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Some fixed point theorems for (ξ, α)-expansive mappings in complete bipolar metric spaces

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ABSTRACT: In this paper, we will introduce the notion of (ξ, α)-expansive covariant and contravariant mappings in bipolar metric spaces. In addition, we prove some fixed point theorems for existence and uniqueness of fixed points for (ξ, α)-expansive covariant and contravariant mappings in complete bipolar metric space. At the end, we shall also provide some examples to support the theorems.

Key Words: Fixed point, bipolar metric space, expansive mappings.

Contents

1. Introduction

The fixed point theory has tremendous applications in various branches of Mathematics and sciences. In the last ten decades, researchers have provided a number of fixed point theorems in various metric spaces with applications. In 1989, Zeidler [13] introduced fixed point theorems with applications in differential and integral equations. Further, fixed point theory has a lot of applications in game theory relevant to military, sports, and medical sciences as well as in Economics [2]. To obtain new fixed point theorems, a number of generalizations [3,6,7,9] have been made based on the Banach [1] contraction principle. In 1984, Wang *et al.* [12] introduced some fixed point theorems for expansive mappings in complete metric spaces. Following this, many researchers generalized their research work for expansive mappings [4,11]. In 2012, Priya Shahi *et al.* [10] generalized expansive mappings and introduced a new notion of (ξ, α)-expansive mappings and proved some fixed point theorems for such kind of mappings.

2. Preliminaries

To prove fixed point theorems for (ξ, α)-expansive mappings in complete bipolar metric spaces, we need the following definitions and results.

Definition 2.1. [10] Let χ denote the set of all functions $\xi : [0, +\infty) \to [0, +\infty)$ which satisfy the following conditions:

(i) ξ is non-decreasing;

(ii) $\sum_{n=1}^{\infty} \xi_n(a) < +\infty$ for each $a > 0$, where ξ_n is the nth iteration of ξ;

(iii) $\xi(a + b) = \xi(a) + \xi(b)$ for all $a, b \in [0, +\infty)$.

Definition 2.2. [5] Let X and Y be two non-empty sets and $d : X \times Y \to [0, \infty)$ be a function satisfying the followings:

(i) $d(x, y) = 0$ if and only if $x = y$, where $x \in X, y \in Y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X \cap Y$;

(iii) $d(x_1, y_2) \leq d(x_1, y_1) + d(x_2, y_1) + d(x_2, y_2)$ for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$.

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Then d is called a bipolar metric, and (X, Y, d) is called a bipolar metric space.
If $X \cap Y = \emptyset$, then the space is called disjoint otherwise joint. The set X is called the left pole, and Y is called the right pole of the bipolar metric space (X, Y, d). Any element of the left pole (X), right pole (Y), or $X \cap Y$ is called a left element, right element, and central element, respectively.

Definition 2.3. [5] Let (X, Y, d) be a bipolar metric space.Then, any sequence $(x_n) \subset X$ is called a left sequence and is said to be convergent to a right element, say y, if $d(x_n, y) \to 0$ as $n \to \infty$. Similarly, a right sequence $(y_n) \subset Y$ is said to be convergent to a left element, say x, if $d(x, y_n) \to 0$ as $n \to \infty$.

Definition 2.4. [5] Let (X, Y, d) be a bipolar metric space.

(i) A sequence (x_n, y_n) on the set $X \times Y$ is called a bisequence on (X, Y, d).

(ii) If both the sequences (x_n) and (y_n) converge, then the bisequence (x_n, y_n) is said to be convergent. If both the sequences (x_n) and (y_n) converge to the same point $v \in X \cap Y$, then this bisequence is said to be biconvergent.

(iii) A bisequence (x_n, y_n) on (X, Y, d) is said to be a Cauchy bisequence if, for each $\epsilon > 0$, there exists a positive integer $N \in \mathbb{N}$ such that $d(x_n, y_m) < \epsilon$ for all $n, m \geq N$.

(iv) A bipolar metric space is said to be complete if every Cauchy bisequence is convergent in this space.

