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Fixed point theorems in modular $b-$metric spaces and application

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ABSTRACT: In this paper, we prove a general fixed point theorem for a mapping satisfying a new type of implicit relation in modular $b-$metric spaces and we give an application to establish the existence of a solution for a system of nonlinear integral equations.

Key Words: Modular $b-$metric spaces, fixed point, implicit relation.

Contents

1. Introduction and Preliminary

Since the famous Banach fixed point theorem (1922), the study of fixed point theory in metric spaces has several applications in mathematics, especially in solving differential and functional equations. Many authors have introduced a new class of generalized metric space, in particular M.E. Ege and C. Alaca [2] they provide the required definitions and theorems about modular and $b-$metric spaces. Next, they define the modular $b-$metric space and give definitions to prove Banach contraction principle in the new space.

In this paper, we are interested to prove a general fixed point theorem in modular b-metric spaces. The results in this paper give us particular results and illustrated by examples. To show the significance of our result an application is presented to establish the existence of a solution for system of nonlinear integral equations.

Definition 1.1 *[1] Let X be a non-empty set. A map $\omega : (0, \infty) \times X \times X \longrightarrow [0, \infty]$ is called a modular metric, if the following statements hold for all $x, y, z \in X$,*
(i) $\omega_\lambda(x, y) = 0$ *for all* $\lambda > 0 \Leftrightarrow x = y$,
(ii) $\omega_\lambda(x, y) = \omega_\lambda(y, x)$ *for all* $\lambda > 0$,
(iii) $\omega_{\lambda+\mu}(x, y) \leq \omega_\lambda(x, z) + \omega_\mu(z, y)$ *for all* $\lambda, \mu > 0$.
Then we say that ω is a modular metric.

Definition 1.2 *[2] Let X be a non-empty set and let $s \geq 1$ be a real number. A map $\omega : (0, \infty) \times X \times X \longrightarrow [0, \infty]$ is called a modular b-metric, if the following statements hold for all $x, y, z \in X$,*
(i) $\omega_\lambda(x, y) = 0$ *for all* $\lambda > 0 \Leftrightarrow x = y$,
(ii) $\omega_\lambda(x, y) = \omega_\lambda(y, x)$ *for all* $\lambda > 0$,
(iii) $\omega_{\lambda+\mu}(x, y) \leq s\,[\omega_\lambda(x, z) + \omega_\mu(z, y)]$ *for all* $\lambda, \mu > 0$.
Then we say that (X, ω) is a modular b-metric space.

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The modular b-metric space could be seen as a generalization of the modular metric space.

Example 1.1 *[2] Consider the space* $l_p = \{(x_n) \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$ *,*
$0 < p < 1$*,* $\lambda \in (0, \infty)$ *and* $\omega_\lambda(x, y) = \frac{d(x,y)}{\lambda}$ *such that* $d(x, y) = (\sum_{n=1}^{\infty} |x_n - y_n|^p)^{\frac{1}{p}}$ *,*
$x = x_n, y = y_n \in l_p$*.*
It could be easily seen that (X, ω) *is a modular b-metric space.*

Definition 1.3 *[2] Let* ω *be a modular b-metric on a set* X*. For* $x, y \in X$*, the binary relation* $\overset{\omega}{\sim}$ *on* X *defined by*

$$x \overset{\omega}{\sim} y \Leftrightarrow \lim_{\lambda \to \infty} \omega_\lambda(x, y) = 0$$

is an equivalence relation. A modular set is defined by

$$X_\omega = \{y \in X : y \overset{\omega}{\sim} x\}.$$

We define state the set

$$X_\omega^* = \{x \in X : \exists \lambda = \lambda(x) > 0 \text{ such that } \omega_\lambda(x, x_0) < \infty\} \quad (x_0 \in X)$$

