Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–14.
ISSN-0037-8712
doi:10.5269/bspm.64536

Parameter estimation in Reflected Vasicek model: Least-squares approach

Fateh Merahi* and Abdelouahab Bibi

ABSTRACT: In this paper, we investigate least-squares estimation (LSE) for continuous-time reflected Ornstein-Uhlenbeck (ROU) processes with one and two-sided barriers. So, we derive explicit formulas for the estimators, and then we prove their strong consistency and asymptotic normality. We also illustrate the asymptotic properties of the estimators through a simulation study.

Key Words: Reflected Ornstein-Uhlenbeck processes, Brownian motion, Least-squares estimation.

Contents

1. Introduction

In recent years, reflected diffusion processes have been widely used to model various phenomena in different fields such as queueing systems, financial engineering, mathematical biology, insurance, and so on. Indeed, Ward and Glynn [20] showed that a queueing system with reneging can be approximated via an appropriate Markovian reflected Ornstein-Uhlenbeck (ROU) process.The ROU processes behave like the standard Ornstein-Uhlenbeck (OU) processes in the interior of their domain, making them an appealing tool for approximating Makovian queue models (which can be characterized as birth-death continuous-time Markov chains) when reneging is present. Moreover, among others, Harrison [8] and Bo et al. [2] model the exchange rates in the European monetary system using a ROU model. Krugman [12] has also proposed a ROU model with two reflecting barriers to model the dynamics of currency exchange rates in a target zone. In the field of mathematical biology, the application of the ROU model is discussed by Ricciardi and Sacerdote [17] and in ecology by Ricciardi [16]. Bo et al. [4,3] have presented the ROU model for modeling the so-called regulated financial market. We refer the interested reader to [8] and [22] for more details on reflected processes and their broad applications.

Statistical inference for the ROU model has been considered by several authors, who have proposed some methods to estimate the parameters involved in such models. For instance, Prakasa Rao [18] and Bo et al. [5] have studied the asymptotic properties of the maximum likelihood estimator (MLE), while Lee et al. [13] have studied the sequential MLE for the drift parameter based on continuous time observations. Hu et al. [10] studied the MLE for a ROU processes with discrete observations under the assumption that only the state process itself (not the local time process) is observed. Valdivieso et al. [19] investigated the MLE for the OU type processes driven by Lévy process. More recently, Zhu [23] studied the asymptotic properties of the MLE for the ROU model with two sided barriers. In the

* Corresponding author
Submitted July 28, 2022. Published May 23, 2025
2010 *Mathematics Subject Classification:* Primary 40A05, 40A25, Secondary 45G05.

present paper, we focus on the problem of parameter estimation involved in the ROU process using least-squares estimation (LSE) for more general cases in which multiple parameters are present in one and/or two-sided barriers.

The remainder of the paper is organized as follows. In the next section, we give some preliminary results related to the model with one barrier and the asymptotic properties of its least-squares method for estimating ROU processes. In section 3, we extend LSE to the two-barriers case. Section 4 is devoted to highlighting the theoretical results with some Monte Carlo simulations. The last section concludes the paper.

2. Reflected Ornstein-Uhlenbeck processes with one-sided barrier

Given a usual filtered probability space $\left(\Omega, \Im, (\Im_t)_{t\geq 0}, P\right)$ and let $X = (X(t))_{t\geq 0}$ be a diffusion process, with infinitesimal variance β^2 and infinitesimal drift $\mu - \alpha x$ (the parameter α carries the physical meaning of customer reneging (or balking) rate from the system). We first deal with ROU with one-sided barrier b^L. According to Ward and Glynn [21], the process X is called a ROU process if X is the strong solution of the following stochastic differential equation (SDE), almost surely ($a.s$) for $t \geq 0$,

$$dX(t) = (-\alpha X(t) + \mu)dt + \beta dW(t) + dL(t) \text{ with } X(t) \geq b^L, \text{ and } X(0) = X_0, \tag{2.1}$$

where μ,$b^L \in \mathbb{R}$, $\alpha, \beta \in (0, \infty)$ and $W = (W(t))_{t\geq 0}$ is a one-dimensional standard Brownian motion defined on a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P\right)$, $L = (L(t))_{t\geq 0}$ is the minimal continuous increasing process (which makes $X(t) \geq b^L$ for all $t \geq 0$) and satisfies $L(0) = 0$, $\int\limits_0^\infty 1_{\{X(t)>b^L\}} dL(t) = 0$, $\int\limits_0^t 1_{\{X(s)=b^L\}} dL(s) = L(t)$ and where 1_Δ denotes the indicator function (see [8], Ata et al. [1] for more discussion). Sometimes L is called the regulator of the point b^L, and has an explicit expression as $L(t) = \max\left\{0, \sup_{u\in[0,t]}\left(-X_0 + \alpha\int\limits_0^u X(s)ds - \mu u - \beta w(u)\right)\right\} = \max\left\{0, \sup_{u\in[0,t]}(L(u) - X(u))\right\}$ and the ROU process (2.1) can be constructed via a Markovian approximation procedure (see,e.g., Bo et al. [6]). Starting from initial position X_0 which is assumed to be not dependent on W, then, a formal solution of (2.1) is given by

$$X(t) = e^{-\alpha t}\left\{X_0 - \frac{\mu}{\alpha}\left(e^{\alpha t} - 1\right) + \int\limits_0^t e^{\alpha s} dL(s) + \beta \int\limits_0^t e^{\alpha s} dW(s)\right\}.$$

The existence of a unique strong solution to (2.1) is ensured by a careful extension of the results of Lions et al. [15] and Ward et al. [21].

