(m(x)-1)r(x)}\ln(1+|u_n|)\cdot\left(1+\sum_{j=1}^N|D_ju_n|^{p_j(x)}\right)^{1-\frac{1}{p_i(x)}}dx, \end{aligned}$$

By dropping the positif term, the fact that $|u_n|^{r(\cdot)}\ge 2^{1-r^+}(1+|u_n|)^{r(\cdot)}-1$, Proposition 2.1, and Young inequality, we find

$$\begin{aligned} &\sum_{i=1}^N\int_\Omega \frac{|D_iu_n|^{p_i(x)}}{(1+|u_n|)^{\sigma_i(x)+1-\gamma(x)-(m(x)-1)r(x)}}\,dx+\frac12\int_\Omega(1+|u_n|)^{\gamma(x)+m(x)r(x)}\,dx\\ &\le C_{16}+C_{16}\max\big(\|f\|^{m^+}_{L^{m(\cdot)}(\Omega)},\|f\|^{m^-}_{L^{m(\cdot)}(\Omega)}\big)\\ &\quad + C_{16}\beta\sum_{i=1}^N\int_\Omega(1+|u_n|)^{\gamma(x)+(m(x)-1)r(x)}\ln(1+|u_n|).\left(1+\sum_{j=1}^N|D_ju_n|^{p_j(x)}\right)^{1-\frac{1}{p_i(x)}}dx, \end{aligned}\tag{4.7}$$

We can estimate the last term in (4.7) by application of Young's inequality

$$\begin{aligned} &(1+|u_n|)^{\gamma(x)+(m(x)-1)r(x)}\ln(1+|u_n|)\times\left(1+\sum_{j=1}^N|D_ju_n|^{p_j(x)}\right)^{1-\frac{1}{p_i(x)}}\\ &=(1+|u_n|)^{\sigma_i(x)+1-\frac{(1-\gamma(x)-(m(x)-1)r(x)+\sigma_i(x))}{p_i(x)}}\ln(1+|u_n|)\\ &\times\left(1+\sum_{j=1}^N|D_ju_n|^{p_j(x)}\right)^{1-\frac{1}{p_i(x)}}(1+|u_n|)^{-\frac{(1-\gamma(x)-(m(x)-1)r(x)+\sigma_i(x))(p_i(x)-1)}{p_i(x)}}\\ &\le C_{17}(1+|u_n|)^{\gamma(x)+(m(x)-1)r(x)+(p_i(x)-1)(\sigma_i(x)+1)}(\ln(1+|u_n|))^{p_i(x)}+\frac{1}{4C_{16}\beta}\\ &+\frac{1}{4C_{16}\beta}\sum_{i=1}^N\frac{|D_iu_n|^{p_i(x)}}{(1+|u_n|)^{\sigma_i(x)+1-\gamma(x)-(m(x)-1)r(x)}}, \end{aligned}\tag{4.8}$$

We combine (4.7) and (4.8), we obtain

$$\begin{aligned} &\frac{3}{4}\sum_{i=1}^{N}\int_\Omega \frac{|D_iu_n|^{p_i(x)}}{(1+|u_n|)^{\sigma_i(x)+1-\gamma(x)-(m(x)-1)r(x)}}\,dx+\frac{1}{2}\int_\Omega (1+|u_n|)^{\gamma(x)+m(x)s(x)}\,dx\\ &\le C_{18}+C_{19}\sum_{i=1}^{N}\int_\Omega (1+|u_n|)^{\gamma(x)+(m(x)-1)r(x)+(p_i(x)-1)(\sigma_i(x)+1)}\\ &\quad\times \ln(1+|u_n|)^{p_i(x)}\,dx=J. \end{aligned} \tag{4.9}$$

In view of (3.4), we observe that

$$(p_i(x)-1)(\sigma_i(x)+1)<(p_i(x)-1)(\sigma_i(x)+1+\gamma(x))<r(x),\ \text{for all } x\in\overline{\Omega} \text{ and for all } i=1,\dots,N.$$

which yield,

$$(p_i(x)-1)(\sigma_i(x)+1)-r(x)\le \Big((p_i(x)-1)(\sigma_i(x)+1)-r(x)\Big)^+=d_i<\frac{d_i}{2}<0.$$

so that, $(1+|u_n|)^{(p_i(x)-1)(\sigma_i(x)+1)-r(x)-\frac{d_i}{2}}\ln(1+|u_n|)^{p_i(x)}$ is bounded for all $x\in\overline{\Omega}$. This concludes that