Definition 2.5. [5] Let (X_1, Y_1, d_1) and (X_2, Y_2, d_2) be two bipolar metric spaces. Let $T : X_1 \cup Y_1 \to X_2 \cup Y_2$ be a function:

(i) If $T(X_1) \subseteq X_2$ and $T(Y_1) \subseteq Y_2$, then T is called a covariant map and is denoted by $T : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$.

(ii) If $T(X_1) \subseteq Y_2$ and $T(Y_1) \subseteq X_2$, then T is called a contravariant map and is denoted by $T : (X_1, Y_1, d_1) \rightleftarrows (X_2, Y_2, d_2)$.

Definition 2.6. [5] Let (X_1, Y_1, d_1) and (X_2, Y_2, d_2) be two bipolar metric spaces.

(i) A map $T : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ is called left continuous at a point $x_0 \in X_1$ if, for every $\epsilon > 0$, there exists $\delta > 0$ such that $d_2(Tx_0, Ty) < \epsilon$ whenever $d_1(x_0, y) < \delta$.

(ii) A map $T : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ is called right continuous at a point $y_0 \in Y_1$ if, for every $\epsilon > 0$, there exists $d > 0$ such that $d_2(Tx, Ty_0) < \epsilon$ whenever $d_1(x, y_0) < d$.

(iii) A map T is called continuous if it is left continuous at each $x_0 \in X_1$ and right continuous at each $y_0 \in Y_1$.

(iv) A contravariant map $T : (X_1, Y_1, d_1) \rightleftarrows (X_2, Y_2, d_2)$ is continuous if and only if it is continuous as a covariant map $T : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$.

Definition 2.7. [6] Let $T : (X, Y) \rightrightarrows (X, Y)$ and $\alpha : X \times Y \to [0, +\infty)$. Then T is called α-admissible if $\alpha(x, y) \geq 1$ implies that $\alpha(Tx, Ty) \geq 1$ for all $x, y \in X$.

Definition 2.8. [6] Let $T : (X, Y) \rightleftarrows (X, Y)$ and $\alpha : X \times Y \to [0, +\infty)$. Then T is called α-admissible if $\alpha(x, y) \geq 1$ implies that $\alpha(Ty, Tx) \geq 1$ for all $x, y \in X$..

Definition 2.9. [8] Let (X, Y, d) be a bipolar metric space, $a \in X$, $p \in Y$, and $F : (X \times Y, Y \times X) \rightrightarrows (X, Y)$ be a covariant mapping. Then (a, p) is said to be a coupled fixed point of F if $F(a, p) = a$ and $F(p, a) = p$.

Lemma 2.10. [8] Let $F : (X \times Y, Y \times X) \rightrightarrows (X, Y)$ be a covariant mapping. If we define the covariant mapping $T : (X \times Y, Y \times X) \rightrightarrows (X \times Y, Y \times X)$ with

$$T(x, y) = (F(x, y), F(y, x)) \quad \text{for all } (x, y) \in X \times Y. \tag{2.1}$$

Then (x, y) is a coupled fixed point of F if and only if (x, y) is a fixed point of T.

3. Main Results

In this section, we will prove fixed point theorems for $(\xi-\alpha)$-expansive mapping in complete bipolar metric space.

Definition 3.1. Let (X,Y,d) be a bipolar metric space and $T:(X,Y,d)\to(X,Y,d)$ be a given covariant mapping. We say that T is $(\xi-\alpha)$-expansive mapping if there exists $\xi\in\chi$ and $\alpha: X\times Y\to[0,\infty)$ such that

$$\xi(d(Tx,Ty))\geq\alpha(x,y)d(x,y) \text{ for all } (x,y)\in X\times Y. \tag{3.1}$$

Definition 3.2. Let (X,Y,d) be a bipolar metric space and $T:(X,Y,d)\to(X,Y,d)$ be a given contravariant mapping. We say that T is $(\xi-\alpha)$-expansive mapping if there exists $\xi\in\chi$ and $\alpha: X\times Y\to[0,\infty)$ such that

$$\xi(d(Ty,Tx))\geq\alpha(x,y)d(x,y) \text{ for all } (x,y)\in X\times Y. \tag{3.2}$$

Theorem 3.3. Let (X,Y,d) be a complete bipolar metric space and $T:(X,Y,d)\to(X,Y,d)$ be a bijective covariant $(\xi-\alpha)$-expansive mapping satisfying the followings:

1. T^{-1} is α-admissible;
2. There exist $(x_0,y_0)\in X\times Y$ such that $\alpha(x_0,y_0)\geq 1$ and $\alpha(x_0,T^{-1}y_0)\geq 1$;
3. T is continuous.