Remark 2.1 In case $b^L = 0$, we refer to Linetsky [9], Ward et al. [21]. When $\alpha = 0$ (balking case), the corresponding ROU process X reduces to the so-called reflected Brownian motion (RBM) process and we refer to Harrison [8] for a rigorous definition of such processes and their properties of interest.

In the sequel, we shall note that $\phi(u) = \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{u^2}{2}\right\}$ and $\Phi(y) = \int\limits_{-\infty}^y \phi(u)du$ are the density and the distribution function associated with $\mathcal{N}(0,1)$. The following lemma due to Hu et al. [10], which states an ergodic theorem and will be used frequently.

Lemma 2.1 (Hu et al.) *Consider the ROU process X defined by (2.1), then, for any $x \geq 0$*

1. *The process X has a unique ergodic and stationary distribution π with an associated invariant density $p(x) = \sqrt{\frac{2\alpha}{\beta^2}}\frac{\phi(\overline{x})}{1 - \Phi(\overline{0})}$ with $\overline{x} = \sqrt{\frac{2\alpha}{\beta^2}}\left(x - \frac{\mu}{\alpha}\right)$.*

2. *For any integrable function f, the following mean ergodic theorem holds*

$$\lim_{T\to+\infty}\frac{1}{T}\int_0^T f\left(X(s)\right)ds=\int_0^\infty f(x)p(x)dx \ \ a.s.$$

Moreover, we have

$$E\left\{X^k\right\}=\begin{cases}\dfrac{\mu}{\alpha}+\dfrac{\phi\left(\overline{0}\right)}{1-\Phi\left(\overline{0}\right)}\sqrt{\dfrac{\beta^2}{2\alpha}}, \ k=1,\\ \dfrac{\beta^2}{2\alpha}+\dfrac{\mu^2}{\alpha^2}+\dfrac{\phi\left(\overline{0}\right)}{1-\Phi\left(\overline{0}\right)}\sqrt{\dfrac{\beta^2}{2\alpha}}\dfrac{\mu}{\alpha}, \ k=2.\end{cases}$$

Proof: See Hu et al. [10]. □

Remark 2.2 If $b^L\neq 0$, the unique invariant density can be rewritten as $p(x)=\sqrt{\dfrac{2\alpha}{\beta^2}}\dfrac{\phi\left(\overline{x}\right)}{1-\Phi\left(\overline{b}^L\right)}$, $x\in[b^L,\infty)$ and hence the first and second order moments are

$$E\left\{X^k\right\}=\begin{cases}\dfrac{\mu}{\alpha}+\dfrac{\phi\left(\overline{b}^L\right)}{1-\Phi\left(\overline{b}^L\right)}\sqrt{\dfrac{\beta^2}{2\alpha}}, \ k=1,\\ \dfrac{\beta^2}{2\alpha}+\dfrac{\mu^2}{\alpha^2}+2\dfrac{\phi\left(\overline{b}^L\right)}{1-\Phi\left(\overline{b}^L\right)}\sqrt{\dfrac{\beta^2}{2\alpha}}\dfrac{\mu}{\alpha}+\dfrac{\phi\left(\overline{b}^L\right)\overline{b}^L}{1-\Phi\left(\overline{b}^L\right)}, \ k=2.\end{cases}$$

Remark 2.3 It follows from the above expressions that the parameter β^2 can be expressed in terms of α, μ, $E\left\{X\right\}$ and the second-order moments as follows

$$\beta^2=2\alpha\left(E\left\{X^2\right\}-\frac{\mu^2}{\alpha^2}-\left(b^L+\frac{\mu}{\alpha}\right)\left(E\left\{X\right\}-\frac{\mu}{\alpha}\right)\right). \tag{2.2}$$

2.1. Least squares estimation for the ROU processes with one-sided barrier

In this section, we investigate the estimation of the unknown parameters α,μ and β of the ROU process (2.1) from continuous observations. That is, we assume that the ROU process $X=\left(X(t)\right)_{t\geq 0}$ is observed and $T\to\infty$. For the purpose, we assume that β is known since the process can be observed continuously (see, e.g., Prakasa Rao [18], p.15). So we mainly focus on the estimation of the parameters α and μ gathered in vector $\underline{\theta}=(\alpha,\mu)'$ its true value is denoted by $\underline{\theta}_0$. The least squares estimator $\widehat{\underline{\theta}}_T$ of $\underline{\theta}_0$ is defined as any measurable $\widehat{\underline{\theta}}_T$ of $\widehat{\underline{\theta}}_T=Arg\min_{\underline{\theta}}Q_T\left(\underline{\theta}\right)$ where $Q_T\left(\underline{\theta}\right)=\int_0^T\left(X^{(1)}(s)+\alpha X(s)-\mu-L^{(1)}(s)\right)^2ds$, in which the superscript $^{(j)}$ denotes $j-$fold differentiation with respect to t. Rewriting $Q_T\left(\underline{\theta}\right)$ as

$$\begin{aligned}&\int_0^T\left(X^{(1)}(s)+\alpha X(s)-\mu-L^{(1)}(s)\right)^2ds\\ &=\int_0^T\left(X^{(1)}(s)-L^{(1)}(s)\right)\left(\left(X^{(1)}(s)-L^{(1)}(s)\right)+2\alpha X(s)-2\mu\right)ds+\alpha^2T\widehat{m}_2-2\alpha\mu T\widehat{m}_1+\mu^2T,\end{aligned}$$

where $\widehat{m}_k = \frac{1}{T}\int_0^T X^k(s)ds$. It is easy to see that the minimum is attained when α is given by

$$\widehat{\alpha}_T = \frac{(X(T)-L(T))\,T\widehat{m}_1 + T\int_0^T X(s)dL(s) - T\int_0^T X^{(1)}(s)X(s)ds}{T\widehat{\sigma}_T^2} \tag{2.3}$$
$$= \frac{(X(T)-L(T))\,T\widehat{m}_1 + Tb^L L(T) - T\int_0^T X(s)dX(s)}{T\widehat{\sigma}_T^2},$$

