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Fixed Point Results for G-F-Contractive Mappings of Hardy-Rogers Type

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Abstract: In this paper, we present the notation of G-F-Contractive mappings of Hardy-Rogers type and give some fixed point results of Hardy-Rogers type for self-mappings in complete G-metric spaces.

Key Words: G-metric space, F-contraction.

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1. Introduction

We know by the Banach contraction principle [1], which is a classical and powerful tool in nonlinear analysis, that a self-mapping for a complete metric space (X, d) such that $d(fx, fy) \leq cd(x, y)$ for all $x, y \in X$, where $c \in [0, 1)$ has a unique fixed point. Since then, the Banach contraction principle has been generated in several directions.(see [2,5,7])

The concept of a generalized metric space, or a G-metric space, was introduced by Mustafa and Sims[14]. Many authors have recently obtained different fixed point theorems for mappings satisfying various constructive conditions on G-metric spaces.

Recently, Wardowski [13] introduced a new contraction concept, F-contraction, and obtained some fixed point results by using this Contraction. In 2014, Monica Cosentino. et al.[4] have got some results on F-Contractive mappings of Hardy-Rogers type.

2. Preliminaries

Definition 2.1. *[13] $F : R^+ \to R$ satisfying the following properties:*
(F_1) is strictly increasing;
(F_2) for each sequence a_n of positive numbers, we have
$\lim_{n\to\infty} a_n = 0$ *if and only if* $\lim_{n\to\infty} F(a_n) = -\infty$;
(F_3) there exists $k \in (0, 1)$ such that $\lim_{a\to 0^+} a^k.F(a) = 0$.
We denote with $\mathfrak{F}$ the family of all functions F that satisfy the conditions $(F_1) - (F_3)$.

Definition 2.2. *[13] Let (X, d) be a metric space. A self-mapping T on X is called an F-contraction if there exists $F \in \mathfrak{F}$, $\tau \in \mathfrak{R}^+$ and such that*
$\tau + F(d(Tx, Ty)) \leq F(d(x, y))$ $\forall x, y \in X$ *with* $d(Tx, Ty) > 0$.

Definition 2.3. *[4] Let (X, d) be a metric space. A self-mapping T, X is called a F-contraction of Hardy-Rogers type if there exists $F \in \mathfrak{F}$ and $\tau \in \mathfrak{R}^+$ such that*
$\tau+F(d(Tx, Ty)) \leq F(\alpha d(x, y)+\beta d(x, Tx)+\gamma d(y, Ty)+\delta d(x, Ty)+Ld(y, Tx))$ $\forall x, y \in X$ *with* $d(Tx, Ty) > 0$, *where* $\alpha + \beta + \gamma + 2\delta = 1, \gamma \neq 1$ *and* $L \geq 0$.

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Definition 2.4. *Let (X, G) be G-metric space. A self-mapping T, X is called a $G - F$-contraction of Hardy-Rogers type if there exists $F \in \mathfrak{F}$ and $\tau \in \mathfrak{R}^+$ such that*

$$\tau + F(G(Tx, Ty, Tz)) \leq F(G(x, y, z)) \tag{2.1}$$

for all $x, y, z \in X$ with $G(Tx, Ty, Tz) > 0$.

Definition 2.5. *Let (X, G) be G-metric space. A self-mapping T, X is called a $G - F$-contraction of Hardy-Rogers type if there exists $F \in \mathfrak{F}$ and $\tau \in \mathfrak{R}^+$ such that*

$$\begin{aligned} \tau + F(G(Tx, Ty, Tz)) \leq F(\alpha G(x, y, z) + \beta G(x, Tx, Tx) + \gamma G(y, Ty, Ty) + hG(z, Tz, Tz) \\ + \delta G(x, Ty, Ty) + \Delta G(y, Tz, Tz) + eG(z, Tx, Tx)) \end{aligned} \tag{2.2}$$

for all $x, y, z \in X$ with $G(Tx, Ty, Tz) > 0$, $\alpha + \beta + \gamma + h + \Delta + 2\delta = 1$, $h \neq 1$ and $e \geq 0$.

Example 2.6. *Let $F : R^+ \to R$ be given by $F(x) = ln(x)$. It is clear that F satisfies $(F_1) - (F_3)$ any $c \in (0, 1)$. Each mapping $T : X \to X$ satisfying(2.1) is a $G - F-$Contraction such that*

$$G(Tx, Ty, Tz) \leq e^{-\tau} G(x, y, z)$$

for all $x, y, z \in X$ with $G(Tx, Ty, Tz) > 0$.