Then T has a fixed point.

Proof Let $x_0\in X$ and $y_0\in Y$ such that $\alpha(x_0,T^{-1}y_0)\geq 1$. We define the bisequence (x_n,y_n) by $x_n=Tx_{n+1}$ and $y_n=Ty_{n+1}$ for all $n\in\mathbb{N}$.
Since T^{-1} is α-admissible and $\alpha(x_0,y_0)\geq 1$, this implies

$$\begin{aligned} 1 &\leq \alpha(T^{-1}x_0,T^{-1}y_0),\\ 1 &\leq \alpha(x_1,y_1). \end{aligned}$$

Using mathematical induction, we obtain

$$1\leq\alpha(x_n,y_n) \text{ all for } n\in\mathbb{N}. \tag{3.3}$$

Similarly, T^{-1} is α-admissible and $\alpha(x_0,T^{-1}y_0)\geq 1$, so this implies

$$\begin{aligned} 1 &\leq \alpha(T^{-1}x_0,T^{-1}y_1),\\ &= \alpha(x_1,y_2),\\ 1 &\leq \alpha(x_1,y_2). \end{aligned}$$

Again, by mathematical induction, we get

$$1\leq\alpha(x_n,y_{n+1}) \text{ all for } n\in\mathbb{N}. \tag{3.4}$$

Putting $x=x_{n+1}, y=y_{n+1}$ in equation (3.1) and using equation (3.3), we get

$$\begin{aligned} d(x_{n+1},y_{n+1})\leq\alpha(x_{n+1},y_{n+1})d(x_{n+1},y_{n+1}) &\leq \xi(d(Tx_{n+1},Ty_{n+1})),\\ &\leq \xi(d(x_n,y_n)). \end{aligned}$$

Continuing this process, we get

$$d(x_{n+1},y_{n+1})\leq\xi^n(d(x_0,y_0)). \tag{3.5}$$

Similarly, putting $x=x_n, y=y_{n+1}$ in equation (3.1) and using equation (3.4), we get

$$\begin{aligned} d(x_n,y_{n+1})\leq\alpha(x_n,y_{n+1})d(x_n,y_{n+1}) &\leq \xi(d(Tx_n,Ty_{n+1})),\\ &\leq \xi(d(x_n,y_{n-1})). \end{aligned}$$

By repeating the same process, we get

$$d(x_n, y_{n+1}) \leq \xi^n(d(x_0, y_1)). \tag{3.6}$$

Since, $\sum_{n=1}^{+\infty} \xi^n(a) < +\infty$ for each $a >$. So, for every $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that

$$\sum_{n\geq N} \xi^n(d(x_0, y_1)) < \frac{\epsilon}{2} \text{ and } \sum_{n\geq N} \xi^{n+1}(d(x_0, y_0)) < \frac{\epsilon}{2} \tag{3.7}$$

Now for $n, m \in \mathbb{N}$ with $m > n \geq N$, applying the condition (iii) of Definition 2.2, we get

$$\begin{aligned} d(x_n, y_m) &\leq d(x_n, y_{n+1}) + d(x_{n+1}, y_{n+1}) + d(x_{n+1}, y_{n+2}) + \cdots + d(x_{m-1}, y_{m-1}) \\ &+ d(x_{m-1}, y_m), \\ &= \sum_{k=n}^{m-1} d(x_k, y_{k+1}) + \sum_{k=n}^{m-2} d(x_{k+1}, y_{k+1}), \end{aligned}$$