where $\widehat{\sigma}_T^2 = T\left(\widehat{m}_2 - (\widehat{m}_1)^2\right)$. An estimate of the parameter μ is given by $\widehat{\mu}_T = \frac{1}{T}(X(T)-L(T)) + \widehat{\alpha}_T\widehat{m}_1$. Now, we can use the expression (2.2) to estimate β^2 as follows

$$\widehat{\beta}_T^2 = 2\widehat{\alpha}_T\left(\widehat{m}_2 - \frac{\widehat{\mu}_T^2}{\widehat{\alpha}_T^2} - \left(b^L + \frac{\widehat{\mu}_T}{\widehat{\alpha}_T}\right)\left(\widehat{m}_1 - \frac{\widehat{\mu}_T}{\widehat{\alpha}_T}\right)\right). \tag{2.4}$$

We are now in a position to state the main results concerning the asymptotic properties of the estimators $\widehat{\alpha}_T$, $\widehat{\mu}_T$ and $\widehat{\beta}_T^2$.

Theorem 2.1 *The estimators $\widehat{\alpha}_T$ and $\widehat{\mu}_T$ are strongly consistent estimators of α and μ i.e., a.s,*

$$\lim_{T\to\infty}\widehat{\alpha}_T = \alpha \text{ and } \lim_{T\to\infty}\widehat{\mu}_T = \mu. \tag{2.5}$$

Proof: To show that the least squares estimator $\widehat{\alpha}_T$ converges to α, we rewrite the expression (2.3) as

$$\widehat{\alpha}_T = \alpha - \frac{\beta}{\widehat{\sigma}_T^2}\int_0^T X(s)dW(s). \tag{2.6}$$

Since $\int_0^T X(s)dW(s)$ is a martingale with bracket $T\widehat{m}_2$ and when $T\to\infty$, $\int_0^T X(s)dW(s)$ is of the order $T\widehat{m}_2$. Moreover, as $T\to\infty$, $\widehat{m}_k$ converges to $E\{X^k\}$ (this is an immediate consequence of the ergodic theorem in Lemma 2.1 by setting $f(x)=x^n$, $n=1,2$). Thus $\frac{1}{\widehat{\sigma}_T^2}\int_0^T X(s)dW(s)$ converges to 0 with the order $\frac{1}{\sqrt{T}}$. To prove the strong consistency of $\widehat{\mu}_T$ we can see that $\widehat{\mu}_T$ can be rewritten as follows

$$\widehat{\mu}_T - \mu = (\widehat{\alpha}_T - \alpha)\,\widehat{m}_1 + \frac{\beta}{T}W(T). \tag{2.7}$$

The convergence of $\widehat{\mu}_T$ to μ follows immediately from the expression (2.7) and the fact that $\widehat{\alpha}_T\to\alpha$ and $\frac{1}{T}\int_0^T X(s)ds \to E\{X\}$ by the Lemma 2.1. □

Theorem 2.2 *The estimator $\widehat{\alpha}_T$ of α admits asymptotic normality, i.e.,*

$$\widehat{\sigma}_T^2\left(\frac{\alpha-\widehat{\alpha}_T}{\beta}\right)(T\widehat{m}_2)^{-1/2} \rightsquigarrow \mathcal{N}(0,1) \text{ as } T\to\infty. \tag{2.8}$$

Proof: From the expression (2.6), we have

$$\widehat{\sigma}_T^2\left(\frac{\alpha-\widehat{\alpha}_T}{\beta}\right) = \int_0^T X(s)dW(s), \tag{2.9}$$

and the fact that $E\left\{\int_0^T X(s)dW(s)\right\} = 0$, $E\left\{\left(\int_0^T X(s)dW(s)\right)^2\right\} = TE\{\widehat{m}_1\}$. Then applying the Central Limit Theorem (Theorem B.10, p.313 of Prakasa Rao [18]), we obtain (2.8). □

The following theorem shows the strong consistency of the estimator $\widehat{\beta}^2_T$.

Theorem 2.3 *The estimator $\widehat{\beta}^2_T$ of β^2 is strongly consistent, i.e., a.s.* $\lim_{T\to\infty}\widehat{\beta}^2_T = \beta^2$ *as* $T \to \infty$.

Proof: The strong consistency of the estimator $\widehat{\beta}^2_T$ follows immediately from the expression (2.2) and the Theorem 2.1, where we prove that both estimators $\widehat{\alpha}_T$ and $\widehat{\mu}_T$ are strong consistency for α and μ respectively and the strong convergence of $\widehat{m}_j$ for $j = 1, 2$ by the ergodic theorem (see Lemma 2.1). □

3. Reflected Ornstein-Uhlenbeck processes with two-sided barriers

Following the motivations by Ward et al. [20,21] for one-sided barrier *ROU* processes and from the point of view of queuing systems, it is natural to suggest a flexible model with finite buffer capacity. This leads us to consider a *ROU* process $(X(t))_{t\geq 0}$ with two-sided barriers b^L and b^U defined as

$$dX(t) = (-\alpha X(t)+\mu)dt+\beta dW(t)+dL(t)-dU(t),\ X(t) \in \left[b^L, b^U\right],\ \text{for all } t \geq 0 \text{ and } X(0) = X_0. \quad (3.1)$$

By the standard definition in Harrison [8] or Ata et al. [1], the processes $L = (L(t))_{t\geq 0}$ is uniquely determined and associated with the lower barrier b^L and the upper barrier b^U, respectively. Both processes L and U are minimal continuous increasing processes which ensure that $X(t) \in \left[b^L, b^U\right]$ for all $t \geq 0$ with $L_0 = U_0 = 0$ and satisfy.

