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Stochastic differential equations driven by relative martingales

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ABSTRACT: This paper contributes to the study of relative martingales. Specifically, for a closed random set H, they are processes null on H which decompose as $M = m + v$, where m is a cádlág uniformly integrable martingale and, v is a continuous process with integrable variations such that $v_0 = 0$ and dv is carried by H. First, we extend this notion to stochastic processes not necessarily null on H, where m is considered local martingale instead of a uniformly integrable martingale. Thus, we provide a general framework for the new larger class of relative martingales by presenting some structural properties. Second, as applications, we construct solutions for skew Brownian motion equations using continuous stochastic processes of the above mentioned new class. In addition, we investigate stochastic differential equations driven by a relative martingale.

Key Words: Relative martingales, Skew Brownian motion, class (Σ), Stochastic differential equations, Signed measure theory.

Contents

1. Introduction

In the theory of zeros of continuous martingales [1], Azéma and Yor have introduced two remarkable classes of processes respectively named: $\mathcal{R}(\mathcal{H})$ and $\mathcal{R}$. More precisely, they are processes satisfying the next both definitions:

Definition 1.1 (Definition 2.1 of Azéma and Yor [1]) Let $\mathcal{H}$ be a random optional closed set. We call $\mathcal{R}(\mathcal{H})$ the class of processes $(X_t; t \geq 0)$ vanishing on $\mathcal{H}$ and admitting a decomposition of the form

$$X_t = M_t + A_t,$$

where $(M_t; t \geq 0)$ is a right continuous uniformly integrable martingale, $(A_t; t \geq 0)$ is a continuous and adapted variation integrable process such that dA is carried by $\mathcal{H}$.

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Definition 1.2 (Definition 2.2 of Azéma and Yor [1]) We call $\mathcal{R}$ the class of processes $(X_t; t \geq 0)$ admitting a decomposition of the form

$$X_t = M_t + A_t,$$

where $(M_t; t \geq 0)$ is a right continuous uniformly integrable martingale, $(A_t; t \geq 0)$ is a continuous and adapted variation integrable process such that dA is carried by $H = \{t \geq 0 : X_t = 0\}$.

Meyer called processes of the class $\mathcal{R}(\mathcal{H})$, relative martingales because they are true martingales outside of the random set $\mathcal{H}$. Remark from Definition 1.1 that all relative martingales vanish on $\mathcal{H}$. This allows to see that $\mathcal{R}(\mathcal{H}) \subset \mathcal{R}$. These both classes have been extensively studied in [1].

On another hand, Yor has extended the notion of class $\mathcal{R}$ to semi-martingales by introducing in [11], an another remarkable larger class (Σ) of processes. Specifically, they are processes X which decompose as $X = M + A$, where M is a cádlág local martingale with $M_0 = 0$ and A is a finite variation process such that the signed measure dA is carried by $\{t \geq 0 : X_t = 0\}$. Such stochastic processes are strongly related to many studies in probability theory. For instance, they play a capital role in the theory of Azéma-Yor martingales, the study of zeros of continuous martingales [1], the study of Brownian local times, the balayage formulas for the progressive case [8]. They are used to resolve Skorokhod's reflection equation and embedding problem . This class has been studied extensively in several studies, enriching the general framework by deriving characterization results, by studying their main properties, presenting their applications, and relaxing more and more the original hypotheses. Note that the class (Σ) contains the other two above mentioned classes. However, remark that $\mathcal{R}(\mathcal{H})$ is include in the class $\mathcal{R}$ and the class (Σ) only because the fact that all elements of the class $\mathcal{R}(\mathcal{H})$ vanish on $\mathcal{H}$. Thus, if we remove this cancellation condition on $\mathcal{H}$ we lose the inclusion of $\mathcal{R}(\mathcal{H})$ in (Σ).

The aim of this paper is to extend the notion of class $\mathcal{R}(\mathcal{H})$ to cádlág processes not necessarily null on $\mathcal{H}$ and whose the martingale part is not necessarily uniformly integrable. We do this by considering a new class that we term $\mathcal{M}(\mathcal{H})$ and define as follows:

Definition 1.3 We shall say that a process M is a relative martingale ($M \in \mathcal{M}(H)$) if it decomposes as $M = m + v$, where

1. m is a cádlág local martingale, with $m_0 = 0$;
2. v is an adapted continuous process with finite variations such that $v_0 = 0$;
3. $\int 1_{\mathcal{H}^c}(s) dv_s = 0$.

Admittedly, this new class is not a subset of the class (Σ) and reciprocally. But, it also contains interesting examples playing a key role in the stochastic analysis. For instance, if $\mathcal{H}$ is the set of zeros of a standard Brownian motion D, hence for an other Brownian motion B independent of D, the geometric Itô-Mckean skew Brownian motion with Azzalini skew normal distribution

$$X^\delta = \sqrt{1-\delta^2} B + \delta |D|$$

is a process of the class $\mathcal{M}(\mathcal{H})$. This process is used by several authors. For instance, Corns and Satchell [4] and Zhu and He [24] worked on this type of skew Brownian motion and priced European style options. Recently, in the preprint (), the authors consider an asset evolving as X^δ to formulate the wealth function under continuous time investment strategy of insurance companies. In this last mentioned reference, the authors investigate the next stochastic differential equation:

$$dS_t = \left(\mu + \frac{\sigma^2}{2} \right) S_t dt + \sigma S_t dX_t^\delta.$$

Thus, it would be useful to provide a general framework and develop techniques to manipulate the processes of this new class of relative martingales. This could open new perspectives in applications and in other areas of probability theory.

The remainder of this paper is organized as follows. In Section 2, we present some useful preliminaries. Section 3 is devoted to the study of the class $\mathcal{M}(\mathcal{H})$, where we give some examples and explore some general properties. Section 4 focuses on the construction of solutions for skew Brownian motion equations using stochastic processes of the class $\mathcal{M}(\mathcal{H})$. Finally, in Section 5, we investigate stochastic differential equations driven by a relative martingale.

2. Notations and recall of some useful results

In this section, we provide notations and recall some useful results that will be used throughout this work. Thus, we start by giving some notations.

2.1. Notations

Throughout we fix a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}, \mathbb{P})$ satisfying the usual conditions. We shall always consider that H is the zero set of a continuous martingale D. And we shall use the following notations:

- $\forall t \geq 0$, $\gamma_t = \sup\{s \leq t : D_s = 0\}$;
- $\gamma = \sup\{t \geq 0 : D_s = 0\}$;
- For any other process X, we will denote $g_t = \sup\{s \leq t : X_s = 0\}$ and $g = \sup\{t \geq 0 : X_s = 0\}$;
- $\mathbb{Q} = \frac{|D_\infty|}{\mathbb{E}(|D_\infty|)}\mathbb{P}$.

We consider in this paper that

$$\mathbb{P}(0 < \gamma < \infty) = 1.$$

Remark that the random time γ is not a stopping time with respect to $(\mathcal{F}_t)_{t\geq 0}$ but an honest time. Hence, we shall denote $(\mathcal{G}_t)_{t\geq 0}$ the smallest right continuous filtration containing $(\mathcal{F}_t)_{t\geq 0}$ for which γ is a stopping time.

On another hand, it is known that for any continuous semi-martingale Y, the set $\mathcal{W} = \{t \geq 0; Y_t = 0\}$ cannot be ordered. However, the set $\mathbb{R}_+ \setminus \mathcal{W}$ can be decomposed as a countable union $\cup_{n\in\mathbb{N}} J_n$ of intervals J_n. Each interval J_n corresponds to some excursion of Y. In other words, if $J_n =]g_n, d_n[$, $Y_t \neq 0$ for all $t \in]g_n, d_n[$ and $Y_{g_n} = Y_{d_n} = 0$. For any constant $\alpha \in [0, 1]$, we consider a sequence (ξ_n) of i.i.d. Bernoulli variables such that

$$\mathbb{P}(\zeta_n = 1) = \alpha \text{ and } P(\zeta_n = -1) = 1 - \alpha.$$

Now, let us define the process Z^Y as follows.

$$Z_t^Y = \sum_{n=0}^{+\infty} \zeta_n 1_{]g_n,d_n[}(t). \tag{2.1}$$

If we assume that α is a piecewise constant function associated with a partition $(0 = t_0 < t_1 < \cdots < t_{n-1} < t_m)$, i.e., α is of the form

$$\alpha(t) = \sum_{i=0}^{m} \alpha_i 1_{[t_i,t_{i+1})}(t),$$

where $\alpha_i \in [0, 1]$ for all $i = 0, 1, \cdots, m$, then we shall consider the process

$$\mathcal{Z}_t^Y = \sum_{n=0}^{+\infty} \sum_{i=0}^{m} \zeta_n^i 1_{]g_n,d_n[\cap[t-i,t_{i+1})}(t), \tag{2.2}$$

where $(\zeta_n^i)_{n\geq 0}$, $i = 1, 2, \cdots, m$, are m independent sequences of independent variables such that

$$\mathbb{P}(\zeta_n^i = 1) = \alpha_i \text{ and } \mathbb{P}(\zeta_n^i = -1) = 1 - \alpha_i.$$

2.2. Some useful results of enlargement filtrations

Now, we shall recall some results of the theory of enlargement filtrations which are useful in the current work.