$$\begin{aligned} &(1+|u_n|)^{\gamma(x)+(m(x)-1)r(x)+(p_i(x)-1)(\sigma_i(x)+1)}\ln(1+|u_n|)^{p_i(x)}=(1+|u_n|)^{\gamma(x)+m(x)r(x)+\frac{d_i}{2}}\\ &\times(1+|u_n|)^{(p_i(x)-1)(\gamma_i(x)+1)-r(x)-\frac{d_i}{2}}\ln(1+|u_n|)^{p_i(x)}\\ &\le C_{20}(1+|u_n|)^{\gamma(x)+m(x)r(x)+\frac{d_i}{2}}. \end{aligned}$$

We get by Young's inequality,

$$J\le \frac{1}{8}\int_\Omega (1+|u_n|)^{\gamma(x)+m(x)r(x)}\,dx+C_{21}. \tag{4.10}$$

Finally by (4.9) and (4.10), we obtain the estimation (4.6). □

Lemma 4.3 *Let $p_i(\cdot)$, $\sigma_i(\cdot)$, $r(\cdot)$, $\gamma(\cdot)$ and $m(\cdot)$ be restricted as in Theorem 3.2. Then, there exists a constant $C>0$ such that*

$$\|D^iu_n\|_{L^{q_i(\cdot)}(\Omega)}\le C, \tag{4.11}$$

for all continuous functions $q_i(\cdot)$ on $\overline{\Omega}$ satisfying (3.5).

Proof: Remark that $r(x)<\frac{\sigma_+(x)+1+\gamma(x)}{m(x)-1}$ and (3.5) imply $q_i(x)<p_i(x)$. Using Young's inequality, we obtain

$$\begin{aligned} &\int_\Omega |D_iu_n|^{q_i(x)}\,dx\\ &=\int_\Omega \frac{|D_iu_n|^{q_i(x)}}{(1+|u_n|)^{(\sigma_i(x)+1-\gamma(x)-(m(x)-1)r(x))\frac{q_i(x)}{p_i(x)}}}(1+|u_n|)^{(\sigma_i(x)+1-\gamma(x)-(m(x)-1)r(x))\frac{q_i(x)}{p_i(x)}}\,dx\\ &\le \int_\Omega \Big(\frac{q_i(x)}{p_i(x)}\Big)\frac{|D_iu_n|^{p_i(x)}}{(1+|u_n|)^{\sigma_i(x)+1-\gamma(x)-(m(x)-1)r(x)}}\,dx\\ &\quad+\int_\Omega\Big(1-\frac{q_i(x)}{p_i(x)}\Big)(1+|u_n|)^{\frac{(\sigma_i(x)+1-\gamma(x)-(m(x)-1)r(x))q_i(x)}{p_i(x)-q_i(x)}}\,dx, \end{aligned}$$

and by (3.5) , we get

$$\begin{aligned} &\int_\Omega |D_iu_n|^{q_i(x)}\,dx\\ &\le C_1\int_\Omega \frac{|D_iu_n|^{p_i(x)}}{(1+|u_n|)^{\sigma_i(x)+1-\gamma(x)-(m(x)-1)r(x)}}\,dx+C_2\int_\Omega (1+|u_n|)^{\gamma(x)+m(x)s(x)}\,dx. \end{aligned} \tag{4.12}$$

Furthermore, by (4.12) and (4.6), we deduce the desert result. □

Lemma 4.4 *Let $p_i : \overline{\Omega} \to (1, \infty)$, $\sigma_i : \overline{\Omega} \to [0, \infty)$, $i = 1, \cdots, N$, $r : \overline{\Omega} \to (0, \infty)$, and $\gamma : \overline{\Omega} \to (0, 1)$ be continuous functions. Then, there exists a constant $C > 0$ such that*

$$\sum_{i=1}^{N} \int_\Omega \frac{|D^i u_n|^{p_i(x)}}{(1 + |u_n|)^{\sigma_i(x)+\gamma^- + \nu}}\, dx \leq C, \quad \forall \nu > 1, \tag{4.13}$$

$$\int_\Omega |D^i T_k(u_n)|^{p_i(x)}\, dx \leq \frac{k}{\alpha C_2}(1 + k)^{\sigma_+^+}\Big(M\|f\|_{L^1(\Omega)} + \|\mu\|_{L^1(\Omega)}\Big), \quad i = 1, \dots, N. \tag{4.14}$$