$$\int_0^\infty 1_{\{X(t)>b^L\}}dL(t) = \int_0^\infty 1_{\{X(t)<b^U\}}dU(t) = 0. \quad (3.2)$$

Actually, the *ROU* process with two-sided barriers can be constructed via a Markovian approximation procedure (see,e.g., Bo et al. [6], Ward and Glynn [20,21] for details). We can also refer to Linetsky [9] to see that the stationary density of X is given by $\Pi(x) = M\exp\left\{-\overline{x}^2\right\}, x \in \left[b^L, b^U\right]$, where M is a constant such that $\int_{b^L}^{b^U}\Pi(x)dx = 1$. In order to obtain the moments of the process $X =(X(t))_{t\geq 0}$, we rewrite $\Pi(x)$ as $\Pi(x) = \sqrt{\frac{2\alpha}{\beta^2}}\frac{\phi\left(\overline{x}\right)}{\Phi\left(\overline{b}^U\right)-\Phi\left(\overline{b}^L\right)}, x \in \left[b^L, b^U\right]$ and for any $t \in \mathbb{R}$, the generator function has the form

$$E\left\{e^{tX}\right\} = \frac{\exp\left\{\frac{\mu}{\alpha}t+\frac{\beta^2}{4\alpha}t^2\right\}\left(\Phi\left(\overline{b}^U - t\sqrt{\frac{\beta^2}{2\alpha}}\right)-\Phi\left(\overline{b}^L - t\sqrt{\frac{\beta^2}{2\alpha}}\right)\right)}{\Phi\left(\overline{b}^U\right)-\Phi\left(\overline{b}^L\right)}, \quad (3.3)$$

where we have used the identity $\int_b^\infty \phi(z)e^{az}dz = e^{\frac{a^2}{2}}\left[1-\Phi(b-a)\right]$. Set, $g(t) = \Phi\left(\overline{b}^U - t\sqrt{\frac{\beta^2}{2\alpha}}\right) - \Phi\left(\overline{b}^L - t\sqrt{\frac{\beta^2}{2\alpha}}\right)$ and $f(t) = exp\left\{\frac{\mu}{\alpha}t+\frac{\beta^2}{4\alpha}t^2\right\}$, then we can compute the following quantities which are used to obtain the moments of the third order:

$$f(0)=1,\ g(0)=\Phi\left(\bar{b}^{U}\right)-\Phi\left(\bar{b}^{L}\right),\ f'(t)=\left(\frac{\mu}{\alpha}+\frac{\beta^2}{2\alpha}t\right)f(t\ )\text{ and }\ f'(0)=\frac{\mu}{\alpha}$$

$$g'(t)=-\sqrt{\frac{\beta^2}{2\alpha}}\left(\phi\left(\bar{b}^{U}-t\sqrt{\frac{\beta^2}{2\alpha}}\right)-\phi\left(\bar{b}^{L}-t\sqrt{\frac{\beta^2}{2\alpha}}\right)\right)\text{ and } g'(0)=-\sqrt{\frac{\beta^2}{2\alpha}}\left(\phi\left(\bar{b}^{U}\right)-\phi\left(\bar{b}^{L}\right)\right),$$

$$f''(t)=\left(\frac{\beta^2}{2\alpha}+\left(\frac{\mu}{\alpha}+\frac{\beta^2}{2\alpha}t\right)^2\right)f(t)\text{ and } f''(0)=\frac{\beta^2}{2\alpha}+\frac{\mu^2}{\alpha^2},$$

$$g''(t)=-\frac{\beta^2}{2\alpha}\left(\bar{b}^{U}-t\sqrt{\frac{\beta^2}{2\alpha}}\right)\phi\left(\bar{b}^{U}-t\sqrt{\frac{\beta^2}{2\alpha}}\right)+\frac{\beta^2}{2\alpha}\left(\bar{b}^{L}-t\sqrt{\frac{\beta^2}{2\alpha}}\right)\phi\left(\bar{b}^{L}-t\sqrt{\frac{\beta^2}{2\alpha}}\right),$$

$$g''(0)=\sqrt{\frac{\beta^2}{2\alpha}}\left(-\left(b^{U}-\frac{\mu}{\alpha}\right)\phi\left(\bar{b}^{U}\right)+\left(b^{L}-\frac{\mu}{\alpha}\right)\phi\left(\bar{b}^{L}\right)\right),$$

$$f'''(t)=2\frac{\beta^2}{2\alpha}f'(t)+\left(\frac{\mu}{\alpha}+\frac{\beta^2}{2\alpha}t\right)f''(t)\text{ and } f'''(0)=\frac{3\beta^2\mu}{2\alpha^2}+\frac{\mu^3}{\alpha^3},$$

$$g'''(t)=\frac{\beta^2}{2\alpha}\sqrt{\frac{\beta^2}{2\alpha}}\phi\left(\bar{b}^{U}-t\sqrt{\frac{\beta^2}{2\alpha}}\right)\left(1-\left(\bar{b}^{U}-t\sqrt{\frac{\beta^2}{2\alpha}}\right)^2\right)-\frac{\beta^2}{2\alpha}\sqrt{\frac{\beta^2}{2\alpha}}\phi\left(\bar{b}^{L}-t\sqrt{\frac{\beta^2}{2\alpha}}\right)\left(1-\left(\bar{b}^{L}-t\sqrt{\frac{\beta^2}{2\alpha}}\right)^2\right)$$

$$g'''(0)=\frac{\beta^2}{2\alpha}\sqrt{\frac{\beta^2}{2\alpha}}\left(\phi\left(\bar{b}^{U}\right)\left(1-\bar{b}^{U2}\right)-\phi\left(\bar{b}^{L}\right)\left(1-\bar{b}^{L2}\right)\right).$$