Using equation (3.5) in (3.6), we get

$$\begin{aligned} d(x_n, y_m) &\leq \sum_{k=n}^{m-1} \xi^k d(x_0, y_1) + \sum_{k=n}^{m-2} \xi^{k+1} d(x_0, y_0), \\ &\leq \sum_{k=N}^{m-1} \xi^k d(x_0, y_1) + \sum_{k=N}^{m-2} \xi^{k+1} d(x_0, y_0), \\ &\leq \sum_{n\geq N}^{m-1} \xi^n d(x_0, y_1) + \sum_{n\geq N} \xi^{n+1} d(x_0, y_0). \end{aligned} \tag{3.8}$$

Using equation (3.7) in (3.8), we get

$$d(x_n, y_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{3.9}$$

Similarly, one can prove easily for $n, m \in \mathbb{N}$ with $n > m \geq N$ that

$$d(x_n, y_m) < \epsilon. \tag{3.10}$$

From equations (3.9) and (3.10), we can say that (x_n, y_n) is Cauchy bisequence.
Since (X, Y, d) is a complete bipolar metric space, so (x_n, y_n) is convergent and thus biconverges to a point $v \in X \cap Y$.
Since T is continuous, so $x_n \to v$ implies

$$(Tx_n) \to Tv.$$

as $n \to \infty$.
Hence $Tv = v$.
So, T has a fixed point.

Theorem 3.4. Let (X, Y, d) be a complete bipolar metric space and $T : (X, Y, d) \to (X, Y, d)$ be a bijective contravariant $(\xi - \alpha)$-expansive mapping satisfying the followings:

1. T^{-1} is α-admissible;
2. There exist $x_0 \in X$ such that $\alpha(x_0, T^{-1}x_0) \geq 1$;
3. T is continuous.

Then T has a fixed point.
Proof Let $x_0 \in X$ such that $\alpha(x_0, T^{-1}x_0) \geq 1$. We define the bisequence (x_n, y_n) as $x_n = Ty_n$ and $y_n = Tx_{n+1}$ for all $n \in \mathbb{N}$.
But T^{-1} is α-admissible, using equation (2.2), we get

$$\alpha(T^{-1}y_0, T^{-1}x_0) = \alpha(x_1, y_0),$$

Similarly,
As $\alpha(x_1, y_0) \geq 1$ implies that

$$\alpha(T^{-1}y_0, T^{-1}x_1) = \alpha(x_1, y_1) \geq 1,$$

Continuing this process, we get

$$\alpha(x_n, y_n) \geq 1 \text{ and } \alpha(x_{n+1}, y_n) \geq 1. \tag{3.11}$$

Putting $x = x_n$ and $y = y_n$ in equation (3.2) and using (3.11), we get

$$\begin{aligned} d(x_n, y_n) \leq \alpha(x_n, y_n)d(x_n, y_n) &\leq \xi(d(Ty_n, Tx_n)), \\ &\leq \xi(d(x_n, y_{n-1})), \end{aligned}$$

Continuing like this, we get

$$d(x_n, y_n) \leq \xi^n(d(x_1, y_0)). \tag{3.12}$$

Again, taking $x = x_{n+1}$ and $y = y_n$ in equation (3.2) and using (3.11), we get

$$\begin{aligned} d(x_{n+1}, y_n) \leq \alpha(x_{n+1}, y_n)d(x_{n+1}, y_n) &\leq \xi(d(Ty_n, Tx_{n+1})), \\ &\leq \xi(d(x_n, y_n)), \end{aligned} \tag{3.13}$$

Repeating this process, we get

$$d(x_{n+1}, y_n) \leq \xi^{n+1}(d(x_0, y_0)). \tag{3.14}$$

Since, $\sum_{n=1}^{+\infty} \xi^n(a) < +\infty$ for each $a > 0$. So, for every $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that

$$\sum_{n \geq N} \xi^n(d(x_1, y_0)) < \frac{\epsilon}{2} \text{ and } \sum_{n \geq N} \xi^{n+1}(d(x_0, y_0)) < \frac{\epsilon}{2} \tag{3.15}$$