Proof: Let $\nu > 1$, we define the function $\varrho_\nu(\cdot) : \mathbb{R} \longmapsto \mathbb{R}$ given by

$$\varrho_\nu(t) = \int_0^t \frac{dx}{(1 + |x|)^{\gamma^- + \nu}},$$

It is clear that

$$\varrho_\nu(t) = \frac{1}{1 - \gamma^- - \nu}\Big((1 + |t|)^{1-\gamma^- - \nu} - 1\Big)\, \mathrm{sign}(t).$$

Note that ϱ_ν is a continuous function satisfies $\varrho_\nu(0) = 0$ and $|\varrho_\nu'(\cdot)| \leq 1$. We take $\varrho_\nu(u_n)$ as a test function in (3.21) and using the assumptions (1.2), (1.4), and (1.7), we obtain

$$\sum_{i=1}^{N} \int_\Omega \frac{|D^i u_n|^{p_i(x)}}{(1 + |u_n|)^{\sigma_i(x)+\gamma^- + \nu}}\, dx \leq \frac{1}{\alpha C_2}\Big(M\|f_n\|_{L^1(\Omega)} + \|\mu_n\|_{L^1(\Omega)}\Big).$$

Thanks to this estimate, (3.18), and (3.19), we get (4.13).

In particular, there exists $C > 0$ such that

$$\int_\Omega \frac{|D_i u_n|^{p_i^-}}{(1 + |u_n|)^{\sigma_+^+ + \gamma^- + \nu}}\, dx \leq C_3, \quad \forall i = 1, \dots, N. \tag{4.15}$$

Taking $\varphi = T_k(u_n)$ in (3.21), we get

$$\int_\Omega \frac{|D^i T_k(u_n)|^{p_i(x)}}{(1 + |u_n|)^{\sigma_i(x)}}\, dx \leq \frac{k}{\alpha C_2}\Big(M\|f\|_{L^1(\Omega)} + \|\mu_n\|_{L^1(\Omega)}\Big).$$

So that

$$\begin{aligned} \int_\Omega |D^i T_k(u_n)|^{p_i(x)}\, dx &= \int_\Omega \frac{|D^i T_k(u_n)|^{p_i(x)}}{(1 + |T_k(u_n)|)^{\sigma_i(x)}}(1 + |T_k(u_n)|)^{\sigma_i(x)}\, dx \\ &\leq \frac{k}{\alpha C_2}(1 + k)^{\sigma_+^+}\Big(M\|f\|_{L^1(\Omega)} + \|\mu\|_{L^1(\Omega)}\Big). \end{aligned}$$

which yields (4.14). □

The summability of u depends on the summability of f , which is stated in the next lemma.

Lemma 4.5 *Let $p_i(\cdot)$, $\sigma_i(\cdot)$, $r(\cdot)$ and $\gamma(\cdot)$ be restricted as in Theorem 3.3. Then, there exists a constant $C > 0$ such that*

$$\|D^i u_n\|_{L^{q_i(\cdot)}(\Omega)} \leq C, \tag{4.16}$$

for all continuous functions $q_i(\cdot)$ on $\overline{\Omega}$ satisfying (3.8).

Proof: Observe that (3.8) implies that $q_i(x) < p_i(x)$ for all $x \in \overline{\Omega}$, $i = 1, \dots, N$. Let us write

$$\int_\Omega |D_i u_n|^{q_i(x)}\, dx = \int_\Omega \frac{|D_i u_n|^{q_i(x)}}{(1 + |u_n|)^{\frac{q_i(x)}{p_i(x)}(\sigma_i(x)+\gamma^- + \nu)}}(1 + |u_n|)^{\frac{q_i(x)}{p_i(x)}(\sigma_i(x)+\gamma^- + \nu)}\, dx.$$

Then, by (4.13) and Young's inequality, we obtain

$$\int_\Omega |D_i u_n|^{q_i(x)}\,dx \le C_1 + C_2 \int_\Omega (1+|u_n|)^{\frac{q_i(x)(\sigma_i(x)+\gamma^-+\nu)}{p_i(x)-q_i(x)}}\,dx. \tag{4.17}$$

The condition (3.8) guarantees that $\frac{r(x)(p_i(x)-q_i(x))}{q_i(x)} - \sigma_i(x) - \gamma^- > 1$. Choosing

$$\nu = \min_{1\le i\le N}\min_{x\in\overline{\Omega}}\Big(\frac{r(x)(p_i(x)-q_i(x))}{q_i(x)} - \sigma_i(x) - \gamma^-\Big) > 1\,.$$