Using the relationship between the generator function and the moments, we can obtain the third-order moments as follows.

$$E\left\{X\right\}=\frac{\mu}{\alpha}-\frac{1}{g(0)}\left(\phi\left(\bar{b}^{U}\right)-\phi\left(\bar{b}^{L}\right)\right)\sqrt{\frac{\beta^2}{2\alpha}},\tag{3.4}$$

$$\begin{aligned}E\left\{X^2\right\}&=\frac{\beta^2}{2\alpha}+\frac{\mu^2}{\alpha^2}-\frac{2}{g(0)}\left(\phi\left(\bar{b}^{U}\right)-\phi\left(\bar{b}^{L}\right)\right)\sqrt{\frac{\beta^2}{2\alpha}}\frac{\mu}{\alpha}\\&-\frac{1}{g(0)}\left(\left(b^{U}-\frac{\mu}{\alpha}\right)\phi\left(\bar{b}^{U}\right)-\left(b^{L}-\frac{\mu}{\alpha}\right)\phi\left(\bar{b}^{L}\right)\right)\sqrt{\frac{\beta^2}{2\alpha}}.\end{aligned}\tag{3.5}$$

$$\begin{aligned}E\left\{X^3\right\}&=\frac{3\beta^2\mu}{2\alpha^2}+\frac{\mu^3}{\alpha^3}-\frac{3}{g(0)}\left(\phi\left(\bar{b}^{U}\right)-\phi\left(\bar{b}^{L}\right)\right)\sqrt{\frac{\beta^2}{2\alpha}}\left(\frac{\beta^2}{2\alpha}+\frac{\mu^2}{\alpha^2}\right)\\&-\frac{3}{g(0)}\left(\left(b^{U}-\frac{\mu}{\alpha}\right)\phi\left(\bar{b}^{U}\right)-\left(b^{L}-\frac{\mu}{\alpha}\right)\phi\left(\bar{b}^{L}\right)\right)\sqrt{\frac{\beta^2}{2\alpha}}\frac{\mu}{\alpha}\\&+\frac{1}{g(0)}\left(\phi\left(\bar{b}^{U}\right)\left(1-\bar{b}^{U2}\right)-\phi\left(\bar{b}^{L}\right)\left(1-\bar{b}^{L2}\right)\right)\sqrt{\frac{\beta^2}{2\alpha}}\frac{\beta^2}{2\alpha}.\end{aligned}\tag{3.6}$$

From the above expressions, the parameter β^2 can be written as

$$\beta^2=\frac{2\alpha\left\{M1\left(\alpha,\mu\right)-M2\left(\alpha,\mu\right)\right\}}{M3\left(\alpha,\mu\right)},\tag{3.7}$$

where

$$M1\left(\alpha,\mu\right)=\left(E\left\{X^2\right\}-\frac{\mu^2}{\alpha^2}-\left(E\left\{X\right\}-\frac{\mu}{\alpha}\right)\left(b^{L}+\frac{\mu}{\alpha}\right)\right)\left(b^{L}+b^{U}+\frac{\mu}{\alpha}\right),$$

$$M2\left(\alpha,\mu\right)=E\left\{X^3\right\}-\frac{\mu^3}{\alpha^3}-\left(E\left\{X\right\}-\frac{\mu}{\alpha}\right)\left(\left(b^{L}+\frac{\mu}{\alpha}\right)^2-b^{L}\frac{\mu}{\alpha}\right),$$

$$M3\left(\alpha,\mu\right)=b^{L}+b^{U}+\frac{\mu}{\alpha}-\frac{\mu}{2\alpha^2}-\frac{E\left\{X\right\}}{\alpha},$$

which depends on α, μ, b^U, b^L and on $E\left\{X^k\right\}$, $k=1,2,3$.

3.1. Least squares estimation for the ROU processes with two-sided barriers

In this subsection, we intend to estimate the parameters of the *ROU* process (3.1) from continuous observations,i.e., we suppose that the *ROU* process $(X(t))_{t\geq 0}$ is observed and $T\to\infty$. For this purpose, we assume that β is known since the process can be observed continuously (see, e.g., Prakasa Rao [18], p.15). So we mainly focus on the estimation of the parameters α and μ. The Least squares estimate (LSE) of α,μ is obtained by minimizing the quadratic function

$$\begin{aligned}&\int_0^T\left(X^{(1)}(s)+\alpha X(s)-\mu-L^{(1)}(s)+U^{(1)}(s)\right)^2ds=\int_0^T\left(X^{(1)}-L^{(1)}(s)+U^{(1)}(s)\right)^2ds\\&+2\alpha\int_0^T X(s)\left(X^{(1)}(s)-L^{(1)}(s)+U^{(1)}(s)\right)ds+\alpha^2\int_0^T X^2(s)ds\\&-2\mu\int_0^T\left(X^{(1)}(s)-L^{(1)}(s)+U^{(1)}(s)\right)ds-2\alpha\mu\int_0^T X(s)ds+\mu^2T.\end{aligned}\tag{3.8}$$

It is easy to see that the minimum is attained when α is given by

$$\begin{aligned}\widehat{\alpha}_T&=\frac{(X(T)-L(T)+U(T))\int_0^T X(s)ds+T\int_0^T X(s)dL(s)-T\int_0^T X(s)dU(s)-T\int_0^T X^{(1)}(s)X(s)ds}{T\widehat{\sigma}_T^2}\\&=\frac{(X(T)-L(T)+U(T))\,T\widehat{m}_1+Tb^LL(t)-Tb^UU(t)-T\int_0^T X(s)dX(s)}{T\widehat{\sigma}_T^2},\end{aligned}\tag{3.9}$$

and hence the parameter μ is estimated by $\widehat{\mu}_T=\frac{1}{T}\left(X(T)-L(T)+U(T)\right)+\widehat{\alpha}_T\widehat{m}_1$ where the equality (3.9) follows from (3.2). Now, we can use the expression (3.7) to estimate the parameter β^2 as follows.

$$\widehat{\beta}_T^2=\frac{2\widehat{\alpha}_T\left\{M1\left(\widehat{\alpha}_T,\widehat{\mu}_T\right)-M2\left(\widehat{\alpha}_T,\widehat{\mu}_T\right)\right\}}{M3\left(\widehat{\alpha}_T,\widehat{\mu}_T\right)}.\tag{3.10}$$

We now state the main results concerning the consistency and the asymptotic normality of the *LSE* of $\widehat{\mu}_T$ and $\widehat{\alpha}_T$.