Now for $n, m \in \mathbb{N}$ with $m > n \geq N$, applying the condition (iii) of Definition 2.2, we get

$$\begin{aligned} d(x_n, y_m) &\leq d(x_n, y_n) + d(x_{n+1}, y_n) + d(x_{n+1}, y_{n+1}) + \cdots + d(x_m, y_{m-1}) \\ &+ d(x_m, y_m), \\ &= \sum_{k=n}^{m} d(x_k, y_k) + \sum_{k=n}^{m-1} d(x_{k+1}, y_k), \end{aligned}$$

Now, using equations (3.12) and (3.13), we get

$$\begin{aligned} d(x_n, y_m) &\leq \sum_{k=n}^{m} \xi^k d(x_1, y_0) + \sum_{k=n}^{m-1} \xi^{k+1} d(x_0, y_0), \\ &\leq \sum_{k=N}^{m} \xi^k d(x_1, y_0) + \sum_{k=N}^{m-1} \xi^{k+1} d(x_0, y_0), \\ &\leq \sum_{n \geq N}^{m-1} \xi^n d(x_1, y_0) + \sum_{n \geq N} \xi^{n+1} d(x_0, y_0). \end{aligned}$$

Using equation (3.15), we get

$$d(x_n, y_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{3.16}$$

Similarly, one can prove easily for $n, m \in \mathbb{N}$ with $n > m \geq N$ that

$$d(x_n, y_m) < \epsilon. \tag{3.17}$$

From equations (3.16) and (3.17), it is clear that (x_n, y_n) is a Cauchy bisequence. Since (X, Y, d) is a complete bipolar metric space, so (x_n, y_n) is convergent and thus biconverges to a point $u \in X \cap Y$ and this guarantees that (x_n) and (y_n) have unique limit u. Since T is continuous $(x_n) \to u$ implies that $(y_n) = (Tx_n) = (Tu)$ and combining this with $(y_n) \to u$ gives $T(u) = u$.
So, T has a fixed point.
In the next two Theorems, we replace the hypothesis of continuity by new condition in Theorems 3.3 and 3.4.

Theorem 3.5. Let (X, Y, d) be a complete bipolar metric space and $T : (X, Y, d) \to (X, Y, d)$ be a bijective covariant $(\xi - \alpha)$-expansive mapping satisfying the followings:

1. T^{-1} is α-admissible;
2. There exist $(x_0, y_0) \in X \times Y$ such that $\alpha(x_0, y_0) \geq 1$ and $\alpha(x_0, T^{-1}y_0) \geq 1$;
3. If (x_n, y_n) is a bisequence such that $\alpha(T^{-1}x_n, T^{-1}y_n) \geq 1$ for all n and $(x_n) \to u, (y_n) \to u$ as $n \to \infty$ where $u \in X \cap Y$, then $\alpha(T^{-1}u, T^{-1}y_n) \geq 1$ for all n.

Then T has a fixed point.
Proof Following the proof of Theorem 3.3, we obtain that the bisequence (x_n, y_n) defined by $x_n = Tx_{n+1}$ and $y_n = Ty_{n+1}$ for all $n \in \mathbb{N}$, is a Cauchy bisequence and as the space (X, Y, d) is a complete bipolar metric space and converges to a point $u \in X \cap Y$. So, $(x_n) \to u$ and $(y_n) \to u$ as $n \to \infty$.
Now,

$$d(T^{-1}u, u) \leq d(T^{-1}u, T^{-1}y_n) + d(T^{-1}x_n, T^{-1}y_n) + d(T^{-1}x_n, u),$$

using condition (iii), we get

$$d(T^{-1}u, u) \leq \alpha(T^{-1}u, T^{-1}y_n)d(T^{-1}u, T^{-1}y_n) + \alpha(T^{-1}x_n, T^{-1}y_n)d(T^{-1}x_n, T^{-1}y_n) + d(T^{-1}x_n, u)$$

Using equation (3.1), we get

$$d(T^{-1}x_n, u) \leq \xi(d(u, y_n)) + \xi(d(x_n, y_n)) + d(x_{n+1}, u). \tag{3.18}$$

Letting $n \to \infty$ in equation (3.18), we get

$$d(T^{-1}u, u) = 0,$$

$T^{-1}u = u$, which implies that

$$Tu = u.$$

So, T has a fixed point.