Definition 1.4 *[2] Let* (X, ω) *be a modular b-metric space.*
- A sequence $(x_n)_{n\in\mathbb{N}}$ *in* X_ω^* *is called* ω*-convergent to* $x \in X_\omega^*$ *if* $\omega_\lambda(x_n, x) \to 0$*, as* $n \to \infty$ *for all* $\lambda > 0$*.*
- A sequence $(x_n)_{n\in\mathbb{N}} \subset X_\omega^*$ *is said to be* ω*-Cauchy if and only if for all* $\epsilon > 0$ *there exists* $n(\epsilon) \in \mathbb{N}$ *such that for each* $n, m \geq n(\epsilon)$ *and* $\lambda > 0$ *we have* $\omega_\lambda(x_n, x_m) < \epsilon$*.*
- A modular b-metric space X_ω^* *is* ω*-complete if each* ω*-Cauchy sequence in* X_ω^* *is* ω*-convergent and its limit is in* X_ω^**.*

2. Main Results

Definition 2.1 *Let* $s \geq 1$ *and* $\mathcal{F}_s$ *be the set of all functions* $\phi(t_1, t_2, ..., t_6) : \mathbb{R}_+^6 \longrightarrow \mathbb{R}$ *satisfying the following conditions :*
(ϕ_1) ϕ *is continuous on* $\mathbb{R}_+^6$*, non increasing in variables* t_4 *and* t_5*.*
(ϕ_2) $\exists k \in [0, \frac{1}{s})$ *such that* $\forall u, v \in \mathbb{R}_+$*,*
$\phi(u, v, v, su, sv + s^2u, 0) \leq 0$ *or* $\phi(u, 0, 0, sv, sv, 0) \leq 0 \Rightarrow u \leq kv$*.*
(ϕ_3) $\forall u \in \mathbb{R}_+$*,* $\phi(u, u, 0, 0, su, u) \leq 0 \Rightarrow u = 0$*.*

Example 2.1 $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - rt_2$*, with* $sr < 1$*.*

Example 2.2 $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \alpha[t_3 + \frac{t_4}{s}]$*, with* $s\alpha < \frac{1}{2}$*.*

Example 2.3 $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - k[\frac{t_5}{s^2} + t_6]$*, with* $sk < \frac{1}{2}$*.*

Example 2.4 $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = \eta t_1 - (\alpha t_2 + \beta t_3 + \gamma t_4)$*, where* $\eta, \alpha, \beta, \gamma \in \mathbb{R}_+$*, with* $s(\alpha + \beta + \gamma) < \eta$ *and* $\gamma s^2 < \eta$*.*

Example 2.5 $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = \eta t_1 - (\alpha t_2 + \beta t_3 + \gamma t_4 + \mu[t_5 + t_6])$*, where* $\eta, \alpha, \beta, \gamma, \mu \in \mathbb{R}_+$*, with* $s(\alpha + \beta + \gamma) + 2s^2\mu \leq \eta$ *and* $(\gamma + \mu)s^2 \leq \eta$*.*

Example 2.6

$$\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - r \max\left\{t_2, t_3, t_4, \frac{t_5 + t_6}{2s}\right\},$$

with $0 \leq r < \frac{1}{s}$*.*

Example 2.7 $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - r \max\{t_2, t_3, t_4\}$*. with* $0 \leq r < \frac{1}{s}$*.*

Example 2.8 $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - (\alpha t_5 + \beta t_6)$*, with* $\alpha s^2 < \frac{1}{2}$ *and* $\alpha + \beta < 1$*.*

Example 2.9 $\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - (\alpha t_2 + \beta t_3 + \gamma t_4 + \delta t_5 + \lambda t_6)$, *with* $\gamma + \delta < \frac{1}{s^2}$, $s(\alpha+\beta+\gamma) + 2s^2\delta < 1$ *and* $\alpha + \delta + \lambda < 1$.

Theorem 2.1 *Let* (X,ω) *be a* $\omega-$*complete modular* $b-$*metric space with coefficient* $s \geq 1$ *and* $f : X^*_\omega \longrightarrow X^*_\omega$ *satisfying*

$$\phi\left(\omega_\lambda(fx,fy), \omega_\lambda(x,y), \omega_\lambda(x,fx), \omega_{2\lambda}(y,fy), \omega_{3\lambda}(x,fy), \omega_\lambda(fx,y)\right) \leq 0, \quad (2.1)$$

for all $\lambda > 0$, $x,y \in X^*_\omega$, *where* $\phi \in \mathcal{F}_s$. *Then* f *has a unique fixed point in* X^*_ω.