Proposition 2.1 (Proposition 3.1 of Azéma and Yor [1]) *Let H be a random optional closed set. Denote $g = \sup H$ and represent by $(\mathcal{G}_t)_{t\geq 0}$, the progressive enlargement of the filtration $(\mathcal{F}_t)_{t\geq 0}$ with respect to g. Let $(V_t)_{t\geq 0}$ be a $(\mathcal{G}_{g+t})_{t\geq 0}-$ optional process. There exists a unique $(\mathcal{F}_t)_{t\geq 0}-$ optional process $(U_t)_{t\geq 0}$ which vanishes on H such that $\forall t \geq 0$, $U_{g+t} = V_t$ and $U_0 = V_0$ on $\{g = 0\}$. That defines a function $\rho : V \longmapsto U$. ρ is linear, non-negative and preserves products.*

Theorem 2.1 *[**Quotient theorem: Theorem 3.2 of Azéma and Yor** [1]]*

1. *If $(X_t; t \geq 0)$ is a stochastic process of the class $\mathcal{R}(H)$, hence, the process $(\chi_t; t > 0)$ defined by $\chi_t = \frac{X_{g+t}}{Y_{g+t}}$ is a $(Q, (\mathcal{G}_{g+t})t > 0)$ uniformly integrable martingale.*

2. *Reciprocally, let $(\chi_t; t > 0)$ a $(Q, (\mathcal{G}_{g+t})t > 0)$ uniformly integrable martingale; the stochastic process $X = (Y_t \rho(\chi_\cdot)_t; t \geq 0)$ is the unique process of $\mathcal{R}(H)$ such that $\chi_t = \frac{X_{g+t}}{Y_{g+t}}$ for all $t > 0$.*

Theorem 2.2 *[**Theorem 4.1.2 of Azéma and Yor** [1]] Let $X = m + v$ be a $(\mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$- semimartingale, where m is a local martingale and v is a process with finite variations such that dv is carried by H. The process $\widetilde{X}$ defined by*

$$\widetilde{X}_t = X_{\gamma+t} - X_\gamma - \int_\gamma^{\gamma+t} \frac{d\langle X, |D|\rangle_s}{|D_s|}$$

is then a $(\mathbb{Q}, (\mathcal{G}_{\gamma+t})_{t\geq 0})$- local martingale and $\langle \widetilde{X}, \widetilde{X}\rangle_t = \langle X, X\rangle_{\gamma+t} - \langle X, X\rangle_\gamma$.

Remark that $\widetilde{X}$ holds the next:

$$\widetilde{X}_t = X_{\gamma+t} - X_\gamma - \int_\gamma^{\gamma+t} \frac{d\langle X, D\rangle_s}{D_s}.$$

Lemma 2.1 (Lemma 5.7 of Jeulin [6]) *Let g be an honest variable with respect to $(\mathcal{F}_t)_{t\geq 0}$. Let $(\mathcal{G}_t)_{t\geq 0}$ be the progressive enlargement of the filtration $(\mathcal{F}_t)_{t\geq 0}$ with respect to g. If τ is a stopping time with respect to $(\mathcal{G}_t)_{t\geq 0}$ such that $g < \tau$ on $\{g < \infty\}$, hence*

$$\mathcal{G}_\tau = \mathcal{F}_\tau.$$

2.3. Recall of some useful balayage formulas

Balayage formulas are power tools in this work. In next, we recall some balayage results we use in this paper. Thus, we start by the predictable case for continuous semimartingales.

Proposition 2.2 *Let X be a continuous semimartingale and define $g_t = \sup\{s \leq t : X_s = 0\}$. If k is a locally bounded predictable process, then*

$$k_{g_t} X_t = k_{g_0} X_0 + \int_0^t k_{g_s} dX_s.$$

In next, we provide the result for cádlág semimartingales.

Proposition 2.3 *Let X be a continuous semimartingale and define $g_t = \sup\{s \leq t : X_s = 0\}$. If k is a bounded predictable process, then*

$$k_{g_t} X_t = k_{g_0} X_0 + \int_0^t k_{g_s} dX_s.$$

The balayage formulas for continuous semi-martingales in the progressive case, are critical tools in this study. We recall these results below.

Proposition 2.4 *Let Y be a continuous semi-martingale and $\gamma_t^{'} = \sup\{s \leq t : Y_s = 0\}$. Let k be a bounded progressive process, where ${}^p k.$ denotes its predictable projection. Then,*

$$k_{\gamma_t^{'}} Y_t = k_0 Y_0 + \int_0^t {}^p k_{\gamma_s^{'}} dY_s + R_t,$$

where R is an adapted, continuous process with bounded variations such that dR_t is carried by the set $\{Y_s = 0\}$.

Proposition 2.4 is a powerful and interesting tool. However, the fact that we know nothing about the form of the process R can be limiting. The processes Z^Y and $\mathcal{Z}^Y$ are critical to this study. Bouhadou and Ouknine [2] identified the process R of Proposition 2.4 when the progressive process k is equal to Z^α or $\mathcal{Z}^\alpha$. We recall these results below.

Proposition 2.5 (Ouknine and Bouhadou [2]) *Let Y be a continuous semi-martingale and Z^Y be the process defined in (2.1). Then,*

$$Z_t^Y Y_t = \int_0^t Z_s^Y dY_s + (2\alpha - 1) L_t^0(Z^Y Y),$$

where $L_.^0(Z^Y Y)$ is the local time of the semi-martingale $Z^Y Y$.

Proposition 2.6 (Ouknine and Bouhadou [2]) *Let Y be a continuous semi-martingale and Z^Y be the process defined in (2.1). Then,*

$$\mathcal{Z}_t^Y Y_t = \int_0^t \mathcal{Z}_s^Y dY_s + \int_0^t (2\alpha(s) - 1) dL_s^0(\mathcal{Z}^Y Y),$$

where $L_.^0(\mathcal{Z}^Y Y)$ is the local time of the semi-martingale $\mathcal{Z}^Y Y$.

3. A general framework for a larger family of relative martingales

In this section, we bring contributions to the study of stochastic processes of the form: $M = m + v$, where m is a martingale and v is a process with finite variations such that dv is carried by H. Note that a known subfamily of such processes is the class of relative martingales, $\mathcal{R}(H)$. Here, we extend this notion of relative martingales to semimartingales which don't necessary vanish on H and whose the martingale part is not necessary uniformly integrable. More precisely, we provide a general framework to a larger class of processes that we shall name, class $\mathcal{M}(H)$.

3.1. Some examples

Now, we shall provide some examples of the class $\mathcal{M}(H)$. First remark a natural example which is, the process $M = |D|$. In fact, all processes of the class $\mathcal{R}(H)$ and all elements, X of the class (Σ) such that $\{t \geq 0 : X_t = 0\} \subset H$, are in the class $\mathcal{M}(H)$. However, there also exist stochastic processes which don't necessary vanish on H. In next, we provide some such examples.

Example 3.1 Any semimartingale $M = m + v$ such that $M_0 = 0$ and $DM - \langle D, M\rangle$ is a local martingale, is an element of the class $\mathcal{M}(H)$. Indeed, We have from integration by parts that

$$D_t M_t = \int_0^t M_s dD_s + \int_0^t D_s dm_s + \int_0^t D_s dv_s + \langle D, M\rangle_t.$$

Hence, it follows that $\int_0^t D_s dv_s = 0$. That is, dv is carried by H. This proves that $M \in \mathcal{M}(H)$.

Example 3.2 Let m be a cádlág local martingale which vanishes at zero and X be a continuous process of the class (Σ) such that $\{t \geq 0 : X_t = 0\} \subset H$. Hence, the following processes are in the class $\mathcal{M}(H)$:

- $X^1 = \min(m, m - X)$;

- $X^2 = \min(m, m + X)$;
- $X^3 = \min(m - X, m + X)$;
- $X^4 = \max(m, m - X)$;
- $X^5 = \max(m, m + X)$,
- $X^6 = \max(m - X, m + X)$.

3.2. Some structural properties

Here, we shall explore some general properties satisfied by stochastic processes of the class $\mathcal{M}(H)$. Hence, we start by the next remark:

Remark 3.1 The class $\mathcal{M}(H)$ is a vector space.

In what follows, we derive some properties related to the notion of stochastic integral.

Lemma 3.1 *Let $M = m + v$ be a process of the class $\mathcal{M}(H)$. The following hold:*

1. *For any locally bounded predictable process h, $\int_0^{\cdot} h_s dM_s$ is an element of the class $\mathcal{M}(H)$.*
2. *If h is a locally bounded predictable process null on H. Hence, $\int_0^{\cdot} h_s dM_s$ and $\int_0^{\cdot} h_{s-} dM_s$ are local martingales.*

Proof: We have $\forall t \geq 0$,

$$\int_0^t h_s dM_s = \int_0^t h_s dm_s + \int_0^t h_s dv_s$$

and

$$\int_0^t h_{s-} dM_s = \int_0^t h_{s-} dm_s + \int_0^t h_{s-} dv_s.$$

But, $\int_0^t h_{s-} dv_s = \int_0^t h_s dv_s$ because h is continuous.

1. Hence, $\int_0^{\cdot} h_s dM_s \in \mathcal{M}(H)$ since $\int_0^{\cdot} h_s dm_s$ is a local martingale and it is obvious to see that $A = \int_0^{\cdot} h_s dv_s$ is a process with finite variations such that dA is carried by H.
2. Since h vanishes on H and dv is carried by H, we obtain that $\int_0^t h_s dv_s = 0$. Consequently, $\int_0^{\cdot} h_s dM_s$ and $\int_0^{\cdot} h_{s-} dM_s$ are local martingales.

□

Lemma 3.2 *For any processes M and W of the class $\mathcal{M}(H)$, $MW - [M, W]_.$ is also an element of the class $\mathcal{M}(H)$.*

Proof: Through integration by parts, we have:

$$M_t W_t = \int_0^t M_{s-} dW_s + \int_0^t W_{s-} dM_s + [M, W]_t.$$

Hence,

$$M_t W_t - [M, W]_t = \int_0^t M_{s-} dW_s + \int_0^t W_{s-} dM_s.$$

Then, we obtain the result from Remark 3.1 and Lemma 3.1. □

In what follows, we derive a series of corollaries of Lemma 3.2 which show that the process $MW - [M, W]_.$ can be a local martingale under some assumptions.