Theorem 3.1 *The estimator $(\widehat{\alpha}_T,\widehat{\mu}_T)'$ of $(\alpha,\mu)'$ is strongly consistent.*

$$\lim_{T\to\infty}\widehat{\alpha}_T=\alpha\ ,\quad a.s\ \text{and}\ \lim_{T\to\infty}\widehat{\mu}_T=\mu,\quad a.s.\tag{3.11}$$

Proof: The proof follows essentially the same arguments as in the proof of theorem 2.1. □

Theorem 3.2 *The estimator $\widehat{\alpha}_T$ of α is asymptotically normal, i.e.,*

$$\sqrt{T}\left(\frac{\widehat{\alpha}_T-\alpha}{\beta}\right)\rightsquigarrow\mathcal{N}\left(0,\Sigma_\alpha\right),\ \ as\ T\to\infty,\tag{3.12}$$

where $\Sigma_\alpha=\dfrac{E\left\{X^2\right\}}{\left(\left(E\left\{X\right\}\right)^2-E\left\{X^2\right\}\right)^2}$.

Proof: One can easily observe that

$$\left(\frac{\alpha-\widehat{\alpha}_T}{\beta}\right)T^{-1}\widehat{\sigma}_T^2=\int_0^T X(s)dW(s),\tag{3.13}$$

and the fact that $E\left\{\int_0^T X(s)dW(s)\right\} = 0$, $E\left\{\left(\int_0^T X(s)dW(s)\right)^2\right\} = TE\{\widehat{m}_1\}$. Then applying the Central Limit Theorem (Theorem B.10, p.313 of Prakasa Rao [18]), we obtain

$$\sqrt{T}\left(\frac{\alpha-\widehat{\alpha}_T}{\beta}\right)\frac{T^{-1}\widehat{\sigma}_T^2}{\sqrt{\widehat{m}_2}} \rightsquigarrow \mathcal{N}(0,1) \text{ as } T\to\infty. \tag{3.14}$$

Now, we use the continuous-time ergodic theorem (Theorem 9.8, page 161, Kallenberg [11]) to see that

$$\lim_{T\to\infty}\frac{-T^{-1}\widehat{\sigma}_T^2}{\sqrt{\widehat{m}_2}} = \frac{(E\{X\})^2 - E\{X^2\}}{\sqrt{E\{X^2\}}} \text{ a.s },\tag{3.15}$$

where $E\{X\}$ and $E\{X^2\}$ are given respectively by the expressions (3.4) and (3.5), it follows from (3.14) – (3.15) that $\sqrt{T}\left(\frac{\widehat{\alpha}_T-\alpha}{\beta}\right) \rightsquigarrow \mathcal{N}(0,\Sigma_\alpha)$, as $T\to\infty$, with $\Sigma_\alpha = \frac{E\{X^2\}}{\left[(E\{X\})^2 - E\{X^2\}\right]^2}$. □

The following theorem shows the strong consistency of the estimator $\widehat{\beta}_T^2$ of β^2.

Theorem 3.3 *The estimator $\widehat{\beta}_T^2$ is strongly consistent, i.e., almost surely $\lim_{T\to\infty}\widehat{\beta}_T^2 = \beta^2$.*

Proof: The strong consistency of the estimator $\widehat{\beta}_T^2$ follows immediately from the expression (3.10) and the Theorem 3.1, where we have proved that both estimators $\widehat{\alpha}_T$ and $\widehat{\mu}_T$ are strongly consistent to α and μ respectively, and the strong convergence of $\widehat{m}_j$ for $j = 1,2,3$ by the ergodic theorem (Theorem 9.8, page 161, Kallenberg [11]). □

4. Numerical results

As already pointed out in the introduction, the class of birth-death continuous time Markov chains can be approximated (in a distribution sense) by a *ROU* process. For illustration purposes, consider a birth-death process $(Z(t))_{t\geq 0}$ with a birth rate $\lambda_n = \lambda$ $(n\geq 0)$ and a death rate $\mu_n = \mu + (n-1)\gamma$ $(n\geq 1)$, where λ, μ, and γ are positive parameters. The process is described by the following equations.

$$\begin{cases} \frac{dp_n(t)}{dt} = -(\lambda_n+\mu_n)\,p_n(t) + \lambda_{n-1}p_{n-1}(t) + \mu_{n+1}p_{n+1}(t), \text{ if } n\geq 1 \\ \frac{dp_1(t)}{dt} = -\lambda_0 p_0(t) + \mu_1 p_1(t),\ n=1 \\ \sum_{n\geq 0} p_n(t) = 1, t\geq 0, \end{cases}$$

where $p_n(t) = P(Z(t)=n)$ is the probability that the population must be in state n at time t. The approximation for the particular case $\alpha = 0.3, \mu = 5.5$ and $\beta = 1.5$ is shown in Figure 1 and their descriptive statistics are summarized in Table 1.