Theorem 3.6. Let (X, Y, d) be a complete bipolar metric space and $T : (X, Y, d) \to (X, Y, d)$ be a bijective contravariant $(\xi - \alpha)$-expansive mapping satisfying the followings:

1. T^{-1} is α-admissible;
2. There exist $x_0 \in X$ such that $\alpha(x_0, T^{-1}x_0) \geq 1$;
3. If (x_n, y_n) is a bisequence such that $\alpha(T^{-1}y_n, T^{-1}x_n) \geq 1$ for all n and $(y_n) \to u$ as $n \to \infty$ where $u \in X \cap Y$, then $\alpha(T^{-1}u, T^{-1}x_n) \geq 1$ for all n.

Then T has a fixed point.
Proof Following the proof of Theorem 3.4, we obtain that the bisequence (x_n, y_n) defined as $x_n = Ty_n$ and $y_n = Tx_{n+1}$ for all $n \in \mathbb{N}$, is a Cauchy bisequence and as the space (X, Y, d) is a complete bipolar metric space and converges to a point $u \in X \cap Y$. So, $(x_n) \to u$ and $(y_n) \to u$ as $n \to \infty$.
Now,

$$d(T^{-1}u, u) \leq d(T^{-1}u, T^{-1}x_n) + d(T^{-1}y_n, T^{-1}x_n) + d(T^{-1}y_n, u),$$

using condition (iii), we get

$$d(T^{-1}u, u) \leq \alpha(T^{-1}u, T^{-1}x_n)d(T^{-1}u, T^{-1}x_n) + \alpha(T^{-1}y_n, T^{-1}x_n)d(T^{-1}y_n, (T^{-1}x_n) + d(T^{-1}y_n, u)$$

Using equation (3.2), we get

$$d(T^{-1}x_n, u) \leq \xi(d(x_n, u)) + \xi(d(x_n, y_n)) + d(x_{n+1}, u). \tag{3.19}$$

Letting $n \to \infty$ in equation (3.19), we get

$$d(T^{-1}u, u) = 0,$$

$T^{-1}u = u$, which implies that

$$Tu = u.$$

So, T has a fixed point.

Now, we provide a hypothesis to obtain the unique fixed point.
P: There exists $z \in X \cap Y$ such that $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$ for all $(x, y) \in X \cap Y$.

Theorem 3.7. If we add the hypothesis **P** to the Theorem 3.3 and Theorem 3.5 (resp., Theorem 3.4 and Theorem 3.6), then we obtain that covariant mapping (resp., contravariant mapping) T has a unique fixed point.
Proof To prove the uniqueness of fixed point of covariant mapping (resp., contravariant mapping) T , let us suppose, if possible, u and v are two distinct fixed point of T.
Then by the hypothesis **P** there exists $z \in X \cap Y$ such that

$$\alpha(u, z) \geq 1 \text{ and } \alpha(z, v) \geq 1. \tag{3.20}$$

Now,
By using the fact that T is α-admissible, from equation (3.20), we get

$$\alpha(u, T^{-1}z) \geq 1 \text{ and } \alpha(T^{-1}z, v) \geq 1. \tag{3.21}$$

Continuing the same argument, we get

$$\alpha(u, T^{-n}z) \geq 1 \text{ and } \alpha(T^{-n}z, v) \geq 1.$$

$$d(u, T^{-n}z) \leq \alpha(u, T^{-n}z)d(u, T^{-n}z) \leq \xi(d(u, T^{-n+1}z)). \tag{3.22}$$

This implies,

$$d(u, T^{-n}z) \leq \xi^n(d(u, z)). \tag{3.23}$$

Similarly, we can show that

$$d(T^{-n}z, v) \leq \xi^n(d(z, v)). \tag{3.24}$$

Letting $n \to \infty$ in equations (3.23) and (3.24), we get

$$T^{-n}z \to v \text{ and } T^{-n}z \to u.$$

a contradiction to uniqueness of limit.
So, $u = v$.

Hence, T has a unique fixed point.
If T is the contravariant mapping, then proof is similar.