Again, thanks to the choice of ν and (3.8), we find

$$\frac{q_i(x)(\sigma_i(x)+\gamma^-+\nu)}{p_i(x)-q_i(x)} \le r(x), \quad \forall x\in\overline{\Omega},\ \forall i=1,\dots,N. \tag{4.18}$$

Hence, it follows from (4.17), (4.18), and (3.23) that (4.16) holds. □

Lemma 4.6 *Suppose that the assumptions of Theorem 3.4 are satisfied. Then, there exists a constant $C>0$ such that for all continuous functions $q_i(\cdot)$, $i=1,\dots,N$ on $\overline{\Omega}$ as in (3.12), we have*

$$\|D^i u_n\|_{L^{q_i(\cdot)}(\Omega)} \le C, \tag{4.19}$$

$$\|u_n\|_{L^{\overline{q}^\star(\cdot)}(\Omega)} \le C. \tag{4.20}$$

Proof: First of all, the condition (3.10) give us

$$1 < \frac{N(\overline{p}(x)-1-\sigma_+^+ +\gamma^-)p_i(x)}{\overline{p}(x)(N-1-\sigma_+^+ +\gamma^-)}, \quad \forall x\in\overline{\Omega},$$

Thanks to (3.12) and (1.10), we get $q_i(x) < p_i(x)$ for all $x\in\overline{\Omega}$, $i=1,\dots,N$.

Case (a): In the first step, let q_i^+ be a constant satisfying

$$q_i^+ < \frac{N(\overline{p}^- -1-\sigma_+^+ +\gamma^-)p_i^-}{\overline{p}^-(N-1-\sigma_+^+ +\gamma^-)}, \quad i=1,\dots,N,\ \frac{1}{\overline{p}^-} = \frac{1}{N}\sum_{i=1}^N \frac{1}{p_i^-}. \tag{4.21}$$

We can assume that $\frac{q_i^+}{p_i^-} = \frac{\overline{q}^+}{\overline{p}^-}$ where $\frac{1}{\overline{q}^+} = \frac{1}{N}\sum_{i=1}^N \frac{1}{q_i^+}$. If not, we set

$$\Theta = \max\{q_i^+/p_i^-,\ \ i=1,\dots,N\}$$

and replace q_i^+ by Θp_i^-. Observe that, since $\Theta p_i^- \ge q_i^+$, the fact that $(D_i u_n)$ remains in a bounded set of $L^{\Theta p_i^-}(\Omega)$ implies the result.

From now on, we set $q_i^+ = \Theta p_i^-$, $\Theta = \frac{\overline{q}^+}{\overline{p}^-} \in (0, \frac{N(\overline{p}^- -1-\sigma_+^+ +\gamma^-)}{\overline{p}^-(N-1-\sigma_+^+ +\gamma^-)}) \subseteq (0,1)$. So (4.21) is equivalent to

$$(\frac{1-\Theta}{\Theta})\overline{q}^{+\star} - \sigma_+^+ + \gamma^- > 1, \quad \overline{q}^{+\star} = \frac{N\overline{q}^+}{N-\overline{q}^+}.$$

This implies that there exists $\tau>1$ such that

$$(\frac{1-\Theta}{\Theta})\overline{q}^{+\star} - \sigma_+^+ + \gamma^- > \tau > 1.$$

That is

$$(\sigma_+^+ - \gamma^- + \tau)(\frac{\Theta}{1-\Theta}) < \overline{q}^{+\star}. \tag{4.22}$$

Using Hölder's inequality and (4.15), we obtain

$$\begin{aligned}\int_\Omega |D_i u_n|^{q_i^+}\,dx &= \int_\Omega \frac{|D_i u_n|^{q_i^+}}{(1+|u_n|)^{(\sigma_+^+-\gamma^-+\tau)\Theta}}(1+|u_n|)^{(\sigma_+^+-\gamma^-+\tau)\Theta}\,dx\\ &\le \Big(\int_\Omega \frac{|D_i u_n|^{p_i^-}}{(1+|u_n|)^{(\sigma_+^+-\gamma^-+\tau)}}\,dx\Big)^{\Theta}\Big(\int_\Omega (1+|u_n|)^{(\sigma_+^+-\gamma^-+\tau)\frac{\Theta}{1-\Theta}}\Big)^{1-\Theta}\\ &\le C_4\Big(\int_\Omega (1+|u_n|)^{(\sigma_+^+-\gamma^-+\tau)\frac{\Theta}{1-\Theta}}\Big)^{1-\Theta},\end{aligned}$$