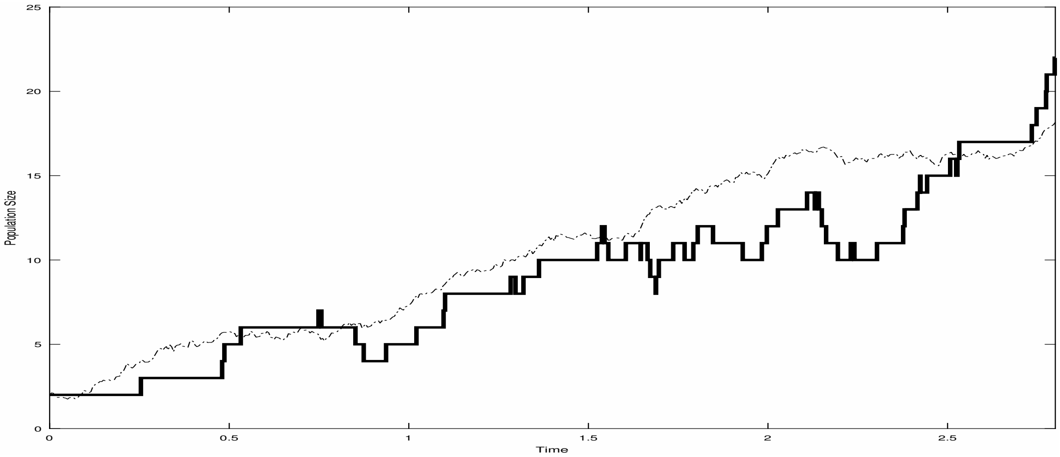

Figure 1: Approximation of birth-death process by *ROU* with one-sided barrier.

The processes	min	max	mean	var	Skew	Kurt
$10^3\times$ Birth-Death	0.0010	0.1420	0.0747	1.5711	-0.0003	0.0021
$10^3\times$ ROU(one sided barrier)	0.0017	0.1415	0.0861	1.5415	-0.0005	0.0019

Table 1: Some descriptive statistics.

In order to investigate the finite sample properties of the LSE method developed in the previous section via Carlo simulation, we simulate 500 independent trajectories according to a ROU process with length $n \in \{500, 1000, 1500\}$. Since, in practice, the observations are recorded at discrete times, then we suppose that the data are observed at times $0 = t_1 < t_2 < ... < t_n = T$ and let $\Delta_i = t_{i+1} - t_i$ and $h = \sup_t \Delta_i$. So, for small h, it seems reasonable to estimate $E\{X\}$, $E\{X^2\}$ and $E\{X^3\}$ by the numerical integrals respectively $\widetilde{m}_1 = \frac{1}{T}\sum_{i=0}^{n-1} x(t_i)\Delta_i$, $\widetilde{m}_2 = \frac{1}{T}\sum_{i=0}^{n-1} x^2(t_i)\Delta_i$ and $\widetilde{m}_3 = \frac{1}{T}\sum_{i=0}^{n-1} x^3(t_i)\Delta_i$. The results of simulation for estimating the vector $\underline{\theta}' = (\alpha, \mu, \beta)$ are reported in tables below in which we have indicated in the line (Mean of) correspond to the average of the parameters estimates over 500 simulations. In order to show the performance of such method, we have reported (results between bracket) the root-mean square errors ($RMSE$) of each estimates. Hence, we define a discrete form of the LSE by replacing the approximation of the integrals and the estimators of the first and second moments in the above expressions (2.3) − (2.4) for the case of one-sided barrier and in the expressions (3.9) − (3.10) for the case of two-sided barriers, it follows that the estimators of the parameters α, μ and β^2 in discrete form are given respectively by

1. Case of one-sided barrier

$$\widetilde{\alpha}_T = \frac{(X(t_n) - L(t_n))\,\widetilde{m}_1 + b^L L(t_n) - \sum_{i=0}^{n-1} x(t_i)\,(x(t_i) - x(t_{i-1}))}{T\,(\widetilde{m}_2 - \widetilde{m}_1^2)},$$

$$\widetilde{\mu}_T = \frac{1}{T}\,(X(t_n) - L(t_n)) + \widetilde{\alpha}\widetilde{m}_1,$$

$$\widetilde{\beta}_T^2 = 2\widehat{\alpha}_T\left(\widetilde{m}_2 - \frac{\widehat{\mu}_T^2}{\widehat{\alpha}_T^2} - \left(b^L + \frac{\widehat{\mu}_T}{\widehat{\alpha}_T}\right)\left(\widetilde{m}_1 - \frac{\widehat{\mu}_T}{\widehat{\alpha}_T}\right)\right).$$

2. Case of two-sided barriers

$$\widetilde{\alpha}_T = \frac{\left(X(t_n) - L(t_n)\right)\widetilde{m}_1 + b^L L(t_n) - \sum_{i=0}^{n-1} x(t_i)\left(x(t_i) - x(t_{i-1})\right)}{T\left(\widetilde{m}_2 - \widetilde{m}_1^2\right)},$$

$$\widetilde{\mu}_T = \frac{1}{T}\left(X(t_n) - L(t_n)\right) + \widetilde{\alpha}\widetilde{m}_1,$$

$$\widetilde{\beta}_T^2 = 2\widehat{\alpha}_T\left(\widetilde{m}_2 - \frac{\widehat{\mu}_T^2}{\widehat{\alpha}_T^2} - \left(b^L + \frac{\widehat{\mu}_T}{\widehat{\alpha}_T}\right)\left(\widetilde{m}_1 - \frac{\widehat{\mu}_T}{\widehat{\alpha}_T}\right)\right).$$

4.1. ROU with one-sided barrier

The results of the simulation of the first model are reported in Table 2 below

Length	500	1000	1500
Mean($\widetilde{\alpha}$)	0.5959	0.5624	0.5407
($RMSE$)	(0.1565)	(0.1195)	(0.0831)
Mean($\widetilde{\mu}$)	2.9902	2.8296	2.7130
($RMSE$)	(0.7838)	(0.6048)	(0.4150)
Mean($\widetilde{\beta}$)	0.7828	0.9091	0.9496
($RMSE$)	(0.1752)	(0.1080)	(0.0603)

design(1): $\alpha = 0.5$, $\mu = 2.5$ and $\beta = 1.0$, $b^L = 0.5$.