Theorem 3.8. Let (X, Y, d) be a complete bipolar metric space and $F : (X \times Y) \times (Y \times X) \to X \times Y$ is a bijective map. Suppose that $\xi \in \chi$ and $\alpha : (X \times Y) \times (Y \times X) \to [0, \infty)$ such that

$$\xi(d(F(x, y), F(u, v))) \geq \frac{1}{2}\alpha((x, y), (u, v))[d(x, u) + d(v, y)] \tag{3.25}$$

for all $(x, y) \in X \times Y$ and $(u, v) \in Y \times X$ and the following conditions are satisfied:

1. If $\alpha((x, y), (u, v)) \geq 1$ then $\alpha(F^{-1}x, F^{-1}u) \geq 1$ for all $(x, y) \in X \times Y$ and $(u, v) \in Y \times X$;
2. There exists $(x_0, y_0) \in X \times Y$ such that
$$\alpha((x_0, y_0), F^{-1}y_0) \geq 1 \text{ and } \alpha(F^{-1}x_0, (y_0, x_0)) \geq 1;$$
3. T is continuous.

Then T has a coupled fixed point.
Proof For the proof we will consider the mapping T given by equation (2.1) is bijective mapping such that $T^{-1}(x, y) = F^{-1}x$.
Also,
(A, B, δ) is a complete bipolar metric space, where $A = X \times Y, B = Y \times X$ and

$$\delta((x, y), (u, v)) = d(x, u) + d(v, y) \text{ for all } (x, y) \in X \times Y \text{ and } (u, v) \in Y \times X.$$

Equation (3.25) implies that

$$\xi(d(F(v, u), F(y, x))) \geq \frac{1}{2}\alpha((v, u), (y, x))[d(x, u) + d(v, y)], \tag{3.26}$$
$$\xi(d(F(x, y), F(u, v))) \geq \frac{1}{2}\alpha((x, y), (u, v))[d(x, u) + d(v, y)], \tag{3.27}$$

Choose $\epsilon = (\epsilon_1, \epsilon_2) \in A, \gamma = (\gamma_1, \gamma_2) \in B$ and $\beta : A \times B \to [0, \infty)$ is the function defined by

$$\beta(\epsilon, \gamma) = \min\{\alpha((\epsilon_1, \epsilon_2), (\gamma_1, \gamma_2)), \alpha((\gamma_2, \gamma_1), (\epsilon_2, \epsilon_1),)\}.$$

Combining (3.26) and (3.27), we get

$$\beta(\epsilon, \gamma)\delta(\epsilon, \gamma) \leq \xi(\delta(T\epsilon, T\gamma)). \tag{3.28}$$

So, clearly T is continuous $(\xi - \alpha)$- expansive mapping.
We take $\epsilon = (\epsilon_1, \epsilon_2) \in A, \gamma = (\gamma_1, \gamma_2) \in B$, such that $\beta(\epsilon, \gamma) \geq 1$.
Now, using condition (i) we get

$$\beta(T^{-1}\epsilon, T^{-1}\gamma) \geq 1.$$

This implies T^{-1} is β-admissible.
Moreover, from the condition (ii) it is clear that there exists $(x_0, y_0) \in A \times B$ such that $\beta(x_0, y_0) \geq 1$ and $\beta(x_0, T^{-1}y_0) \geq 1$.
So, all the conditions of Theorem 3.3 are satisfied. Then T has a fixed point. By Lemma 2.10, F has a coupled fixed point.

In the next Theorem we omit the condition of continuity to get the coupled fixed point of F.

Theorem 3.9. Let (X, Y, d) be a complete bipolar metric space and $F : (X \times Y) \times (Y \times X) \to X \times Y$ is a bijective map. Suppose that $\xi \in \chi$ and $\alpha : (X \times Y) \times (Y \times X) \to [0, \infty)$ such that

$$\xi(d(F(x, y), F(u, v))) \geq \frac{1}{2}\alpha((x, y), (u, v))[d(x, u) + d(v, y)] \tag{3.29}$$

for all $(x, y) \in X \times Y$ and $(u, v) \in Y \times X$ and the following conditions are satisfied:

1. If $\alpha((x,y),(u,v)) \geq 1$ then $\alpha(F^{-1}x, F^{-1}u) \geq 1$ for all $(x,y) \in X \times Y$ and $(u,v) \in Y \times X$;
2. There exists $(x_0, y_0) \in X \times Y$ such that
$$\alpha((x_0,y_0), F^{-1}y_0)) \geq 1 \text{ and } \alpha(F^{-1}x_0, (y_0,x_0)) \geq 1;$$
3. If (x_n, y_n) is a bisequence such that $\alpha(T^{-1}(x_n,y_n), T^{-1}(y_{n+1},x_{n+1})) \geq 1$ for all n and $(x_n) \to y, (y_n) \to x$ as $n \to \infty$ where $x \in X$, $y \in Y$, then $\alpha(T^{-1}(x_n,y_n), T^{-1}(y,x)) \geq 1$ and $\alpha(T^{-1}(x,y), T^{-1}(y_n,x_n)) \geq 1$ for all n .

Then T has a coupled fixed point.
Proof We follow the same notion of Theorem 3.8 to prove the above result. Let (x_n, y_n) is a bisequence such that $\alpha(T^{-1}(x_n,y_n), T^{-1}(y_{n+1},x_{n+1})) \geq 1$ for all n and $(x_n) \to y, (y_n) \to x$ as $n \to \infty$. We get $\beta(T^{-1}(x_n,y_n), T^{-1}(y,x)) \geq 1$ from the condition (iii). Hence all the conditions of Theorem 3.5 are satisfied. So, T has a fixed point. By Lemma 2.10, F has a coupled fixed point.
In the following, we provide a hypothesis to obtain the unique coupled fixed point.

R: There exist $(Z_1, Z_2) \in (X \times Y) \cap (Y \times X))$ such that
$$\alpha((x,y),(z_1,z_2)) \geq 1, \alpha((z_2,z_1),(y,x)) \geq 1,$$
and
$$\alpha((u,v),(z_1,z_2)) \geq 1, \alpha((z_2,z_1),(v,u)) \geq 1,$$
for all $(x,y) \in X \times Y$ and $(u,v) \in Y \times X$.

Theorem 3.10. If we add hypothesis **R** to the Theorem 3.8 and Theorem 3.9 then mapping F has a unique coupled fixed point.
Proof By considering hypothesis **R** one can easily prove that T and β and satisfy hypothesis **P**.
From Theorem 3.7 and Lemma 2.10, clearly, result is obtained.

Example 3.11. Let $X = (-\infty, 0]$ and $Y = [0, +\infty)$
$d(x,y) = |x-y|$ for all $(x,y) \in X \times Y$.
Then, clearly, (X, Y, d) is a complete bipolar metric space.
Define $T : X \cup Y \to X \cup Y$,
$Tx = 7x$. Clearly, T is continuous bijective covariant map.
Taking $\xi(x) = \frac{x}{2}$ and $\alpha(x) = 2$.
Then,
$\xi(d(Tx,Ty)) \geq \alpha(x,y)d(x,y)$ for all $(x,y) \in X \times Y$ becomes
$\frac{7}{2}|x-y| \geq 2|x-y|$,
which is always true, implies that T is $(\xi - \alpha)$- expansive mapping.
Now, as, $2 = \alpha(x,y) \geq 1$ for all $(x,y) \in X \times Y$,
so clearly, T^{-1} is α-admissible.
Also, for all $(x,y) \in X \times Y$, $\alpha(x,y) \geq 1$ and $\alpha(x, T^{-1}y) \geq 1$.
and $X \cap Y = \{0\}$.
Taking $z = 0$, clearly hypothesis **P** is hold.
So, by Theorem 3.7 T has a unique fixed point.
Clearly, 0 is the unique fixed point of T.
Hence, Theorem 3.7 is verified for covariant mapping.

Example 3.12. If in the above example, we take $Tx = -7x$. Then, T is bijective continuous contravariant mapping. One can easily notice that T satisfies all the conditions of Theorem 3.7.
So, by Theorem 3.7 T has a unique fixed point.
Clearly 0 is the unique fixed point of T.
Hence, Theorem 3.7 is verified for contravariant mapping.

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