Example 2.7. *Let $F : R^+ \to R$ be given by $F(x) = ln(x)+x$. That F satisfies $(F_1)-(F_3)$ any $c \in (0, 1)$. Each mapping $T : X \to X$ satisfying(2.1) is a $G - F-$Contraction such that*

$$\frac{G(Tx, Ty, Tz)}{G(x, y, z)} e^{G(Tx,Ty,Tz)-G(x,y,z)} \leq e^{-\tau}$$

for all $x, y, z \in X$ with $G(Tx, Ty, Tz) > 0$.

Remark 2.8. *From (F_1) and (2.1), we deduce that every $G-F$-Contraction T is a contractive mapping, that is $G(Tx, Ty, Tz) \leq G(x, y, z)$, for all $x, y, z \in X$ with $G(Tx, Ty, Tz) > 0$.*
From (F1) and (2.2), we conclude that every $G - F$-contraction of Hardy-Rogers type T satisfies the following condition

$$\begin{aligned} G(Tx, Ty, Tz) \leq \alpha G(x, y, z) + \beta G(x, Tx, Tx) + \gamma G(y, Ty, Ty) + hG(z, Tz, Tz) \\ + \delta G(x, Ty, Ty) + \Delta G(y, Tz, Tz) + eG(z, Tx, Tx) \end{aligned} \tag{2.3}$$

for all $x, y, z \in X$ with $G(Tx, Ty, Tz) > 0$, $\alpha + \beta + \gamma + h + \Delta + 2\delta = 1$, $h \neq 1$ and $e \geq 0$.

3. Fixed Points for G-F-Contraction of Hardy-Rogers-type

Theorem 3.1. *Let (X, G) be complete G-metric space and let T be a self-mapping on X. Assume that there exists $F \in \mathfrak{F}$ and $\tau \in \mathfrak{R}^+$ such that T is a $G - F$-contraction of Hardy-Rogers type, that is*

$$\begin{aligned} \tau + F(G(Tx, Ty, Tz)) \leq F(\alpha G(x, y, z) + \beta G(x, Tx, Tx) + \gamma G(y, Ty, Ty) + hG(z, Tz, Tz) \\ + \delta G(x, Ty, Ty) + \Delta G(y, Tz, Tz) + eG(z, Tx, Tx)) \end{aligned} \tag{3.1}$$

for all $x, y, z \in X$ with $G(Tx, Ty, Tz) > 0$, where $\alpha + \beta + \gamma + h + \Delta + 2\delta = 1$, $h \neq 1$ and $e \geq 0$. Then T has a fixed point. Moreover if $\alpha + \delta + 2e \leq 1$, then the fixed point T is unique.

Proof. Let $x_0 \in X$ be an arbitrary point, and let $\{x_n\}$ be the Picard sequence with initial point x_0 ,that is, $x_n = T^n x_0 = Tx_{n-1}$. If $x_n = x_{n-1}$ for same $n \in N$, then x_n is a fixed point of T.
Now, let $G_n = G(x_n, x_{n+1}, x_{n+1})$ for all $n \in N \cup \{0\}$. If $x_n \neq x_{n-1}$, that is, $Tx_n \neq Tx_{n-1}$ for all $n \in N$. Now, put $x_n = x_{n-1}$, $y = x_n$ and $z = x_n$ in the contractive condition (3.1), we get

$$\begin{aligned}
\tau + F(G_n) &= \tau + F(G(x_n, x_{n+1}, x_{n+1})) \\
&= \tau + F(G(Tx_{n-1}, Tx_n, Tx_n)) \\
&\leq F(\alpha G(x_{n-1}, x_n, x_n) + \beta G(x_{n-1}, Tx_{n-1}, Tx_{n-1}) + \\
&\gamma G(x_n, Tx_n, Tx_n) + hG(x_n, Tx_n, Tx_n) + \\
&\delta G(x_{n-1}, Tx_n, Tx_n) + \Delta G(x_n, Tx_n, Tx_n) + eG(x_n, Tx_{n-1}, Tx_{n-1})) \\
&= F(\alpha G(x_{n-1}, x_n, x_n) + \beta G(x_{n-1}, x_n, x_n) + \\
&\gamma G(x_n, x_{n+1}, x_{n+1}) + hG(x_n, x_{n+1}, x_{n+1}) + \\
&\delta G(x_{n-1}, x_{n+1}, x_{n+1}) + \Delta G(x_n, x_{n+1}, x_{n+1}) + eG(x_n, x_n, x_n)) \\
&= F((\alpha + \beta)G(x_{n-1}, x_n, x_n) + (\gamma + h + \Delta)G(x_n, x_{n+1}, x_{n+1}) + \\
&\delta G(x_{n-1}, x_n, x_n) + \delta G(x_n, x_{n+1}, x_{n+1})) \\
&= F((\alpha + \beta + \delta)G(x_{n-1}, x_n, x_n) + (\gamma + h + \Delta + \delta)G(x_n, x_{n+1}, x_{n+1})) \\
&= F((\alpha + \beta + \delta)G_{n-1} + (\gamma + h + \Delta + \delta)G_n)
\end{aligned}$$

Since F is strictly increasing, we deduce $G_n < (\alpha + \beta + \delta)G_{n-1} + (\gamma + h + \Delta + \delta)G_n$ hence $(1 - \gamma - h - \Delta - \delta)G_n < (\alpha + \beta + \delta)G_{n-1}$ for all $n \in N$.