Proof.

Existence. Let $x_0 \in X^*_\omega$, define the sequence (x_n) of elements from X^*_ω such that: $x_{n+1} = fx_n$ for every $n \in \mathbb{N}$.

According to (1), with $x = x_{n-1}$ and $y = x_n$ we have :

$$\phi\left(\omega_\lambda(fx_{n-1},fx_n), \omega_\lambda(x_{n-1},x_n), \omega_\lambda(x_{n-1},fx_{n-1}), \omega_{2\lambda}(x_n,fx_n), \omega_{3\lambda}(x_{n-1},fx_n), \omega_\lambda(fx_{n-1},x_n)\right) \leq 0.$$

So,

$$\phi\left(\omega_\lambda(x_n,x_{n+1}), \omega_\lambda(x_{n-1},x_n), \omega_\lambda(x_{n-1},x_n), \omega_{2\lambda}(x_n,x_{n+1}), \omega_{3\lambda}(x_{n-1},x_{n+1}), \omega_\lambda(x_n,x_n)\right) \leq 0.$$

So, since $\omega_{2\lambda}(x_n,x_{n+1}) \leq s\omega_\lambda(x_n,x_{n+1})$ and $\omega_{3\lambda}(x_{n-1},x_{n+1}) \leq s\omega_\lambda(x_{n-1},x_n) + s^2\omega_\lambda(x_n,x_{n+1})$, by (ϕ_1) we have

$$\phi\Big(\omega_\lambda(x_n,x_{n+1}), \omega_\lambda(x_{n-1},x_n), \omega_\lambda(x_{n-1},x_n), s\omega_\lambda(x_n,x_{n+1}), s\omega_\lambda(x_{n-1},x_n) + s^2\omega_\lambda(x_n,x_{n+1}), 0\Big) \leq 0.$$

Using (ϕ_2), we deduce that

$$\omega_\lambda(x_{n+1},x_n) \leq k\omega_\lambda(x_n,x_{n-1}) \leq k^n\omega_\lambda(x_1,x_0) \text{ where } n = 1, ...$$

Now, we show that x_n is a Cauchy sequence in X^*_ω. Let $n,m \in \mathbb{N}^*$, with $m > n$, then we have :

$$\begin{aligned}
\omega_\lambda(x_n,x_m) &\leq s\omega_{\frac{\lambda}{m-n}}(x_n,x_{n+1}) + s^2\omega_{\frac{\lambda}{m-n}}(x_{n+1},x_{n+2}) + s^3\omega_{\frac{\lambda}{m-n}}(x_{n+2},x_{n+3}) + \\
&\ldots + s^{m-n-1}\omega_{\frac{\lambda}{m-n}}(x_{m-2},x_{m-1}) + s^{m-n}\omega_{\frac{\lambda}{m-n}}(x_{m-1},x_m) \\
&\leq sk^n\omega_{\frac{\lambda}{m-n}}(x_0,f(x_0)) + s^2k^{n+1}\omega_{\frac{\lambda}{m-n}}(x_0,f(x_0)) + s^3k^{n+2}\omega_{\frac{\lambda}{m-n}}(x_0,f(x_0)) + \\
&\ldots + s^{m-n-1}k^{m-2}\omega_{\frac{\lambda}{m-n}}(x_0,f(x_0)) + s^{m-n}k^{m-1}\omega_{\frac{\lambda}{m-n}}(x_0,f(x_0)) \\
&= sk^n\omega_{\frac{\lambda}{m-n}}(x_0,f(x_0))\left(1 + sk + (sk)^2 + \ldots + (sk)^{m-n-2} + (sk)^{m-n-1}\right) \\
&= sk^n\omega_{\frac{\lambda}{m-n}}(x_0,f(x_0))\left(\frac{1-(sk)^{m-n}}{1-sk}\right) \\
&\leq \frac{sk^n\omega_{\frac{\lambda}{m-n}}(x_0,f(x_0))}{1-sk}.
\end{aligned}$$

from where $\lim_{n\to\infty}\omega_\lambda(x_n,x_m) = 0$. Then (x_n) is a Cauchy sequence in X^*_ω, so there exists $x \in X^*_\omega$ such that $\lim_{n\to\infty}\omega_\lambda(x_n,x) = 0$. Now we show that $f(x) = x$.