Corollary 3.1 *If M and W are processes of the class $\mathcal{M}(H)$ which vanish on H, hence $MW - [M,W].$ is a local martingale.*

Proof: Let M and W be processes of the class $\mathcal{M}(H)$. According to Lemma 3.2, $MW - [M,W].$ is an element of the class $\mathcal{M}(H)$. Moreover, we have:

$$M_tW_t - [M,W]_t = \int_0^t M_{s-}dW_s + \int_0^t W_{s-}dM_s.$$

But, we know from Lemma 3.1 that $\int_0^t M_{s-}dW_s$ and $\int_0^t W_{s-}dM_s$ are local martingales because M and W vanish on H. This completes the proof. □

Corollary 3.2 *If M is a local martingale vanishing on H and W is a process of the class $\mathcal{M}(H)$. Hence, $MW - [M,W].$ is a local martingale.*

Proof: According to Lemma 3.2, $MW - [M,W].$ is also an element of the class $\mathcal{M}(H)$ and $\forall t \geq 0$,

$$M_tW_t - [M,W]_t = \int_0^t M_{s-}dW_s + \int_0^t W_{s-}dM_s.$$

We can remark that $\int_0^{\cdot} W_{s-}dM_s$ is a local martingale. Moreover, we deduce from Lemma 3.1 that $\int_0^{\cdot} M_{s-}dW_s$ is also a local martingale since M vanishes on H. □

Corollary 3.3 *For any process M of the class $\mathcal{M}(H)$, the process $MD - [M,D].$ is a local martingale.*

Proof: It is enough to notice that M is a process of the class $\mathcal{M}(H)$ and D is by definition, a martingale which vanishes on H. Thus, we obtain the result from Corollary 3.2. □

In next corollary, we show that the product of the processes of class $\mathcal{M}(H)$ with vanishing quadratic covariations is again a relative martingale and in particular under some assumptions, a local martingale.

Corollary 3.4 *Let $(X_t^1)_{t\geq 0}, \cdots, (X_t^n)_{t\geq 0}$ be processes of the class $\mathcal{M}(H)$ such that $[X^i, X^j] = 0$ for $i \neq j$. Hence, the following hold:*

1. *$(\Pi_{i=1}^n X_t^i)_{t\geq 0}$ is also of class $\mathcal{M}(H)$.*
2. *If $\forall i \in \{1, \cdots, n\}$, X^i vanishes on H. Hence, $(\Pi_{i=1}^n X_t^i)_{t\geq 0}$ is a local martingale.*
3. *If $\exists l \in \{1, \cdots, n\}$ such that X^l is a local martingale null on H. Hence, $(\Pi_{i=1}^n X_t^i)_{t\geq 0}$ is a local martingale.*

Proof:

1. Let us first take $n = 2$. Through Lemma 3.2, we obtain that $X^1X^2 - [X^1, X^2]$ is a process of the class $\mathcal{M}(H)$. That is, $X^1X^2 \in \mathcal{M}(H)$ since $[X^1, X^2] = 0$. Hence, we obtain by induction that for any family $(X_t^1)_{t\geq 0}, \cdots, (X_t^n)_{t\geq 0}$ of the class $\mathcal{M}(H)$ such that $[X^i, X^j] = 0$ for $i \neq j$, the process $(\Pi_{i=1}^n X_t^i)_{t\geq 0}$ is also of class $\mathcal{M}(H)$.
2. We proceed in the same way as 1) by using Corollary 3.1 instead of Lemma 3.2 to show that $(\Pi_{i=1}^n X_t^i)_{t\geq 0}$ is a local martingale.
3. Now, we assume that there exists $l \in \{1, \cdots, n\}$ such that X^l is a local martingale null on H. Remark that:
$$\Pi_{i=1}^n X_t^i = X^l \times \Pi_{i=1, i\neq l}^n X_t^i.$$
But, we can see from 1) that $\Pi_{i=1, i\neq l}^n X_t^i \in \mathcal{M}(H)$. Hence, we obtain the result by using Corollary 3.2.

□

Now, we shall derive the result from which Example 3.2 follows.

Lemma 3.3 *Let M and W be processes of the class $\mathcal{M}(H)$ such that W is continuous and $\{t \geq 0 : W_t = 0\} \subset H$. Hence, next processes are elements of the class $\mathcal{M}(H)$:*

1. $X^1 = \min(M, M - W)$;
2. $X^2 = \min(M, M + W)$;
3. $X^3 = \min(M - W, M + W)$;
4. $X^4 = \max(M, M - W)$;
5. $X^5 = \max(M, M + W)$;
6. $X^6 = \max(M - W, M + W)$.

Proof: Firstly, we obtain by using formulas $\min(x, y) = \frac{x+y-|x-y|}{2}$, $\max(x, y) = \frac{x+y+|x-y|}{2}$ that:

1. $X^1_t = M_t - \frac{1}{2}W_t - \frac{1}{2}|W_t|$;
2. $X^2 = M_t + \frac{1}{2}W_t - \frac{1}{2}|W_t|$;
3. $X^3 = M_t - |W_t|$;
4. $X^4 = M_t - \frac{1}{2}W_t + \frac{1}{2}|W_t|$;
5. $X^5 = M_t + \frac{1}{2}W_t + \frac{1}{2}|W_t|$;
6. $X^6 = M_t + |W_t|$.

Moreover, we have from Tanaka's formula that

$$|W_t| = \int_0^t sign(W_s)dW_s + L^0_t(W).$$

We can see that $dL^0_\cdot(W)$ is carried by H because $\{t \geq 0 : W_t = 0\} \subset H$. Hence, $|W| \in \mathcal{M}(H)$ since according to Lemma 3.1, $\int_0^\cdot sign(W_s)dW_s \in \mathcal{M}(H)$. Consequently, we obtain from Remark 3.1 that the above mentioned processes are elements of the class $\mathcal{M}(H)$. □

Now, we shall derive some properties using the balayage formulas. Hence, we start by the predictable case.

Lemma 3.4 *Let M be a continuous process of the class $\mathcal{M}(H)$, and let $g_t = \sup\{s \leq t : M_s = 0\}$. Then, for any locally bounded predictable process k, $k_{g_\cdot}M$ is also an element of class $\mathcal{M}(H)$.*

Proof: By applying the balayage formula, we obtain the following:

$$k_{g_t}M_t = k_{g_0}M_0 + \int_0^t k_{g_s}dM_s = \int_0^t k_{g_s}dM_s.$$

But, we know from Lemma 3.1 that $\int_0^\cdot k_{g_s}dM_s \in \mathcal{M}(H)$. This completes the proof. □

The following Corollary present us a situation under which relative martingales are also processes of the class (Σ).

Corollary 3.5 *Any non-negative process $M = m + v$ of the class $\mathcal{M}(H)$ which vanishes on H, is an element of the class (Σ).*

Proof: Since M vanishes on H, we obtain from Lemma 3.3 that for any locally bounded borel function f, $(f(v_{\gamma_t})M_t : t \geq 0) \in \mathcal{M}(H)$, where $\gamma_t = \sup\{s \leq t : s \in H\}$. In addition, we have from balayage formula's that

$$f(v_{\gamma_t})M_t = \int_0^t f(v_{\gamma_s})dM_s.$$

But, $v_{\gamma_t} = v_t$ since dv is carried by H. Then,

$$f(v_t)M_t = \int_0^t f(v_s)dm_s + \int_0^t f(v_s)dv_s.$$

Therefore, the process $\left(f(v_t)M_t - \int_0^t f(v_s)dv_s : t \geq 0\right)$ is a local martingale. Consequently, we obtain from Theorem 2.4 of [9] that $M \in (\Sigma)$. This completes the Proof. □

Remark 3.2 In fact, all continuous relative martingales of the class $\mathcal{M}(H)$ which vanish on H, are processes of the class (Σ). Indeed, it suffices to apply the above corollary to $|M|$ and to recall that $|M| \in (\Sigma) \Leftrightarrow M \in (\Sigma)$.

Now, we shall use the balayage formula in progressive case to construct processes of the class (Σ) from relative martingales.

Lemma 3.5 *Let M be a continuous process of class $\mathcal{M}(H)$, and let $g_t = \sup\{s \leq t : M_s = 0\}$. Then, for any cádlág bounded progressive process k which vanishes on H, $k_{g.}M$ is an element of class (Σ).*

Proof: The balayage formula in progressive case through that $\forall t \geq 0$,

$$k_{g_t}M_t = \int_0^t {}^p(k_{g_s})dM_s + R_t,$$

where ${}^p(k_{g_s})$ is the predictable projection of k_{g_s} and R is a continuous process with finite variations such dR is carried by $\{t \geq 0 : M_t = 0\}$. Since k is cádlág, we have ${}^p(k_{g_s}) = k_{s-}$. Hence, we obtain:

$$k_{g_t}M_t = \int_0^t k_{s-}dM_s + R_t$$

Which implies the following:

$$k_{g_t}M_t = \int_0^t k_s dM_s + R_t$$

because M is continuous. But, we have from Lemma 3.1 that $\int_0^{\cdot} k_s dM_s$ is a local martingale because h vanishes on H. Consequently, the result holds. □

3.3. Relationship with the Azéma-Yor relative martingales

Now, we shall state some relationship between the classes $\mathcal{M}(H)$ and $\mathcal{R}(H)$. More precisely, we derive some results which permit to decompose a process M of the class $\mathcal{M}(H)$ as:

$$M = M^1 + M^2, \tag{3.1}$$

where $M^1 \in \mathcal{R}(H)$ and $M^2 \in \mathcal{M}(H)$. Hence, we start by the following proposition:

Proposition 3.1 *Let M be a process of the class $\mathcal{M}(H)$ such that its martingale part is uniformly integrable and $\langle M, D\rangle = 0$. Hence, the process $(M_t - M_{\gamma_t} : t \geq 0)$ is a relative martingale of the class $\mathcal{R}(H)$.*