It follows that

$$\prod_{i=1}^N \Big(\|D_i u_n\|_{L^{q_i^+}(\Omega)}\Big)^{1/N} \le C_4^{\frac{1}{\overline{q}^+}}\Big(\int_\Omega (1+|u_n|)^{(\sigma_+^+-\gamma^-+\tau)\frac{\Theta}{1-\theta}}\Big)^{\frac{1-\Theta}{\overline{q}^+}}.$$

By this estimate, (4.22), and Young's inequality, we get

$$\prod_{i=1}^N \Big(\|D_i u_n\|_{L^{q_i^+}(\Omega)}\Big)^{1/N} \le C_5(\varepsilon) + \varepsilon\Big(\int_\Omega |u_n|^{\overline{q}^{+\star}}\Big)^{\frac{1-\Theta}{\overline{q}^+}}. \tag{4.23}$$

Due to the anisotropic Sobolev inequality (2.7) with $s = \overline{q}^{+\star}$, we see that

$$\|u_n\|_{L^{\overline{q}^{+\star}}(\Omega)} \le C_6 \prod_{i=1}^N \Big(\|D_i u_n\|_{L^{q_i^+}(\Omega)}\Big)^{1/N} \le C_7(\varepsilon) + \varepsilon C_6\big(\|u_n\|_{L^{\overline{q}^{+\star}}(\Omega)}\big)^{\frac{N(1-\Theta)}{N-\overline{q}^+}}. \tag{4.24}$$

Now, choose $\varepsilon = 1/(2C_6)$, we obtain

$$\|u_n\|_{L^{\overline{q}^{+\star}}(\Omega)} \le C_8 + \frac{1}{2}\|u_n\|^{\nu}_{L^{\overline{q}^{+\star}}(\Omega)}, \quad \nu = (1-\Theta)\frac{N}{N-\overline{q}^+}. \tag{4.25}$$

The condition (1.10) guarantees that $\nu \in (0,1)$. Consequently, the estimate (4.25) implies (4.20). Therefore, by the estimate (4.23), we deduce that (4.19) holds. This completes the proof of the case (a).
Case (b): In the second, we suppose that (3.12) holds and

$$q_i^+ \ge \frac{N(\overline{p}^- - 1 - \sigma_+^+ + \gamma^-)p_i^-}{\overline{p}^-(N-1-\sigma_+^+ + \gamma^-)},$$

By the continuity of $p_i(\cdot)$ and $q_i(\cdot)$ on $\overline{\Omega}$, there exists a constant $\lambda > 0$ such that

$$\max_{z\in B(x,\lambda)\cap\Omega} q_i(z) < \min_{z\in B(x,\lambda)\cap\Omega} \frac{N(\overline{p}(z)-1-\sigma_+^+ + \gamma^-)p_i(z)}{\overline{p}(z)(N-1-\sigma_+^+ + \gamma^-)}, \quad \forall\, x \in \Omega, \tag{4.26}$$

where $B(x,\lambda)$ is a cube with center x and diameter λ. Note that $\overline{\Omega}$ is compact and therefore we can cover it with a finite number of cubes $(B_j)_{j=1,\dots,k}$ with edges parallel to the coordinate axes. Moreover, there exists a constant $\eta > 0$ such that

$$\lambda > |\Omega_j| = \text{meas}(\Omega_j) > \eta, \quad \Omega_j = B_j \cap \Omega \quad \text{for all } j = 1,\dots,k.$$

We denote by $q_{i,j}^+$ the local maximum of $q_i(\cdot)$ on $\overline{\Omega_j}$ (respectively $p_{i,j}^-$ the local minimum of $p_i(\cdot)$ on $\overline{\Omega_j}$), such that

$$q_{i,j}^+ < \frac{N(\overline{p}_j^- - 1 - \sigma_+^+ + \gamma^-)p_{i,j}^-}{\overline{p}_j^-(N-1-\sigma_+^+ + \gamma^-)} \quad \text{for all } j = 1,\dots,k,\ \frac{1}{\overline{p}_j^-} = \frac{1}{N}\sum_{i=1}^N \frac{1}{p_{i,j}^-}. \tag{4.27}$$

By the anisotropic Sobolev inequality (2.8), we have

$$\|u_n\|_{L^{\overline{q}_j^{+\star}}(\Omega_j)} \le C_9 \prod_{i=1}^N \Big(\|u_n\|_{L^{q_{i,j}^+}(\Omega_j)} + \|D_i u_n\|_{L^{q_{i,j}^+}(\Omega_j)}\Big)^{1/N}. \tag{4.28}$$