Using inequality (1), we have :

$$\phi\Big(\omega_{\frac{\lambda}{2}}(fx_n,fx), \omega_{\frac{\lambda}{2}}(x_n,x), \omega_{\frac{\lambda}{2}}(x_n,fx_n), \omega_\lambda(x,fx), \omega_{\frac{3}{2}\lambda}(x_n,fx), \omega_{\frac{\lambda}{2}}(fx_n,x)\Big) \leq 0.$$

By (ϕ_1)

$$\phi\Big(\omega_{\frac{\lambda}{2}}(x_{n+1},fx), \omega_{\frac{\lambda}{2}}(x_n,x), \omega_{\frac{\lambda}{2}}(x_n,x_{n+1}), s\omega_\lambda(x,fx), s\omega_{\frac{\lambda}{2}}(x_n,x) + s\omega_\lambda(x,fx), \omega_{\frac{\lambda}{2}}(x_{n+1},x)\Big) \leq 0.$$

Letting $n \to \infty$ we obtain

$$\phi\left(\lim\inf_{n\to\infty} \omega_{\frac{\lambda}{2}}(x_{n+1}, fx), 0, 0, s\omega_\lambda(x, fx), s\omega_\lambda(x, fx), 0\right) \le 0.$$

On the other hand we have

$$\omega_{\frac{\lambda}{2}}(x_{n+1}, fx) \le s\omega_{\frac{\lambda}{4}}(x_{n+1}, x) + s\omega_{\frac{\lambda}{4}}(x, fx)$$

we deduce that the sequence $\left(\omega_{\frac{\lambda}{2}}(x_{n+1}, fx)\right)_n$ is bounded. Then by (ϕ_2), we have

$$\lim\inf_{n\to\infty} \omega_{\frac{\lambda}{2}}(x_{n+1}, fx) \le k\omega_\lambda(x, fx). \tag{2.2}$$

On the other hand we have

$$\omega_\lambda(x, fx) \le s[\omega_{\frac{\lambda}{2}}(x, x_{n+1}) + \omega_{\frac{\lambda}{2}}(x_{n+1}, fx)],$$

and by (2.2) we deduce that

$$\begin{aligned} \omega_\lambda(x, fx) &\le \lim\inf_{n\to+\infty} s[\omega_{\frac{\lambda}{2}}(x, x_{n+1}) + \omega_{\frac{\lambda}{2}}(x_{n+1}, fx)] \\ &= s\lim\inf_{n\to+\infty} \omega_{\frac{\lambda}{2}}(x_{n+1}, fx) \\ &\le sk\omega_\lambda(x, fx), \end{aligned}$$

that is $fx = x$.

Unicity. Suppose that there exists $y \in X_\omega$ an other fixed point of f, then by (1) we have

$$\phi(\omega_\lambda(fx, fy), \omega_\lambda(x, y), \omega_\lambda(x, fx), \omega_{2\lambda}(y, fy), \omega_{3\lambda}(x, fy), \omega_\lambda(fx, y)) \le 0,$$

then

$$\phi(\omega_\lambda(x, y), \omega_\lambda(x, y), 0, 0, s\omega_\lambda(x, y), \omega_\lambda(x, y)) \le 0,$$

So, by (ϕ_3), we have $x = y$.