Proof: According to Corollary 3.3, $DM - \langle D, M\rangle = DM$ is a uniformly integrable martingale. Hence, we obtain from quotient theorem that $(M_{t+\gamma})_{t\geq 0}$ is a uniformly integrable martingale with respect to $(\mathcal{G}_{\gamma+t})_{t\geq 0}$. Which entails that $(M_{t+\gamma} - M_\gamma)_{t\geq 0}$ is also a uniformly integrable martingale with respect to the filtration $(\mathcal{G}_{\gamma+t})_{t\geq 0}$. Hence, there exists a random variable M_∞ such that $M_{\gamma+t} - M_\gamma \to M_\infty$ and $\forall t > 0$, we have:

$$M_{t+\gamma} - M_\gamma = \mathbb{E}[M_\infty|\mathcal{G}_{\gamma+t}].$$

But, we know thanks to Lemma 5.7 of [6] that $\mathcal{G}_{\gamma+t} = \mathcal{F}_{\gamma+t}$. Then,

$$M_{t+\gamma} - M_\gamma = \mathbb{E}[M_\infty|\mathcal{F}_{\gamma+t}].$$

Hence, it follows that

$$\rho(M_{\cdot+\gamma} - M_\gamma)_t = \rho(\mathbb{E}[M_\infty|\mathcal{F}_{\gamma+\cdot}])_t.$$

Now, let $Z_t = M_t - M_{\gamma_t}$ and $Z^{'}_t = \mathbb{E}[M_\infty 1_{\{\gamma<t\}}|\mathcal{F}_t]$. We can remark that Z and $Z^{'}$ vanish on H and $\forall t \geq 0$,

$$Z_{\gamma+t} = M_{t+\gamma} - M_\gamma \text{ and } Z^{'}_{t+\gamma} = \mathbb{E}[M_\infty|\mathcal{F}_{\gamma+t}].$$

Consequently, we obtain from uniqueness that

$$M_t - M_{\gamma_t} = \mathbb{E}[M_\infty 1_{\{\gamma<t\}}|\mathcal{F}_t].$$

Consequently, we conclude from Proposition 2.2 of [1] that the process $(M_t - M_{\gamma_t} : t \geq 0)$ is an element of the class $\mathcal{R}(H)$. □

Remark 3.3 We retain that under assumptions of Proposition 3.1, there exists a random variable M_∞ such that $M_{\gamma+t} - M_\gamma \to M_\infty$ as $t \to \infty$ and for any stopping time $0 < T < \infty$, we have:

$$M_T - M_{\gamma_T} = \mathbb{E}[M_\infty 1_{\{\gamma<T\}}|\mathcal{F}_T].$$

Hence, in the particular case where M vanishes on H, we obtain the representation result given in Proposition 2.2 of [1]. That is,

$$M_T = \mathbb{E}[M_\infty 1_{\{\gamma<T\}}|\mathcal{F}_T].$$

Next corollary permit us to see that under assumptions of Proposition 3.1, the process $(M_{\gamma_t} : t \geq 0)$ is also in the class $\mathcal{M}(H)$.

Corollary 3.6 *Let M be a process of the class $\mathcal{M}(H)$ such that its martingale part is uniformly integrable and $\langle M, D\rangle = 0$. Hence, the process $(M_{\gamma_t} : t \geq 0)$ is a relative martingale of the class $\mathcal{M}(H)$.*

Proof: We have $\forall t \geq 0$, $M_{\gamma_t} = (M_{\gamma_t} - M_t) + M_t$. But, according to Proposition 3.1, $(M_t - M_{\gamma_t} : t \geq 0) \in \mathcal{R}(H)$. Hence, we obtain from Remark 3.1 that $(M_{\gamma_t} : t \geq 0) \in \mathcal{M}(H)$. □

Remark 3.4 We obtain from Proposition 3.1 and Corollary 3.6 that any process M satisfying assumptions of Proposition 3.1 admits the decomposition given in (3.1), where $M^1 = (M_t - M_{\gamma_t} : t \geq 0)$ and $M^2 = (M_{\gamma_t} : t \geq 0)$.

In the following, we denote $\widetilde{M}$ to represent the process defined by $\forall t \geq 0$,

$$\widetilde{M}_t = M_{\gamma+t} - M_\gamma - \int_\gamma^{\gamma+t} \frac{d\langle M, D\rangle_s}{D_s}.$$

Recall that from Theorem 4.1.2 of [1], $\widetilde{M}$ is a local martingale with respect to the filtration $(\mathcal{G}_{\gamma+t})_{t\geq 0}$ when M is a process of the class $\mathcal{M}(H)$. Hence, we derive an another decomposition of the form (3.1) for M in the case where $\widetilde{M}$ is a true martingale.

Proposition 3.2 *Let M be a process of the class $\mathcal{M}(H)$ such that $\widetilde{M}$ is a $(\mathcal{G}_{\gamma+t})_{t\geq 0}$ uniformly integrable martingale. Hence, the process $\left(M_t - M_{\gamma_t} - \int_{\gamma_t}^t \frac{d\langle M, D\rangle_s}{D_s} : t \geq 0\right)$ is an element of the class $\mathcal{R}(H)$.*

Proof: Since $\widetilde{M}$ is a uniformly integrable martingale with respect to the filtration $(\mathcal{G}_{\gamma+t})_{t\geq 0}$. Hence, there exists an integrable random variable M_∞ such that $\widetilde{M_t} \to M_\infty$ as $t \to \infty$ and $\forall t \geq 0$,

$$\widetilde{M_t} = \mathbb{E}[M_\infty|\mathcal{G}_{\gamma+t}] = \mathbb{E}[M_\infty|\mathcal{F}_{\gamma+t}].$$

Hence, it follows that

$$\rho(\widetilde{M})_t = \rho(\mathbb{E}[M_\infty|\mathcal{F}_{\gamma+.}])_t.$$

Now, let

$$Z_t = M_t - M_{\gamma_t} - \int_{\gamma_t}^t \frac{d\langle M, D\rangle_s}{D_s} \text{ and } Z_t^{'} = \mathbb{E}[M_\infty 1_{\{\gamma<t\}}|\mathcal{F}_t].$$

We can remark that Z and $Z^{'}$ vanish on H and $\forall t \geq 0$,

$$Z_{\gamma+t} = \widetilde{M}_t \text{ and } Z_{t+\gamma}^{'} = \mathbb{E}[M_\infty|\mathcal{F}_{\gamma+t}].$$

Hence, we obtain that

$$M_t - M_{\gamma_t} - \int_{\gamma_t}^t \frac{d\langle M, D\rangle_s}{D_s} = \mathbb{E}[M_\infty 1_{\{\gamma<t\}}|\mathcal{F}_t].$$

Consequently, we conclude from Proposition 2.2 of [1] that the process

$$\left(M_t - M_{\gamma_t} - \int_{\gamma_t}^t \frac{d\langle M, D\rangle_s}{D_s} : t \geq 0\right)$$

is an element of the class $\mathcal{R}(H)$. □

Corollary 3.7 *Let M be a process of the class $\mathcal{M}(H)$ such that $\widetilde{M}$ is a $(\mathcal{G}_{\gamma+t})_{t\geq 0}$ uniformly integrable martingale. Hence, the process $\left(M_{\gamma_t} + \int_{\gamma_t}^t \frac{d\langle M,D\rangle_s}{D_s} : t \geq 0\right)$ is also an element of the class $\mathcal{M}(H)$.*

Proof: We have $\forall t \geq 0$,

$$M_{\gamma_t} + \int_{\gamma_t}^t \frac{d\langle M, D\rangle_s}{D_s} = \left(M_{\gamma_t} + \int_{\gamma_t}^t \frac{d\langle M, D\rangle_s}{D_s} - M_t\right) + M_t.$$

But, according to Proposition 3.2,

$$\left(M_{\gamma_t} + \int_{\gamma_t}^t \frac{d\langle M, D\rangle_s}{D_s} - M_t : t \geq 0\right) \in \mathcal{R}(H).$$

Hence, we obtain from Remark 3.1 that $\left(M_{\gamma_t} + \int_{\gamma_t}^t \frac{d\langle M,D\rangle_s}{D_s} : t \geq 0\right) \in \mathcal{M}(H)$. □

Corollary 3.8 *Let M be a process of the class $\mathcal{M}(H)$ such that $\widetilde{M}$ is a $(\mathcal{G}_{\gamma+t})_{t\geq 0}$ uniformly integrable martingale and $d\langle M, D\rangle$ is carried by H. Hence, the following hold:*

1. *$(M_t - M_{\gamma_t} : t \geq 0)$ is an element of the class $\mathcal{R}(H)$;*
2. *$(M_{\gamma_t} : t \geq 0)$ is an element of the class $\mathcal{M}(H)$.*

Proof: We obtain respectively from Proposition 3.2 and Corollary 3.7 that

$$\left(M_t - M_{\gamma_t} - \int_{\gamma_t}^t \frac{d\langle M, D\rangle_s}{D_s} : t \geq 0\right) \in \mathcal{R}(H)$$

and

$$\left(M_{\gamma_t} + \int_{\gamma_t}^t \frac{d\langle M, D\rangle_s}{D_s} : t \geq 0\right) \in \mathcal{M}(H).$$

However, $\forall t \geq 0$, $\int_{\gamma_t}^t \frac{d\langle M,D\rangle_s}{D_s} = 0$ since $d\langle M, D\rangle$ is carried by H. Which completes the proof. □

4. Applications to skew brownian motion equations

In this section, weak solutions to time-homogeneous and time-inhomogeneous skew Brownian motions starting form zero are constructed on the one hand, with the help of a geometric Itô-Mckean skew Brownian motion with Azzalini skew normal distribution $X^\delta = \sqrt{1-\delta^2}B + \delta|W|$ and on the other hand, we do it from arbitrary continuous processes of the class $\mathcal{M}(H)$. More precisely, we talk about of the two next equations:

$$X_t = x + B_t + (2\alpha - 1)L_t^0(X) \tag{4.1}$$

and

$$X_t = x + B_t + \int_0^t (2\alpha(s) - 1)dL_s^0(X), \tag{4.2}$$

where B is a standard Brownian motion and $x = 0$. It must be remark that solutions had already been built from the processes of the class (Σ) (see [5]). This should not be seen as a redundancy because the above mentioned processes are not necessary in the class (Σ). Indeed, it is only when X^δ vanishes on $\{t \geq 0 : W_t = 0\}$ that $X^\delta \in (\Sigma)$. And on another hand, an element X of the class $\mathcal{M}(H)$ is in the class (Σ) only when X vanishes on H.