Using (3.9), (3.23), (4.28), and the fact that $q_{i,j}^{+} < p_{i,j}^{-} \leq r_j^{-} = \min_{x\in\overline{\Omega_j}} r(x)$, we derive the estimate

$$\|u_n\|_{L^{\overline{q}_j^{+\star}}(\Omega_j)} \leq C_{10} \prod_{i=1}^{N} \Big(1 + \|D_i u_n\|_{L^{q_{i,j}^{+}}(\Omega_j)}\Big)^{1/N}. \tag{4.29}$$

Moreover, arguing locally as in (4.23) and (4.24), we get

$$\|u_n\|_{L^{\overline{q}_j^{+\star}}(\Omega_j)} \leq C_{10} \prod_{i=1}^{N} \Big(1 + \|D_i u_n\|_{L^{q_{i,j}^{+}}(\Omega_j)}\Big)^{1/N} \leq C_{11} + \frac{1}{2}\|u_n\|^{\nu_j}_{L^{\overline{q}_j^{+\star}}(\Omega_j)}, \tag{4.30}$$

with

$$\nu_j = \Big(1 - \frac{\overline{q}_j^{+}}{\overline{p}_j^{-}}\Big)\frac{N}{N - \overline{q}_j^{+}},$$

The assumption (1.10) implies that $\nu_j \in (0,1)$. Hence, it follows from (4.30) that

$$\begin{aligned} &\int_{\Omega_j} |u_n|^{\overline{q}_j^{+\star}}\, dx \leq C_{12} \quad \text{for } j = 1, \dots, k,\\ &\int_{\Omega_j} |D_i u_n|^{q_{i,j}^{+}}\, dx \leq C_{13} \quad \text{for } j = 1, \dots, k. \end{aligned} \tag{4.31}$$

Since $q_i(x) \leq q_{i,j}^{+}$ and $\overline{q}^{\star}(x) \leq \overline{q}_j^{+\star}$ for all $x \in \overline{\Omega}_j$ and for $j = 1, \dots, k$, we conclude that

$$\int_{\Omega_j} |u_n|^{\overline{q}^{\star}(x)}\, dx + \int_{\Omega_j} |D_i u_n|^{q_i(x)}\, dx \leq C_{14},$$

So that

$$\int_{\Omega} |u_n|^{\overline{q}^{\star}(x)}\, dx + \int_{\Omega} |D_i u_n|^{q_i(x)}\, dx \leq \sum_{j=1}^{k}\Big(\int_{\Omega_j} |u_n|^{\overline{q}^{\star}(x)}\, dx + \int_{\Omega_j} |D_i u_n|^{q_i(x)}\, dx\Big) \leq C.$$

where C is a constant independent of n. Thus, the proof is finished. □

Lemma 4.7 *Let $p_i(\cdot)$, $\sigma_i(\cdot)$, $r(\cdot)$, $m(\cdot)$ and $\gamma(\cdot)$ be restricted as in Theorem 3.5. Then, there exists a constant $C > 0$ such that for all continuous functions $q_i(\cdot)$, $i = 1, \dots, N$ on $\overline{\Omega}$ as in (3.16), we have*

$$\begin{aligned} \|D^i u_n\|_{L^{q_i(\cdot)}(\Omega)} &\leq C,\\ \|u_n\|_{L^{\overline{q}^{\star}(\cdot)}(\Omega)} &\leq C. \end{aligned}$$

Proof: The proof is based on the similar arguments as in the proof of Lemma 4.6, with the application of some additional techniques. Notably, the condition (3.14) ensures that

$$1 < \frac{Nm\Big(\overline{p}(x) - 1 - \sigma_{+}^{+} + \gamma^{-}\Big)p_i(x)}{\overline{p}(x)\Big(N - m(\sigma_{+}^{+} + 1 - \gamma^{-})\Big)}, \quad \forall x \in \overline{\Omega},\ i = 1, \dots, N.$$

□

Lemma 4.8 *Let $g_n \in L^{\infty}(\Omega)$ be a sequence of functions satisfies*

$$g_n \longrightarrow g \quad \textit{strongly in } L^1(\Omega) \quad \textit{as} \quad n \to +\infty.$$

and assume that u_n a nonnegative weak solution of the problem

$$\begin{aligned} -\sum_{i=1}^{N} D^i\Big(b_i(x,T_n(u_n))a_i(x,Du_n)\Big) &= g_n \quad \text{in } \Omega, \\ u_n &> 0 \quad \text{in } \Omega, \\ u_n &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{4.32}$$

Suppose that:

(i) u_n *is such that* $T_k(u_n) \in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$ *for all* $k > 0$.