From $\alpha + \beta + \gamma + h + \Delta + 2\delta = 1$ and $h \neq 1$.
We deduce that $1 - \gamma - h - \Delta - \delta > 0$ and so $G_n < \frac{(\alpha+\beta+\delta)}{1-\gamma-h-\Delta-\delta} G_{n-1} = G_{n-1}$ for all $n \in N$.
Consequently,
$\tau + F(G_n) \leq F(G_{n-1})$, for all $n \in N$. This implies

$$F(G_n) \leq F(G_{n-1}) - \tau \leq \ldots F(G_0) - n\tau \tag{3.2}$$

for all $n \in N$ and so $\lim_{n\to\infty} F(G_n) = -\infty$. From the property (F_2), we get that $G_n \to 0$ as $n \to \infty$. Now, let $k \in (0, 1)$ such that $\lim_{n\to\infty} G_n^k F(G_n) = 0$ by (3.2), the following holds for all $n \in N$.

$$G_n^k F(G_n) - G_n^k F(G_0) \leq G_n^k (F(G_0) - n\tau) - G_n^k (F(G_0) = -n\tau G_n^k \leq 0 \tag{3.3}$$

letting $n \to \infty$ in (3.3), we deduce that $\lim_{n\to\infty} nG_n^k = 0$ and hence $\lim_{n\to\infty} n^{1/k} G_n = 0$.

This implies that the series $\sum_{n=1}^{+\infty}$ is convergent. This x_n means it is a G-Cauchy sequence. X is a complete G-metric space, there exists $u \in X$ such that $x_n \to u$ if $u = Tu$ the proof is finished, assuming that $u \neq Tu$. If $Tx_n = Tu$ for infinite values of $n \in N \cup \{0\}$, then the sequence x_n has a subsequence that converges to Tu and the uniqueness of the limit implies $u = Tu$. Then we can $Tx_n \neq Tu$ take that all $n \in N \cup \{0\}$.
Now by (2.3), we have

$$\begin{aligned}
G(u, Tu, Tu) &\leq G(u, x_{n+1}, x_{n+1}) + G(x_{n+1}, Tu, Tu) \\
&\leq G(u, x_{n+1}, x_{n+1}) + G(Tx_n, Tu, Tu) \\
&\leq G(u, x_{n+1}, x_{n+1}) + \alpha G(x_n, u, u) + \beta G(x_n, Tx_n, Tx_n) + \\
&\gamma G(u, Tu, Tu) + hG(x_n, Tx_n, Tx_n) + \\
&\delta G(x_n, Tu, Tu) + \Delta G(u, Tu, Tu) + eG(u, Tx_n, Tx_n) \\
&= G(u, x_{n+1}, x_{n+1}) + \alpha G(x_n, u, u) + \beta G(x_n, x_{n+1}, x_{n+1}) + \\
&\gamma G(u, Tu, Tu) + hG(x_n, x_{n+1}, x_{n+1}) + \\
&\delta G(x_n, Tu, Tu) + \Delta G(u, Tu, Tu) + eG(u, x_{n+1}, x_{n+1})
\end{aligned}$$

letting $n \to \infty$ the previous inequality, we get
$G(u,Tu,Tu) \leq (\gamma+\Delta)G(u,Tu,Tu) < G(u,Tu,Tu)$
which is a contradiction, hence $Tu = u$.

Now, we prove the uniqueness of the fixed point.Assume that $w \in X$ in another fixed point of T, different from u. This means that $G(u,w,w) > 0$. Taking $x = u$, $y = w$ and $z = w$ in (3.1), we have

$$\begin{aligned}
\tau + F(G(u,w,w)) &= \tau + F(G(Tu,Tw,Tw)) \\
&= \tau + F(G(Tx_{n-1},Tx_n,Tx_n)) \\
&\leq F(\alpha G(u,w,w) + \beta G(u,u,u) + \\
&\gamma G(w,w,w) + hG(w,w,w) + \\
&\delta G(u,w,w) + \Delta G(w,w,w) + eG(w,u,u)) \\
&= F(\alpha G(u,w,w) + \delta G(u,w,w) + 2eG(w,u,u)) \\
&= F((\alpha+\delta+2e)G(u,w,w))
\end{aligned}$$

which is a contradiction if $\alpha + \delta + 2e \leq 1$. Hence $u = w$. □

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