4.1. Construction of solution from Itô-Mckean skew brownian motion

Recall that we presented X^δ in Section 2 as an element of the class $\mathcal{M}(H)$. In fact, this is true only when W vanishes on H. In this subsection, we shall consider X^δ in general case. That is, W does not necessarily vanish on H. Thus, under these assumptions, we construct from $X^\delta = \sqrt{1-\delta^2}B + \delta|W|$, solutions for Equations 4.1 and 4.2. For this purpose, we shall set Z^W and Z^1 to represent processes constructed in (2.1) with respect to W and $(Z_{g_t}^W X_t^\delta; t \geq 0)$ respectively and Z^2 is the process defined in (2.2) with respect to $(Z_{g_t}^W X_t^\delta; t \geq 0)$. We shall also set $g_t = \sup\{t \geq 0 : X_t^\delta = 0\}$.

Proposition 4.1 *The process $Y^{\delta,1}$ defined by $\forall t \geq 0$, $Y_t^{\delta,1} = Z_t^1 Z_{g_t}^W X_t^\delta$ is a weak solution of (4.1) with the parameter α and starting from 0.*

Proof: By applying the balayage formula in the progressive case, we get

$$Z_{g_t}^W X_t^\delta = \int_0^t {}^p(Z_{g_s}^W) dX_s^\delta + R_t,$$

where R is a continuous process with finite variations such that dR is carried by $\{t \geq 0 : X_t^\delta = 0\}$. Since W is continuous, we have: ${}^p(Z_{g_s}^W) = Z_{g_{s-}}^W = Z_{s-}^W$. Thus, it follows from the continuity of X^δ that

$$Z_{g_t}^W X_t^\delta = \int_0^t Z_s^W dX_s^\delta + R_t.$$

Now, remark from Tanaka's formula that

$$dX_s^\delta = \sqrt{1-\delta^2}dB_s + \delta sign(W_s)dW_s + \delta dL_s^0(W).$$

Hence, we obtain:

$$Z_{g_t}^W X_t^\delta = \sqrt{1-\delta^2}\int_0^t Z_s^W dB_s + \delta\int_0^t Z_s^W sign(W_s)dW_s + \delta\int_0^t Z_s^W dL_s^0(W) + R_t.$$

Which becomes

$$Z_{g_t}^W X_t^\delta = \sqrt{1-\delta^2}\int_0^t Z_s^W dB_s + \delta\int_0^t Z_s^W sign(W_s)dW_s + R_t$$

since $dL^0(W)$ is carried by $\{t \geq 0 : W_t = 0\} = \{t \geq 0 : Z_t^W = 0\}$. Hence, through Proposition 2.5, we get

$$Y_t^{\delta,1} = \sqrt{1-\delta^2}\int_0^t Z_s^1 Z_s^W dB_s + \delta\int_0^t Z_s^1 Z_s^W sign(W_s)dW_s + \int_0^t Z_s^1 dR_s + (2\alpha - 1)L_0^0(Y^{\delta,1}).$$

But, dR is carried by $\{t \geq 0 : X_t^\delta = 0\}$ and $\{t \geq 0 : X_t^\delta = 0\} \subset \{t \geq 0 : Z_{g_t}^W X_t^\delta = 0\} = \{t \geq 0 : Z_t^1 = 0\}$. Therefore,

$$Y_t^{\delta,1} = \sqrt{1-\delta^2}\int_0^t Z_s^1 Z_s^W dB_s + \delta\int_0^t Z_s^1 Z_s^W sign(W_s)dW_s + (2\alpha-1)L_t^0(Y^{\delta,1}).$$

Now, remark that the process M defined by $\forall t \geq 0$,

$$M_t = \sqrt{1-\delta^2}\int_0^t Z_s^1 Z_s^W dB_s + \delta\int_0^t Z_s^1 Z_s^W sign(W_s)dW_s$$

is a continuous local martingale. In addition, we have thanks to the continuity of processes B and W:

$$M_t = \sqrt{1-\delta^2}\int_0^t k_s^1 k_s^W dB_s + \delta\int_0^t k_s^1 k_s^W sign(W_s)dW_s,$$

where k^1 and k^W are progressive processes constructed in (2.1) with respect to W and $(Z_{g_t}^W X_t^\delta; t \geq 0)$ respectively. On another hand, we have:

$$\langle M, M\rangle_t = (1-\delta^2)\int_0^t (k_s^1 k_s^W)^2 ds + \delta^2\int_0^t (k_s^1 k_s^W sign(W_s))^2 ds.$$

Which implies: $\langle M, M\rangle_t = t$ because $k_s^1 \in \{-1,1\}$, $k_s^W \in \{-1,1\}$ and $sign(W_s) \in \{-1,1\}$. Consequently, M is a Brownian motion. This completes the proof. □

Proposition 4.2 *The process $Y^{\delta,2}$ defined by $\forall t \geq 0$, $Y_t^{\delta,2} = Z_t^2 Z_{g_t}^W X_t^\delta$ is a weak solution of (4.2) with the parameter α and starting from 0.*

Proof: We have yet showed in the above last proof that

$$Z_{g_t}^W X_t^\delta = \sqrt{1-\delta^2}\int_0^t Z_s^W dB_s + \delta\int_0^t Z_s^W sign(W_s)dW_s + R_t.$$

Hence, from Proposition 2.6, we get

$$Y_t^{\delta,2} = \sqrt{1-\delta^2}\int_0^t Z_s^2 Z_s^W dB_s + \delta\int_0^t Z_s^2 Z_s^W sign(W_s)dW_s + \int_0^t Z_s^2 dR_s + \int_0^t (2\alpha(s)-1)dL_s^0(Y^{\delta,2}).$$

But, dR is carried by $\{t \geq 0 : X_t^\delta = 0\}$ and $\{t \geq 0 : X_t^\delta = 0\} \subset \{t \geq 0 : Z_{g_t}^W X_t^\delta = 0\} = \{t \geq 0 : Z_t^2 = 0\}$. Therefore,

$$Y_t^{\delta,2} = \sqrt{1-\delta^2}\int_0^t Z_s^2 Z_s^W dB_s + \delta\int_0^t Z_s^2 Z_s^W sign(W_s)dW_s + \int_0^t (2\alpha(s)-1)dL_s^0(Y^{\delta,2}).$$

Now, remark that the process M defined by $\forall t \geq 0$,

$$M_t^{'} = \sqrt{1-\delta^2}\int_0^t Z_s^2 Z_s^W dB_s + \delta\int_0^t Z_s^2 Z_s^W sign(W_s)dW_s$$

is a continuous local martingale. In addition, we have thanks to the continuity of processes B and W:

$$M_t^{'} = \sqrt{1-\delta^2}\int_0^t k_s^2 k_s^W dB_s + \delta\int_0^t k_s^1 k_s^W sign(W_s)dW_s,$$

where k^2 and k^W are progressive processes constructed in (2.1) with respect to W and $(Z_{g_t}^W X_t^\delta; t \geq 0)$ respectively. On another hand, we have:

$$\langle M^{'}, M^{'}\rangle_t = (1-\delta^2)\int_0^t (k_s^2 k_s^W)^2 ds + \delta^2\int_0^t (k_s^2 k_s^W sign(W_s))^2 ds.$$

Which implies: $\langle M^{'}, M^{'}\rangle_t = t$ because $k_s^2 \in \{-1,1\}$, $k_s^W \in \{-1,1\}$ and $sign(W_s) \in \{-1,1\}$. Consequently, $M^{'}$ is a Brownian motion. This completes the proof. □

4.2. Construction of solutions from relative martingales

Now, we shall derive solutions by using continuous processes of the class $\mathcal{M}(H)$. Thus, for any continuous process X of the last mentioned class, we let $g_t = \sup\{s \leq t : X_t = 0\}$ and $\tau_t = \inf\{s \geq 0 : \langle X, X\rangle_s > t\}$. Let Z^D and Z^1 be progressive processes defined in (2.1) with respect to D and $(Z^D_{g_t}X_t : t \geq 0)$ respectively. Z^2 is the progressive process defined in (2.2) with respect to $(Z^D_{g_t}X_t : t \geq 0)$.

Proposition 4.3 *The process $\mathcal{Y}^1$ defined by $\forall t \geq 0$, $\mathcal{Y}^1_t = Z^1_t Z^D_{g_{\tau_t}} X_{\tau_t}$ is a weak solution of (4.1) with the parameter α and starting from 0.*

Proof: First remark that Z^D is a continuous bounded progressive process which vanishes on H. Hence, we obtain from Lemma 5.1 that $(Z^D_{g_t}X_t : t \geq 0)$ is a continuous process of the class (Σ). Hence, we obtain the result by applying Proposition 8 of [5] on the process $(Z^D_{g_t}X_t : t \geq 0)$. □

Proposition 4.4 *The process $\mathcal{Y}^2$ defined by $\forall t \geq 0$, $\mathcal{Y}^2_t = Z^2_t Z^D_{g_{\tau_t}} X_{\tau_t}$ is a weak solution of (4.2) with the parameter α and starting from 0.*

Proof: We obtain the result by applying Proposition 9 of [5] on the process $(Z^D_{g_t}X_t : t \geq 0)$. □

5. Stochastic differential equations driven by a relative martingale

In this section, we study stochastic differential equations driven by a relative martingale. More precisely, we investigate stochastic differential equations of the form:

$$dX_t = \sigma(t, X_t)dW_t + b(t, X_t)dt,\ 0 \leq t \leq T,\ X_0 = Z \tag{5.1}$$

where $W = B + v$ is a continuous sub-martingale of the class $\mathcal{M}(H)$ such that B is a standard Brownian motion. The study of such equations can have good applications in finance engineering. For instance, one of such equations has recently appeared in [7] to propose an investment strategy for insurance companies. Specifically, that is next equation:

$$dS_t = \left(\mu + \frac{\sigma^2}{2}\right) S_t dt + \sigma S_t dX^\delta_t,$$

where X^δ is the Itô-Mckean skew Brownian motion presented in Section 3 as a process of the class $\mathcal{M}(H)$. In particular, the present investigations will be do under next hypothesis:

Hypothesis 5.1 *Let $T > 0$ and $b : [0, T] \times \mathbb{R} \to \mathbb{R}$, $\sigma : [0, T] \times \mathbb{R} \to \mathbb{R}$ be measurable functions satisfying*

$$|b(t, x)| + |\sigma(t, x)| \leq C(1 + |x|);\ t \in [0, T],\ x \in \mathbb{R} \tag{5.2}$$

for some constant C and such that

$$|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq K|x - y|;\ t \in [0, T],\ x, y \in \mathbb{R} \tag{5.3}$$

for some constant K. Let Z be a random variable which is independent of the $\sigma-$ algebra $\mathcal{F}_\infty$ generated by B_t, $t \geq 0$ and such that

$$\mathbb{E}[|Z|^2] < \infty \text{ and } \mathbb{E}[V_T|Z|^2] < \infty.$$

Under the above hypothesis, the classical stochastic equation:

$$dY_t = \sigma(t, Y_t)dB_t + b(t, Y_t)dt,\ Y_0 = Z \tag{5.4}$$

admits a unique continuous solution (see Theorem 5.2.1 of [10]). Throughout the rest of this paper, we shall denote this solution Y. The study of Equation (5.1) strongly depends on the random set H. Indeed, the novelty in this equation comes from the integral $\int_0^t \sigma(s, X_s)dv_s$ whose behaviour depends on H since dv is carried by H. Hence, the present section consists of two principal subsections. In the first one, we investigate (5.1) according to the structure of H. In the second part, we approach the study in a more general way without taking into account the structure on H.

5.1. Relationship with the classical equation

We first remark that $dW = dB$ outside set H. Hence, under some conditions, the solution X of (5.1) behaves like the solution Y of (5.4). In this subsection, we investigate situations where the solution Y of (5.4) satisfies (5.1). Thus, we start by show that Y is also solution of (5.1) when $t \mapsto \sigma(t, x)$ vanishes on H.

Proposition 5.1 *If in addition to Hypothesis 5.1, the function σ is such that $\forall s \in H$ and $\forall x \in \mathbb{R}$, $\sigma(s, x) = 0$. Hence, the unique solution of the equation:*

$$X_t = Z + \int_0^t \sigma(s, X_s)dB_s + \int_0^t b(s, X_s)ds$$

is also the unique solution of the next equation:

$$X_t = Z + \int_0^t \sigma(s, X_s)dW_s + \int_0^t b(s, X_s)ds.$$

Proof: Firstly, we have: $\forall t \geq 0$,

$$\int_0^t \sigma(s, X_s)dW_s = \int_0^t \sigma(s, X_s)dB_s + \int_0^t \sigma(s, X_s)dv_s.$$

But, $\int_0^t \sigma(s, X_s)dv_s = 0$ because, $s \mapsto \sigma(s, X_s)$ vanishes on H and dv is carried by H. Which proves that the two above equations are equivalent. This completes the proof. □

Now, recall that H is the zero set of a continuous martingale D. Hence, H cannot be ordered. However $\mathbb{R}_+ \setminus H$ can be decomposed as a countable union $\cup_{n\in\mathbb{N}} J_n$ of intervals J_n. Each interval J_n corresponds to some excursion of D. Specifically, if $J_n =]g_n, d_n[$, $dW_t = dB_t$ for all $t \in]g_n, d_n[$ and $g_n, d_n \in H$. In the following, we explore situations where the solution Y of (5.4) satisfies (5.1). Let $\tau_1 = g_0$ be the first zero of D and denote $N = \inf\{n \geq 0 : d_n \neq g_{n+1}\}$ and $\tau = d_N$.

In the following, we show that Equation (5.1) admits a unique solution before the first entry time in H, τ_1. And that this solution is the same which verifies (5.4).

Proposition 5.2 *Let T be a real such that $T < \tau_1$ a.s. Under Assumptions 5.1, there exists a unique continuous process X such that $\forall t \in [0, T]$,*

$$X_t = Z + \int_0^t \sigma(s, X_s)dW_s + \int_0^t b(s, X_s)ds.$$

It is the same solution of (5.4).

Proof: First remark that v is constant on $[0, T]$ because, dv is carried by H and $[0, T] \subset [0, \tau_1[\subset H^c$. That is, we have: $dW_s = dB_s$, $\forall s \leq T$. Hence, (5.1) coincides with the following standard stochastic differential equation:

$$X_t = Z + \int_0^t \sigma(s, X_s)dB_s + \int_0^t b(s, X_s)ds. \tag{5.5}$$

Consequently, we obtain the existence from Theorem 5.2.1 of [10]. Which completes the proof. □

In the next proposition, we show that the above result is again true on $[0, \tau_2[$, where $\tau_2 = \inf\{t > \tau_1 : t \in H\}$.

Proposition 5.3 *For all $T > 0$ such that $\gamma_T = \tau_1$, we have under Assumption 5.1, that there exists a unique continuous process X such that $\forall t \leq T$,*

$$X_t = Z + \int_0^t \sigma(s, X_s)dW_s + \int_0^t b(s, X_s)ds.$$

Proof: We know from Theorem 5.2.1 of [10] that the classic Equation:

$$X_t = Z + \int_0^t \sigma(s, X_s)dB_s + \int_0^t b(s, X_s)ds$$

admits a unique continuous solution X. According to Proposition 5.2, X is also a solution of (5.1) on $[0, \tau_1[$. Furthermore, $\forall t \in [\tau_1, T]$,

$$X_t = Z + \int_0^{\tau_1} \sigma(s, X_s)dB_s + \int_{\tau_1}^t \sigma(s, X_s)dB_s + \int_0^t b(s, X_s)ds.$$

But,$\forall t \in [\tau_1, T]$, $\gamma_t = \gamma_T$ because $\gamma_T = \tau_1$. Hence, we get:

$$X_t = Z + \int_0^{\tau_1} \sigma(s, X_s)dB_s + \int_{\gamma_t}^t \sigma(s, X_s)dB_s + \int_0^t b(s, X_s)ds.$$

However, $dB = dW$ on $[0, \tau_1[$ and on $[\gamma_t, t]$. Which implies:

$$X_t = Z + \int_0^{\tau_1} \sigma(s, X_s)dW_s + \int_{\gamma_t}^t \sigma(s, X_s)dW_s + \int_0^t b(s, X_s)ds.$$

That is,

$$X_t = Z + \int_0^t \sigma(s, X_s)dW_s + \int_0^t b(s, X_s)ds.$$

Consequently, X is also solution of (5.1). □

Now, we show that Equation (5.1) admits a unique solution on $[0, \tau]$ and that this solution is also the same which verifies (5.4).

Proposition 5.4 *Let T be a real such that $T \leq \tau$ a.s. If in addition, $\tau_1 < \tau$. Hence, under Assumptions 5.1, there exists a unique continuous process X such that $\forall t \in [0, T]$,*

$$X_t = Z + \int_0^t \sigma(s, X_s)dW_s + \int_0^t b(s, X_s)ds.$$

It is the same solution of (5.4).

Proof: First remark that $\forall T \leq \tau$, $[0, T] \cap H$ is a finite and countable set. That is, $\forall t \leq T$, there exist an integer d and reals, $t_1, \cdots, t_d$ such that $[0, t] \cap H = \{t_1, \cdots, t_d\}$, where $\tau_1 = t_1 < t_2 < \cdots < t_d = \gamma_t$. Thus, we have:

$$\int_0^t \sigma(s, X_s)dW_s = \int_0^{t_1} \sigma(s, X_s)dW_s + \sum_{k=1}^{d-1} \int_{t_k}^{t_{k+1}} \sigma(s, X_s)dW_s + \int_{\gamma_t}^t \sigma(s, X_s)dW_s.$$

But, v is constant on $[0, t_1[$, $[\gamma_t, t]$ and on $[t_k, t_{k+1}[$, $\forall k \in \{1, \cdots, d-1\}$ because D does not vanish on intervals $[0, t_1[$, $]\gamma_t, t]$ and on $]t_k, t_{k+1}[$. Hence,

$$\int_0^t \sigma(s, X_s)dW_s = \int_0^{t_1} \sigma(s, X_s)dB_s + \sum_{k=1}^{d-1} \int_{t_k}^{t_{k+1}} \sigma(s, X_s)dB_s + \int_{\gamma_t}^t \sigma(s, X_s)dB_s = \int_0^t \sigma(s, X_s)dB_s.$$

Which means that the equation

$$X_t = Z + \int_0^t \sigma(s, X_s)dW_s + \int_0^t b(s, X_s)ds$$

is equivalent to

$$X_t = Z + \int_0^t \sigma(s, X_s)dB_s + \int_0^t b(s, X_s)ds.$$

This completes the proof. □

Now, we shall explore what happens after the honest time $\gamma = \sup\{t \geq 0 : t \in H\}$. In particular, we show that we have the previous result in the enlarged filtration $(\mathcal{G}_{\gamma+t})_{t\geq 0}$.

Proposition 5.5 *Under Assumptions 5.1, there exists a unique continuous process Y, adapted to the filtration*
$(\mathcal{G}_{\gamma+t})_{t\geq 0}$ *such that* $\forall t \geq 0$,

$$Y_t = Z + \int_0^t \sigma(s, Y_s)d\widetilde{W}_s + \int_0^t b(s, Y_s)ds,$$

where $\widetilde{W}_t = W_{\gamma+t} - W_\gamma - \int_\gamma^{\gamma+t} \frac{d\langle W, D\rangle_s}{D_s}$.