(ii) u_n *converges almost everywhere in* Ω *to some measurable function* u *which is finite almost everywhere, and such that* $T_k(u) \in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$ *for all* $k > 0$ *(note that (i) and (ii) imply that* $T_k(u_n)$ *weakly converges to* $T_k(u)$ *in* $\mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$*).*

(iii) u_n *is bounded in* $M^{r_1}(\Omega)$ *for some* $r_1 > 0$ *and* $u \in M^{r_1}(\Omega)$.

(iv) There exists $\theta_i > 0$, $i = 1, \ldots, N$ *such that* $|D_i u_n|^{\theta_i}$ *is bounded in* $L^{r_2}(\Omega)$*, for some* $r_2 > 1$ *and* $|D_i u|^{\theta_i} \in L^{r_2}(\Omega)$.

Then, the previous choice allows to obtaining

$$D_i u_n \to D_i u, \quad \text{almost everywhere in} \quad \Omega, \ \text{for all } i = 1, \ldots, N.$$

Proof: The result in [18] provides the existence of a solution $u_n \in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$ to the problem (4.32). Since our approach involves anisotropic variable exponents with a singular nonlinearity, we adapt some techniques from [5], with modifications that allow us to achieve the desired result. □

Lemma 4.9 *Let u_n be a nonnegative weak solution to the problem 3.20, and suppose that u_n converges to u almost everywhere in Ω. Then,*

$$F(x, u_n) \to F(x, u) \ \text{in } L^1(\Omega). \tag{4.33}$$

Proof: To prove (4.33), we apply Vitali's theorem. In one hand, since u_n converges to u almost everywhere in Ω and F is a Carathéodory function, we have

$$F(x, u_n) \to F(x, u) \text{ a.e. in } \ \Omega.$$

On the other hand, it is sufficient to establish the equi-integrability of $F(\cdot, u_n)$ on Ω. Let us choose an increasing, uniformly bounded Lipschitz function $\nu_\varepsilon(t)$ which satisfies $\nu_\varepsilon \to \chi_{\{|t|>k\}}\,\text{sign}(t)$ $(k > 0)$, as $\varepsilon \to +\infty$. Specifying $\nu_\varepsilon(u_n)$ as a test function in (3.21), we find

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega \nu_\varepsilon'(u_n) b_i(x, T_n(u_n)) a_i(x, Du_n) |D_i u_n|^{p_i(x)}\, dx + \int_\Omega F(x, u_n)\nu_\varepsilon(u_n)\, dx \\ &= \int_\Omega H_n(u_n) f_n \nu_\varepsilon(u_n)\, dx + \int_\Omega \mu_n \nu_\varepsilon(u_n)\, dx \end{aligned}$$

Recalling (1.2) and (3.18), as $\varepsilon \to +\infty$, we get

$$\int_{\{|u_n|>k\}} F(x, u_n)\, dx \le \int_{\{|u_n|>k\}} Mf\, dx + \int_{\{|u_n|>k\}} \mu\, dx \tag{4.34}$$

Again let us take $G \subset \Omega$ as any measurable set. It follows from (1.8) and (4.34) that

$$\begin{aligned} \int_G F(x, u_n)\, dx &= \int_{G\cap\{|u_n|\le k\}} F(x, u_n)\, dx + \int_{G\cap\{|u_n|>k\}} F(x, u_n)\, dx \\ &\le \text{meas}(G)\|g_k\|_{L^1(\Omega)} + \int_{\{G\cap|u_n|>k\}} Mf\, dx + \int_{\{G\cap|u_n|>k\}} \mu\, dx \end{aligned}$$

This inequality gives equi-integrability of $F(\cdot, u_n)$ on Ω, which finish the proof of this Lemma. □

5. Proof of main results

This section is dedicated to utilizing the uniform estimates from Section 4 to prove Theorems 3.1, 3.2, 3.3, 3.4, and 3.5.