3. Consequences

From theorem 2.1 and example 2.1 we obtain :

Corollary 3.1 (Theorem 1 [2]) *Let (X, ω) be a $\omega-$complete modular $b-$metric space with coefficient $s \ge 1$ and $f : X_\omega^* \longrightarrow X_\omega^*$, $0 < sk < 1$ satisfying*

$$\omega_\lambda(fx, fy) \le k\omega_\lambda(x, y)$$

for all $\lambda > 0$, $x, y \in X_\omega^*$. Then f has a unique fixed point in X_ω^*.

From theorem 2.1 and example 2.8 we obtain :

Corollary 3.2 *Let (X, ω) be a $\omega-$complete modular $b-$metric space with coefficient $s \ge 1$ and $f : X_\omega^* \longrightarrow X_\omega^*$, $s\alpha < \frac{1}{2}$ satisfying*

$$\omega_\lambda(fx, fy) \le \alpha\left[\omega_\lambda(x, fx) + \frac{\omega_{2\lambda}(y, fy)}{s}\right],$$

for all $\lambda > 0$, $x, y \in X_\omega^*$. Then f has a unique fixed point in X_ω^*.

From theorem 2.1 and example 2.8 we obtain :

Corollary 3.3 *Let (X,ω) be a $\omega-$complete modular $b-$metric space with coefficient $s\geq 1$ and $f:X_{\omega}^{*}\longrightarrow X_{\omega}^{*}$, $sk<\frac{1}{2}$ satisfying*

$$\omega_{\lambda}(fx,fy)\leq k\left[\frac{\omega_{3\lambda}(x,fy)}{s^{2}}+\omega_{\lambda}(fx,y)\right],$$

for all $\lambda>0$, $x,y\in X_{\omega}^{*}$. Then f has a unique fixed point in X_{ω}^{*}.

From theorem 2.1 and example 2.4 we obtain :

Corollary 3.4 *Let (X,ω) be a $\omega-$complete modular $b-$metric space with coefficient $s\geq 1$ and $f:X_{\omega}^{*}\longrightarrow X_{\omega}^{*}$, $\eta,\alpha,\beta,\gamma\in\mathbb{R}_{+}$, with $s(\alpha+\beta+\gamma)<\eta$ and $\gamma s^{2}<\eta$, satisfying*

$$\eta\omega_{\lambda}(fx,fy)\leq\alpha\omega_{\lambda}(x,y)+\beta\omega_{\lambda}(x,fx)+\gamma\omega_{2\lambda}(y,fy),$$

for all $\lambda>0$, $x,y\in X_{\omega}^{*}$. Then f has a unique fixed point in X_{ω}^{*}.

From theorem 2.1 and example 2.5 we obtain :

Corollary 3.5 *Let (X,ω) be a $\omega-$complete modular $b-$metric space with coefficient $s\geq 1$ and $f:X_{\omega}^{*}\longrightarrow X_{\omega}^{*}$, $\eta,\alpha,\beta,\gamma,\mu\in\mathbb{R}_{+}$, with $s(\alpha+\beta+\gamma)+2s^{2}\mu\leq\eta$ and $(\gamma+\mu)s^{2}\leq\eta$ satisfying*

$$\eta\omega_{\lambda}(fx,fy)\leq\alpha\omega_{\lambda}(x,y)+\beta\omega_{\lambda}(x,fx)+\gamma\omega_{2\lambda}(y,fy)+\mu\left[\omega_{3\lambda}(x,fy)+\omega_{\lambda}(fx,y)\right],$$

for all $\lambda>0$, $x,y\in X_{\omega}^{*}$. Then f has a unique fixed point in X_{ω}^{*}.

From theorem 2.1 and example 2.6 we obtain :

Corollary 3.6 *Let (X,ω) be a $\omega-$complete modular $b-$metric space with coefficient $s\geq 1$ and $f:X_{\omega}^{*}\longrightarrow X_{\omega}^{*}$, $0\leq r<\frac{1}{s}$, satisfying*

$$\omega_{\lambda}(fx,fy)\leq r\max\left\{\omega_{\lambda}(x,y),\omega_{\lambda}(x,fx),\omega_{2\lambda}(y,fy),\frac{\omega_{3\lambda}(x,fy)+\omega_{\lambda}(fx,y)}{2s}\right\},$$

for all $\lambda>0$, $x,y\in X_{\omega}^{*}$. Then f has a unique fixed point in X_{ω}^{*}.