Proof: We obtain from Theorem 2.4.1 of [1] that $\left(W_{\gamma+t} - W_\gamma - \int_\gamma^{\gamma+t} \frac{d\langle W, |D|\rangle_s}{|D_s|} : t \geq 0\right)$ is a local martingale with respect to $(\mathcal{G}_{\gamma+t})_{t\geq 0}$. But, we have:

$$\int_\gamma^{\gamma+t} \frac{d\langle W, |D|\rangle_s}{|D_s|} = \int_\gamma^{\gamma+t} \frac{d\langle W, D\rangle_s}{D_s}.$$

Hence, $\widetilde{W}$ is a local martingale adapted to $(\mathcal{G}_{\gamma+t})_{t\geq 0}$. Moreover,

$$\langle \widetilde{W}, \widetilde{W}\rangle_t = \langle W, W\rangle_{\gamma+t} - \langle W, W\rangle_\gamma = t.$$

Then, $\widetilde{W}$ is a Brownian motion with respect to $(\mathcal{G}_{\gamma+t})_{t\geq 0}$. Consequently, we obtain from Theorem 5.2.1 of [10] that there exists a unique continuous process Y adapted to filtration $(\mathcal{G}_{\gamma+t})_{t\geq 0}$ such that

$$Y_t = Z + \int_0^t \sigma(s, Y_s)d\widetilde{W}_s + \int_0^t b(s, Y_s)ds.$$

□

Corollary 5.1 *Let $W = B + v$ be a $(\mathcal{F}_t)_{t\geq 0}$- adapted and continuous process of the class $\mathcal{M}(H)$ such that B is a standard Brownian motion and $d\langle W, D\rangle$ is carried by H. Under Hypothesis 5.1, there exists a process X such that* $\forall t \geq 0$,

$$X_t = X_{\gamma_t} + \int_{\gamma_t}^t \sigma(X_s)dW_s + \int_{\gamma_t}^t b(X_s)ds. \tag{5.6}$$

Proof: We first remark that $\widetilde{W}_t = W_{t+\gamma} - W_\gamma$ since $\int_\gamma^{\gamma+t} \frac{d\langle W, D\rangle_s}{D_s} = 0$. Indeed, $d\langle W, D\rangle$ is carried by H. Moreover, we know from Proposition 5.5 that there exists a unique continuous process Y adapted to $(\mathcal{G}_{\gamma+t})_{t\geq 0}$ such that $\forall t \geq 0$,

$$Y_t = Y_0 + \int_0^t \sigma(Y_s)d\widetilde{W}_s + \int_0^t b(Y_s)ds.$$

On another hand, we get from Chapter V [6] that there exists a process X adapted to $(\mathcal{F}_t)_{t\geq 0}$ such that $\forall t \geq 0$, $Y_t = X_{\gamma+t}$. Hence, we obtain:

$$X_{\gamma+t} = X_\gamma + \int_0^t \sigma(X_{\gamma+s})d\widetilde{W}_s + \int_0^t b(X_{\gamma+s})ds.$$

Then, it follows:

$$X_{\gamma+t} - X_\gamma = \int_\gamma^{\gamma+t} \sigma(X_s)dW_s + \int_\gamma^{\gamma+t} b(X_s)ds.$$

Which implies that

$$\rho(X_{\gamma+\cdot} - X_\gamma)_t = \rho\left(\int_\gamma^{\gamma+\cdot} \sigma(X_s)dW_s + \int_\gamma^{\gamma+\cdot} b(X_s)ds\right)_t.$$

Now, let us consider processes Z and Z' defined by $\forall t \geq 0$, $Z_t = X_t - X_{\gamma_t}$ and $Z'_t = \int_{\gamma_t}^t \sigma(X_s)dW_s + \int_{\gamma_t}^t b(X_s)ds$. We can see that Z and Z' vanish on H and that $\forall t \geq 0$,

$$Z_{\gamma+t} = X_{\gamma+t} - X_\gamma \text{ and } Z'_{\gamma+t} = \int_\gamma^{\gamma+t} \sigma(X_s)dW_s + \int_\gamma^{\gamma+t} b(X_s)ds.$$

Consequently, we obtain by uniqueness that

$$X_t - X_{\gamma_t} = \int_{\gamma_t}^t \sigma(X_s)dW_s + \int_{\gamma_t}^t b(X_s)ds.$$

This proves the existence of solutions for (5.6). □

Note that any solution of the equation:

$$X_t = X_0 + \int_0^t \sigma(X_s)dW_s + \int_0^t b(X_s)ds, \tag{5.7}$$

verifies also (5.6). But, we cannot always affirm the reciprocal. In next corollary, we give a sufficient condition under which a solution of (5.6) is also a solution of (5.7).

Corollary 5.2 *Let X be a solution of Equation (5.6) such that $\forall t \in H$, $X_t = X_0 + \int_0^t \sigma(X_s)dW_s + \int_0^t b(X_s)ds$. Hence, X is a solution of (5.7) for every $t \geq 0$.*

Proof: We have: $\forall t \geq 0$,

$$X_t = X_{\gamma_t} + \int_{\gamma_t}^t \sigma(X_s)dW_s + \int_{\gamma_t}^t b(X_s)ds$$

since X is solution of (5.6). In addition, $\forall t \in H$,

$$X_t = X_0 + \int_0^t \sigma(X_s)dW_s + \int_0^t b(X_s)ds.$$

Which means that

$$X_{\gamma_t} = X_0 + \int_0^{\gamma_t} \sigma(X_s)dW_s + \int_0^{\gamma_t} b(X_s)ds.$$

This implies that $\forall t \geq 0$,

$$X_t = X_0 + \int_0^{\gamma_t} \sigma(X_s)dW_s + \int_{\gamma_t}^t \sigma(X_s)dW_s + \int_0^{\gamma_t} b(X_s)ds + \int_{\gamma_t}^t b(X_s)ds.$$

Consequently, we obtain that $\forall t \geq 0$,

$$X_t = X_0 + \int_0^t \sigma(X_s)dW_s + \int_0^t b(X_s)ds.$$

This completes the proof. □

5.2. Study in the general case

Now, we shall investigate next equation:

$$X_t = \zeta 1_H(t) + \left[Z + \int_0^t \sigma(s, X_s) dW_s + \int_0^t b(s, X_s) ds \right] 1_{H^c}(t), \tag{5.8}$$

where $W = B + v$ is a sub-martingale of the class $\mathcal{M}(H)$ such that B is a standard Brownian motion. Thus, we shall consider next assumptions:

We prove existence of solutions for (5.8) in what follows.

Proposition 5.6 *Under the above assumptions, there exist solutions for* (5.8).

Proof: Define $Y^{(0)} = X_0$ and $Y^{(p)} = Y^{(p)}(w)$ inductively as follows

$$Y_t^{(p+1)} = \zeta 1_H(t) + \left[Z + \int_0^t \sigma(s, Y_s^{(p)}) dW_s + \int_0^t b(s, Y_s^{(p)}) ds \right] 1_{H^c}(t).$$

We have $\forall t \leq T$ and $p \geq 1$,

$$\mathbb{E}\left[|Y_t^{(p+1)} - Y_t^{(p)}|^2\right] \leq 2\mathbb{E}\left[\int_0^t (\sigma(s, Y_s^{(p)}) - \sigma(s, Y_s^{(p-1)})) dW_s\right]^2 + 2\mathbb{E}\left[\int_0^t (b(s, Y_s^{(p)}) - b(s, Y_s^{(p-1)})) ds\right]^2.$$

Let $\alpha_s = \sigma(s, Y_s^{(p)}) - \sigma(s, Y_s^{(p-1)})$ and $\beta_s = b(s, Y_s^{(p)}) - b(s, Y_s^{(p-1)})$. Thus, we obtain:

$$\mathbb{E}\left[\int_0^t \beta_s ds\right]^2 \leq K^2 t \int_0^t \mathbb{E}[|Y_s^{(p)} - Y_s^{(p-1)}|^2] ds.$$

And,

$$\begin{aligned}\mathbb{E}\left[\int_0^t \alpha_s dW_s\right]^2 &\leq 2\mathbb{E}\left[\int_0^t \alpha_s dB_s\right]^2 + 2\mathbb{E}\left[\int_0^t \alpha_s dv_s\right]^2 \\ &\leq 2\int_0^t \mathbb{E}(\alpha_s)^2 ds + 2\mathbb{E}\left[\int_0^t |\alpha_s| dv_s\right]^2.\end{aligned}$$

Hence, we get by assumptions the following:

$$\mathbb{E}\left[\int_0^t \alpha_s dW_s\right]^2 \leq 2K^2 \int_0^t \mathbb{E}[|Y_s^{(p)} - Y_s^{(p-1)}|^2] ds + 2K^2 \mathbb{E}\left[\int_0^t |Y_s^{(p)} - Y_s^{(p-1)}| dv_s\right]^2.$$

But $\forall s \in H$, $\int_0^t |Y_s^{(p)} - Y_s^{(p-1)}| dv_s = 0$ since $Y_s^{(p)} - Y_s^{(p-1)} = 0$. Which implies:

$$\mathbb{E}\left[\int_0^t \alpha_s dW_s\right]^2 \leq 2K^2 \int_0^t \mathbb{E}[|Y_s^{(p)} - Y_s^{(p-1)}|^2] ds.$$

Therefore, we get:

$$\mathbb{E}\left[|Y_t^{(p+1)} - Y_t^{(p)}|^2\right] \leq 2K^2(2+t) \int_0^t \mathbb{E}[|Y_s^{(p)} - Y_s^{(p-1)}|^2] ds.$$

In addition, we have:

$$\begin{aligned}\mathbb{E}\left[|Y_t^{(1)} - Y_t^{(0)}|^2\right] &\leq \mathbb{E}\left[\left|\zeta + \int_0^t \sigma(s, X_0) dW_s + \int_0^t b(s, X_0) ds\right|^2\right] \\ &\leq 3\mathbb{E}[\zeta^2] + 3\mathbb{E}\left[\left|\int_0^t \sigma(s, X_0) dW_s\right|^2\right] + 3\mathbb{E}\left[\left|\int_0^t b(s, X_0) ds\right|^2\right]\end{aligned}$$

$$\leq 3\mathbb{E}[\zeta^2] + 6\mathbb{E}\left[\left|\int_0^t \sigma(s,X_0)dB_s\right|^2\right] + 6\mathbb{E}\left[\left|\int_0^t \sigma(s,X_0)dv_s\right|^2\right] + 3\mathbb{E}\left[\left|\int_0^t b(s,X_0)ds\right|^2\right]$$

$$\leq 3\mathbb{E}[\zeta^2] + 6\mathbb{E}\left[\int_0^t |\sigma(s,X_0)|^2 ds\right] + 6\mathbb{E}\left[\left|\int_0^t \sigma(s,X_0)dv_s\right|^2\right] + 3t\mathbb{E}\left[\int_0^t |b(s,X_0)|^2 ds\right].$$