5.1. Proof of theorems 3.3, 3.4, and 3.5

By Lemma 4.6 the sequence (u_n) is bounded in $\mathring{W}^{1,\vec{q}(\cdot)}(\Omega)$ where $q_i(\cdot)$ is defined as (3.12). Without loss of generality, we can therefore assume that

$$\begin{gathered} u_n \rightharpoonup u \quad \text{weakly in } \mathring{W}^{1,\vec{q}(\cdot)}(\Omega), \\ u_n \to u \quad \text{strongly in } L^{q_0}(\Omega), \quad q_0 = \min_{1\le i\le N}\min_{x\in\overline{\Omega}} q_i(x), \\ u_n \to u \quad \text{a.e. in} \quad \Omega. \end{gathered} \tag{5.1}$$

It follows from (3.23) thus $|u|^{s(x)} \in L^1(\Omega)$; furthermore, $u \in \mathcal{M}^{s^-}(\Omega)$. Then, there exists $r_1 = s^- > 0$ such that

$$\|u_n\|_{M^{r_1}(\Omega)} \le C \quad \text{and} \quad u \in M^{r_1}(\Omega). \tag{5.2}$$

Let $g_n = H_n(u_n)f_n + \mu_n - T_n(F(x,u_n)) \in L^\infty(\Omega)$, where u_n is a solution of (4.32). Then, from (4.14), (4.19), (5.1), (5.2), and lemma 4.8 we can deduce that

$$D_i u_n \to D_i u \quad \text{a.e. in } \Omega, \text{for all } i = 1,\dots,N.$$

So, by (4.19), we have

$$a_i(x,Du_n) \rightharpoonup a_i(x,Du) \quad \text{weakly in } L^{\frac{q_i(\cdot)}{p_i(\cdot)-1}}(\Omega),\ \forall i = 1,\dots,N, \tag{5.3}$$

where q_i is defined as in (3.12). The choice of $\frac{q_i(\cdot)}{p_i(\cdot)-1} > 1$ is possible since we have (3.10). From (1.7) and (5.1), we obtain

$$b_i(x,T_n(u_n)) \to b_i(x,u) \quad \text{weak}^* \text{ in } L^\infty(\Omega). \tag{5.4}$$

For any given $\phi \in C_0^\infty(\Omega)$, using ϕ as a test function in (3.20), we have

$$\begin{aligned} &\sum_{i=1}^N \int_\Omega b_i(x,T_n(u_n))a_i(x,Du_n)D^i\phi\,dx + \int_\Omega F(x,u_n)\phi\,dx \\ &= \int_\Omega H_n(u_n)f_n\phi\,dx + \int_\Omega \mu_n\phi\,dx, \end{aligned} \tag{5.5}$$

Letting $n \to +\infty$ in (5.5), by (5.3), (5.4),(3.18), (3.25), (3.18), and (4.33), we obtain

$$\begin{aligned} &\sum_{i=1}^N \int_\Omega b_i(x,u)a_i(x,Du)D^i\phi\,dx + \int_\Omega F(x,u)\phi\,dx \\ &= \int_\Omega H(u)f\phi\,dx + \int_\Omega \mu\phi\,dx \end{aligned}$$

In the proof of Theorem 3.3 (respectively Theorem 3.5), we replace (5.3) with

$$a_i(x,Du_n) \rightharpoonup a_i(x,Du) \quad \text{weakly in } L^{\frac{q_i(\cdot)}{p_i(\cdot)-1}}(\Omega),\ \forall i = 1,\dots,N,$$

where q_i is defined as in (3.8) (respectively (3.16)). The condition $\frac{q_i(\cdot)}{p_i(\cdot)-1} > 1$ is satisfied due to (3.6) (respectively (3.14)).

5.2. Proof of theorems 3.1, 3.2

Since the proof of Theorem 3.2 is similar to that of Theorem 3.4, we provide only the proof of Theorem 3.1 here.
By Lemma 4.1, the sequence (u_n) is bounded in $\mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$. Consequently, we can extract a subsequence (denote again by (u_n)), such that

$$u_n \rightharpoonup u \quad \text{weakly in } \mathring{W}^{1,\vec{p}(\cdot)}(\Omega),$$
$$u_n \to u \quad \text{strongly in } L^{p_0}(\Omega), \quad p_0 = \min_{1\le i\le N} \min_{x\in\overline{\Omega}} p_i(x),$$
$$u_n \to u \quad \text{a.e. in } \Omega.$$

Following the argument in the proof of Theorem 3.4 and using (4.1), we conclude that

$$a_i(x, Du_n) \rightharpoonup a_i(x, Du) \quad \text{weakly in } L^{p'_i(\cdot)}(\Omega),\ \forall i = 1, \dots, N.$$

Thus, the proof of Theorem 3.1 is complete.

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