From theorem 2.1 and example 2.7 we obtain :

Corollary 3.7 *Let (X,ω) be a $\omega-$complete modular $b-$metric space with coefficient $s\geq 1$ and $f:X_{\omega}^{*}\longrightarrow X_{\omega}^{*}$, $0\leq r<\frac{1}{s}$, satisfying*

$$\omega_{\lambda}(fx,fy)\leq r\max\left\{\omega_{\lambda}(x,y),\omega_{\lambda}(x,fx),\omega_{2\lambda}(y,fy)\right\},$$

for all $\lambda>0$, $x,y\in X_{\omega}^{*}$. Then f has a unique fixed point in X_{ω}^{*}.

From theorem 2.1 and example 2.8 we obtain :

Corollary 3.8 *Let (X,ω) be a $\omega-$complete modular $b-$metric space with coefficient $s\geq 1$ and $f:X_{\omega}^{*}\longrightarrow X_{\omega}^{*}$, $\alpha s^{2}<\frac{1}{2}$ and $\alpha+\beta<1$, satisfying*

$$\omega_{\lambda}(fx,fy)\leq\alpha\omega_{3\lambda}(x,fy)+\beta\omega_{\lambda}(fx,y),$$

for all $\lambda>0$, $x,y\in X_{\omega}^{*}$. Then f has a unique fixed point in X_{ω}^{*}.

From theorem 2.1 and example 2.9 we obtain :

Corollary 3.9 *Let (X,ω) be a $\omega-$complete modular $b-$metric space with coefficient $s\geq 1$ and $f:X_{\omega}^{*}\longrightarrow X_{\omega}^{*}$, $\gamma+\delta<\frac{1}{s^{2}}$, $s(\alpha+\beta+\gamma)+2s^{2}\delta<1$ and $\alpha+\delta+\lambda<1$, satisfying*

$$\omega_{\lambda}(fx,fy)\leq\alpha\omega_{\lambda}(x,y)+\beta\omega_{\lambda}(x,fx)+\gamma\omega_{2\lambda}(y,fy)+\delta\omega_{3\lambda}(x,fy)+\lambda\omega_{\lambda}(fx,y),$$

for all $\lambda>0$, $x,y\in X_{\omega}^{*}$. Then f has a unique fixed point in X_{ω}^{*}.

4. Application

Let $X = C\big([a,b],\mathbb{R}^n\big)$, $(a,b \in \mathbb{R}_+)$ and let (X,ω_λ) a complete modular $b-$metric space with $s=2$.

$$\omega_\lambda(x,y) = \frac{\left(\sup\limits_{u\in[a,b]} \|x(u)-y(u)\|_1\right)^2}{1+\lambda}, \quad x,y \in X,\ \lambda>0.$$

Consider the following system of nonlinear integral equations :

$$x(u) = g(u) + \int_{(a,\dots,a)}^{\tau(u)} f(v,x(v))dv, \quad u\in[a,b], \tag{4.1}$$

where $\tau = (\tau_1,\dots,\tau_2) : [a,b] \longrightarrow [a,b]^n$, $f = (f_1,f_2,\dots,f_n) : [a,b]\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $g = (g_1,g_2,\dots,g_n) : [a,b] \longrightarrow \mathbb{R}^n$ are continuous functions.

Consider the operator $F : X = X \longrightarrow X$ defined by

$$Fx(u) = g(u) + \int_{(a,\dots,a)}^{\tau(u)} f(v,x(v))dv. \tag{4.2}$$

That's to say

$$\begin{pmatrix} F_1x(u)\\ F_2x(u)\\ .\\ .\\ .\\ F_nx(u)\end{pmatrix} = \begin{pmatrix} g_1(u)+\int_a^{\tau_1(u)} f_1(v,x(v))dv\\ g_2(u)+\int_a^{\tau_2(u)} f_2(v,x(v)))dv\\ .\\ .\\ .\\ g_n(u)+\int_a^{\tau_n(u)} f_n(v,x(v))dv\end{pmatrix}.$$