Then,

$$\mathbb{E}\left[|Y_t^{(1)} - Y_t^{(0)}|^2\right] \leq 3\mathbb{E}[\zeta^2] + 6C^2 t(1+\mathbb{E}[|X_0|^2]) + 6C^2\mathbb{E}\left[|(1+|X_0|)v_T|^2\right] + 3C^2t^2(1+\mathbb{E}[|X_0|^2]).$$

Hence,

$$\mathbb{E}\left[|Y_t^{(1)} - Y_t^{(0)}|^2\right] \leq A_0 + A_1 t,$$

where $A_0 = 3\mathbb{E}[\zeta^2] + 6C^2\mathbb{E}\left[|(1+|X_0|)v_T|^2\right]$ and $A_1 = 6C^2(1+\mathbb{E}[|X_0|^2]) + 3C^2T(1+\mathbb{E}[|X_0|^2])$. So by induction on p we obtain :

$$\mathbb{E}\left[|Y_t^{(p+1)} - Y_t^{(p)}|^2\right] \leq \frac{B_0^{p+1}\times t^p}{p!} + \frac{B_1^{p+1}\times t^{p+1}}{(p+1)!};\ p \geq 0,\ t\in[0,T],$$

where B_0 and B_1 are some suitable constants depending only on C, K, T, Γ and $\mathbb{E}\left[|X_0|^2\right]$. Now, let λ be Lebesgue measure on $[0,T]$ and $m > n \geq 0$. Hence, we get:

$$\begin{aligned}
||Y_t^{(m)} - Y_t^{(n)}||_{L^2(\lambda\times\mathbb{P})} &= \left|\left|\sum_{p=n}^{m-1} Y_t^{(p+1)} - Y_t^{(p)}\right|\right|_{L^2(\lambda\times\mathbb{P})} \\
&\leq \sum_{p=n}^{m-1}\left|\left|Y_t^{(p+1)} - Y_t^{(p)}\right|\right|_{L^2(\lambda\times\mathbb{P})} \\
&\leq \sum_{p=n}^{m-1}\sqrt{\mathbb{E}\left[\int_0^T |Y_t^{(p+1)} - Y_t^{(p)}|^2 dt\right]} \\
&\leq \sum_{p=n}^{m-1}\sqrt{\int_0^T\left(\frac{B_0^{p+1}\times t^p}{p!} + \frac{B_1^{p+1}\times t^{p+1}}{(p+1)!}\right)dt}.
\end{aligned}$$

Hence, we obtain:

$$||Y_t^{(m)} - Y_t^{(n)}||_{L^2(\lambda\times\mathbb{P})} \leq \sum_{p=n}^{m-1}\sqrt{\frac{B_0^{p+1}\times T^{p+1}}{(p+1)!} + \frac{B_1^{p+1}\times T^{p+2}}{(p+2)!}} \longrightarrow 0$$

as $n,m\longrightarrow\infty$. Therefore, $\{Y_t^{(n)} : n\geq 0\}$ is Cauchy sequence in $L^2(\lambda\times\mathbb{P})$. Hence, $\{Y_t^{(n)} : n\geq 0\}$ is convergent in $L^2(\lambda\times\mathbb{P})$. Let

$$X_t = \lim_{n\to\infty} Y_t^{(n)} \text{ in } L^2(\lambda\times\mathbb{P}).$$

Now, we shall show that X_t is solution of (5.8). We have $\forall n\geq 0$, and all $t\in[0,T]$,

$$Y_t^{(n+1)} = \zeta 1_H(t) + \left[Z + \int_0^t \sigma(s,Y_s^{(n)})dB_s + \int_0^t \sigma(s,Y_s^{(n)})dv_s + \int_0^t b(s,Y_s^{(n)})ds\right]1_{H^c}(t).$$

But as $n\to\infty$, we obtain from the Hölder inequality that

$$\int_0^t b(s,Y_s^{(n)})ds \longrightarrow \int_0^t b(s,X_s)ds \ \text{ in } L^2(\lambda\times\mathbb{P}).$$

And through Itô's isometry, we get:

$$\int_0^t \sigma(s, Y_s^{(n)})dB_s \longrightarrow \int_0^t \sigma(s, X_s)dB_s \ \text{ in } L^2(\lambda \times \mathbb{P}).$$

Furthermore, we have:

$$\begin{aligned}\left\|\int_0^t \sigma(s, X_s)dv_s - \int_0^t \sigma(s, Y_s^{(n)})dv_s\right\|_{L^2(\lambda\times\mathbb{P})}^2 &= \mathbb{E}\left[\left|\int_0^t [\sigma(s, X_s) - \sigma(s, Y_s^{(n)})]dv_s\right|^2\right] \\ &\leq K^2\mathbb{E}\left[\left|\int_0^t |X_s - Y_s^{(n)}|dv_s\right|^2\right].\end{aligned}$$

But, $\int_0^t |X_s - Y_s^{(n)}|dv_s = 0$ since $\forall s \in H$, $X_s = Y_s^{(n)} = \zeta$ and dv is carried by H. Then,

$$\int_0^t \sigma(s, Y_s^{(n)})dv_s \longrightarrow \int_0^t \sigma(s, X_s)dv_s \ \text{ in } L^2(\lambda \times \mathbb{P}).$$

Consequently, $\forall t \in [0, T]$ we have

$$X_t = \zeta 1_H(t) + \left[Z + \int_0^t \sigma(s, X_s)dB_s + \int_0^t \sigma(s, X_s)dv_s + \int_0^t b(s, X_s)ds\right] 1_{H^c}(t).$$

□

Lemma 5.1 *Let X and Y be solutions of (5.1) such that $\forall t \in H$, $X_t = Y_t$. If $X_0 = Y_0$ hence, X and Y are indistinguishable.*

Proof: We have $\forall t \geq 0$,

$$\mathbb{E}\left[|Y_t - X_t|^2\right] \leq 3\mathbb{E}\left[|Y_0 - X_0|^2\right] + 3\mathbb{E}\left[\left|\int_0^t (\sigma(s, Y_s) - \sigma(s, X_s))dW_s\right|^2\right] + 3\mathbb{E}\left[\left|\int_0^t (b(s, Y_s) - b(s, X_s))ds\right|^2\right].$$

But,

$$\mathbb{E}\left[\left|\int_0^t (\sigma(s, Y_s) - \sigma(s, X_s))dW_s\right|^2\right] \leq 2\mathbb{E}\left[\left|\int_0^t (\sigma(s, Y_s) - \sigma(s, X_s))dB_s\right|^2\right] + 2\mathbb{E}\left[\left|\int_0^t (\sigma(s, Y_s) - \sigma(s, X_s))dv_s\right|^2\right]$$

We obtain from Itô isometry and Cauchy-Swarz's inequality the following:

$$\mathbb{E}\left[\left|\int_0^t (\sigma(s, Y_s) - \sigma(s, X_s))dW_s\right|^2\right] \leq 2\int_0^t \mathbb{E}\left[|\sigma(s, Y_s) - \sigma(s, X_s)|^2\right] ds + 2\mathbb{E}\left[v_t \int_0^t |\sigma(s, Y_s) - \sigma(s, X_s)|^2 dv_s\right].$$

Hence, according to Lipschitz property, we get:

$$\mathbb{E}\left[\left|\int_0^t (\sigma(s, Y_s) - \sigma(s, X_s))dW_s\right|^2\right] \leq 2K^2\int_0^t \mathbb{E}|Y_s - X_s|^2 ds + 2K^2\mathbb{E}\left[v_t \int_0^t |Y_s - X_s|^2 dv_s\right].$$

Which implies that

$$\mathbb{E}\left[\left|\int_0^t (\sigma(s, Y_s) - \sigma(s, X_s))dW_s\right|^2\right] \leq 2K^2\int_0^t \mathbb{E}|Y_s - X_s|^2 ds.$$

Indeed $\int_0^t |Y_s - X_s|^2 dv_s = 0$ since dv is carried by H and $Y - X = 0$ on H. On another hand, through Cauchy-Shwarz's inequality and Lipschitz property, we get:

$$\mathbb{E}\left[\left|\int_0^t (b(s, Y_s) - b(s, X_s))ds\right|^2\right] \leq tK^2\int_0^t \mathbb{E}|Y_s - X_s|^2 ds.$$

Then, we obtain the following:

$$\mathbb{E}\left[|Y_t - X_t|^2\right] \leq 3\mathbb{E}\left[|Y_0 - X_0|^2\right] + 3K^2(2+t)\int_0^t \mathbb{E}|Y_s - X_s|^2 ds.$$

Thus, the function φ defined by $\forall t \in [0,T]$, $\varphi(t) = \mathbb{E}\left[|Y_t - X_t|^2\right]$ satisfies,

$$\varphi(t) \leq F + A\int_0^t \varphi(s)ds,$$

where $F = 3\mathbb{E}\left[|Y_0 - X_0|^2\right]$ and $A = 3K^2(2+t)$. By, the Gronwall Inequality, we get that

$$\varphi(t) \leq F\exp(At).$$

Now, assume that $X_0 = Y_0$. That is, $F = 0$. And then, $\varphi(t) = 0$ for all $t \in [0,T]$. Consequently, $X = Y$ a.s. □

Corollary 5.3 *Under assumptions H_1, the stochastic differential equation (5.8) has a unique solution.*

Proof: Let Y and X be two solutions of (5.8). Hence, $\forall t \geq 0$, $Y_t = X_t = \zeta$. Hence, X and Y are solutions of (5.1) such that $X = Y$ on H. Consequently, we obtain the result by applying Lemma 5.1 □

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