Theorem 4.1 *Suppose that the following condition are satisfied:*
(H) *There exists* $\theta\in(0,+\infty)$ *such that* $\theta \sup\limits_{u\in[a,b]}\sum\limits_{i=1}^{n}(\tau_i(u)-a)\sqrt{2}<1$ *and*

$$\mid f_i(v,x)-f_i(v,y)\mid \le \ \theta\,\|x-y\|_1 \ \ \forall x,y\in\mathbb{R}^n, \quad \forall v\in[a, \sup_{u\in[a,b]}\tau_i(u)],\ i=1,2,\dots,n.$$

Then the system (4.1) *has a unique solution in* X*.*

Proof:

It is clear that any fixed point of (4.2) is a solution of (4.1). By condition (H), we have

$$
\begin{aligned}
\omega_\lambda(Fx,Fy) &= \frac{\left(\sup\limits_{u\in[a,b]}\|Fx(u)-Fy(u)\|_1\right)^2}{1+\lambda}\\
&= \frac{\left(\sup\limits_{u\in[a,b]}\|\int_{(a,a\dots,a)}^{\tau(u)} f(v,x(v))dv-\int_{(a,a,\dots,a)}^{\tau(u)} f(v,y(v))dv\|_1\right)^2}{1+\lambda}\\
&= \frac{\left(\sup\limits_{u\in[a,b]}\sum_1^n\left|\int_a^{\tau_i(u)} f_i(v,x(v))dv-\int_a^{\tau(u)} f_i(v,y(v))dv\right|\right)^2}{1+\lambda}\\
&\le \frac{\left(\sup\limits_{u\in[a,b]}\sum_1^n\int_a^{\tau_i(u)}|f_i(v,x(v))-f_i(v,y(v))|\,dv\right)^2}{1+\lambda}\\
&\le \frac{\left(\sup\limits_{u\in[a,b]}\sum_1^n\int_a^{\tau_i(u)}\theta\|x(v)-y(v)\|_1dv\right)^2}{1+\lambda}\\
&\le \frac{\left(\sup\limits_{u\in[a,b]}\sum_1^n\int_a^{\tau_i(u)}\theta\sup\limits_{t\in\mathbb{R}}\|x(t)-y(t)\|_1dv\right)^2}{1+\lambda}\\
&\le \frac{\left(\theta\sup\limits_{t\in\mathbb{R}}\|x(t)-y(t)\|_1\sup\limits_{u\in[a,b]}\sum_1^n\int_a^{\tau_i(u)}dv\right)^2}{1+\lambda}\\
&\le \frac{\left(\theta\sup\limits_{t\in\mathbb{R}}\|x(t)-y(t)\|_1\sum_{i=1}^n\sup\limits_{u\in[a,b]}(\tau_i(u)-a)\right)^2}{1+\lambda}\\
&= \left(\theta\sup_{u\in[a,b]}\sum_{i=1}^n(\tau_i(u)-a)\right)^2\frac{\left(\sup\limits_{u\in[a,b]}\|x(v)-y(v)\|_1\right)^2}{1+\lambda}\\
&\le \alpha\omega_\lambda(x,y).
\end{aligned}
$$

with $\alpha=\left(\theta\sup\limits_{u\in[a,b]}\sum\limits_{i=1}^n(\tau_i(u)-a)\right)^2<\frac{1}{2}=\frac{1}{s}$.
Since, let $x_0\in X$, we have

$$
\begin{aligned}
\omega_\lambda(x_0,Fx_0) &= \frac{\left(\sup\limits_{u\in[a,b]}\|x_0(u)-Fx_0(u)\|_1\right)^2}{1+\lambda}\\
&\le \left(\sup_{u\in[a,b]}\|x_0(u)-Fx_0(u)\|_1\right)^2\\
&< \infty,
\end{aligned}
$$

because x_0-Fx_0 is continous function.

Then all conditions of corollary 3.1 are satisfied. So the operator F has a unique fixed point, that is the system (4.1) has a unique solution in X. □

Example 4.1 *The following system of nonlinear integral equations has a solution in* $X = C([0,1], \mathbb{R}^2)$.

$$(S)\begin{cases} x_1(u) = u^2 + \frac{1}{6}\int_0^1 e^{-v^2}\sin(x_1(v) - x_2(v))dv \\ x_2(u) = \sin(u) + \frac{1}{5}\int_0^1 \ln(1+v)\cos(x_1(v) + x_2(v) + v)dv \end{cases},$$

Proof:

Let F: $X \longrightarrow X$ defined by

$$F(x(u)) = g(u) + \int_0^1 f(v, x(v))dv, \quad u \in [0,1],$$

such that $x(t) = (x_1(t), x_2(t))$, $\tau = (\tau_1(t), \tau_2(t)) = (1,1)$, $a = 0$, $g(t) = (g_1(t), g_2(t)) = (t^2, \sin(t))$ and

$f(t, x(t)) = (f_1(t, x(t)), f_2(t, x(t)))$,
$f_1(t, x(t)) = \frac{1}{6}e^{-t^2}\sin(x_1(t) - x_2(t))$,
$f_2(t, x(t)) = \frac{1}{5}\ln(1+t)\cos(x_1(t) + x_2(t) + t)$.

It is clear that the condition (H) in theorem 4.1 is satisfied with $\theta = \frac{1}{5}$. Indeed

$$\begin{aligned} \mid f_1(v, x(v)) - f_1(v, y(v)) \mid &= \frac{1}{6}e^{-v^2} \mid \sin(x_1(v) - x_2(v)) - \sin(y_1(v) - y_2(v)) \mid \\ &\leq \frac{1}{6}e^{-v^2} \mid (x_1(v) - x_2(v)) - (y_1(v) - y_2(v)) \mid \\ &= \frac{1}{6}e^{-v^2} \mid (x_1(v) - y_1(v)) - (x_2(v) - y_2(v)) \mid \\ &\leq \frac{1}{6}e^{-v^2} (\mid x_1(v) - y_1(v) \mid + \mid x_2(v) - y_2(v) \mid) \\ &\leq \frac{1}{6}e^{-v^2}\|x(v) - y(v)\|_1. \end{aligned}$$

So

$$\mid f_1(v, x(v)) - f_1(v, y(v)) \mid \;\leq\; \frac{1}{6}\|x(v) - y(v)\|_1, \ \forall x, y \in X, \ \forall v \in [0,1].$$

And

$$\begin{aligned} \mid f_2(v, x(v)) - f_2(v, y(v)) \mid &= \frac{1}{5}\ln(1+t) \mid \cos(x_1(v) + x_2(v) + v) - \cos(y_1(v) + y_2(v) + v) \mid \\ &\leq \frac{1}{5}\ln(1+v) \mid (x_1(v) + x_2(v) + v) - (y_1(v) + y_2(v) + v) \mid \\ &= \frac{1}{5}\ln(1+v) \mid (x_1(v) - y_1(v)) - (x_2(v) - y_2(v)) \mid \\ &\leq \frac{1}{5}\ln(1+v) (\mid x_1(v) - y_1(v) \mid + \mid x_2(v) - y_2(v) \mid) \\ &\leq \frac{1}{5}\ln(1+v)\|x(v) - y(v)\|_1. \end{aligned}$$

So

$$\mid f_2(v, x(v)) - f_2(v, y(v)) \mid \;\leq\; \frac{1}{5}\|x(v) - y(v))\|_1, \ \forall x, y \in X, \ \forall v \in [0,1].$$

Therefore, all conditions of theorem 4.1 are satisfied, hence the mapping F has a unique fixed point in X, which is a unique solution of the system (S). □

5. Conclusion

In this paper, we have proved a general fixed point theorem for a mapping satisfying a new type of implicit relation in modular $b-$metric spaces and obtained several other results as a special case of general fixed point theorem. To show the significance of our result an application is presented to establish the existence of a solution for system of nonlinear integral equations.

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