Length	500	1000	1500
Mean($\widetilde{\alpha}$)	1.1279	1.1126	1.0926
($RMSE$)	(0.2079)	(0.1584)	(0.1180)
Mean($\widetilde{\mu}$)	1.7250	1.7050	1.6679
($RMSE$)	(0.3341)	(0.2480)	(0.1800)
Mean($\widetilde{\beta}$)	0.9939	1.0073	1.0126
($RMSE$)	(0.0781)	(0.0556)	(0.0370)

design(2): $\alpha = 1$, $\mu = 1.5$ and $\beta = 1.0$,$b^L = 0$.

Table 2: The results of simulation by the Least squares estimator for ROU with one-sided barrier.

The asymptotic distribution of estimated density is shown in Figure 2.

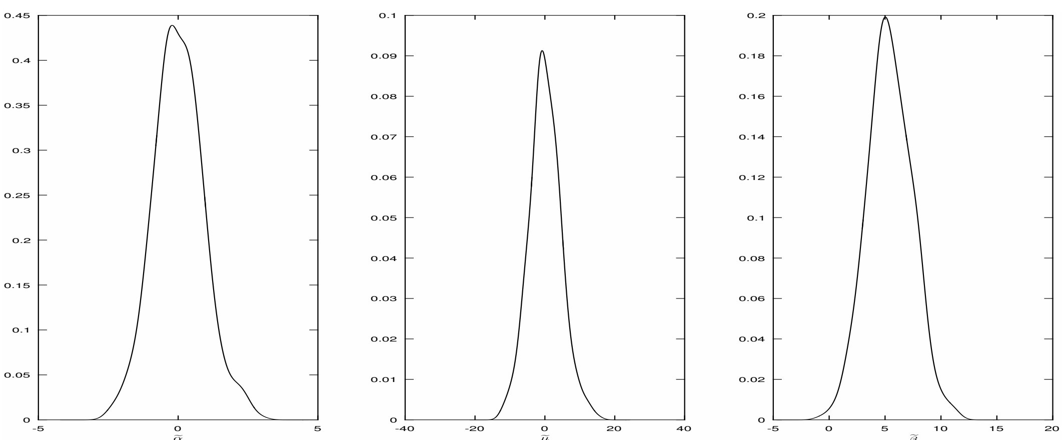

Figure 2: The distribution of LSE for ROU with one-sided barrier.

The box plots summarize the estimates $\widehat{\alpha}_T, \widehat{\mu}_T$ and $\widehat{\beta}_T$, as shown in Figure 3.

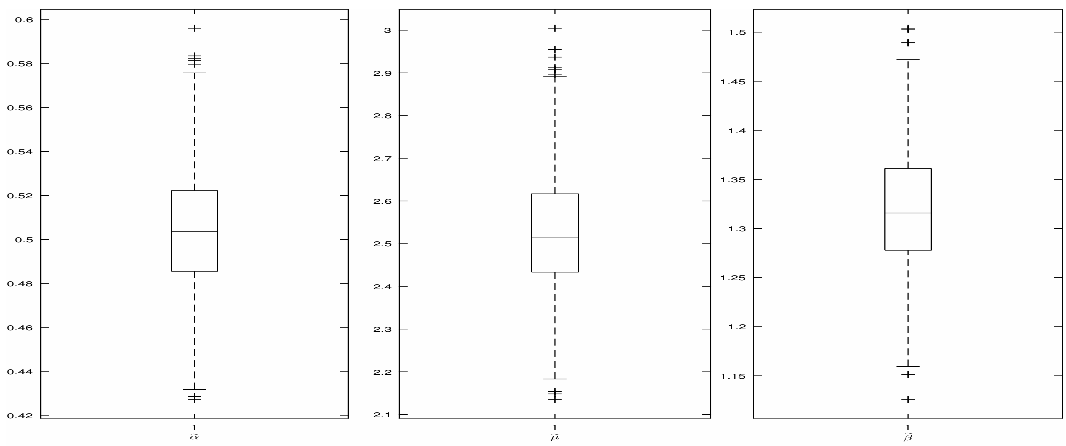

Figure 3: The box plot of LSE for ROU with one-sided barrier.

Now, we examine the sensitivity of the estimator $\tilde{\alpha}$ with respect to the changes in μ. We remark that the percentage changes in the Mean of the estimator are less than those for μ. Table 3 contains some results. The variance tends to increase as μ increases. If $\mu > 0$, the empirical RMSE (root mean squared error) of the estimator is minimized at $\mu \in [8.25, 9.25]$ as shown in Figure 4.

μ	2.0	2.5	3	4	5	6	7	8	8.25	9	9.25	9.5
Mean($\tilde{\alpha}$)	1.1133	1.1096	1.1021	1.0760	1.0685	1.0647	1.0535	1.0498	1.0461	1.0461	1.0461	1.0573
RMSE($\tilde{\alpha}$)	0.1722	0.1705	0.1668	0.1503	0.1443	0.1411	0.1304	0.1263	0.1221	0.1221	0.1221	0.1341

Table 3: Mean ($\tilde{\alpha}$) and $RMSE$ ($\tilde{\alpha}$) against μ for the ROU process with one-sided barrier, $\alpha = 1.0$, $\beta = 1$, and $b^L = 0.5$

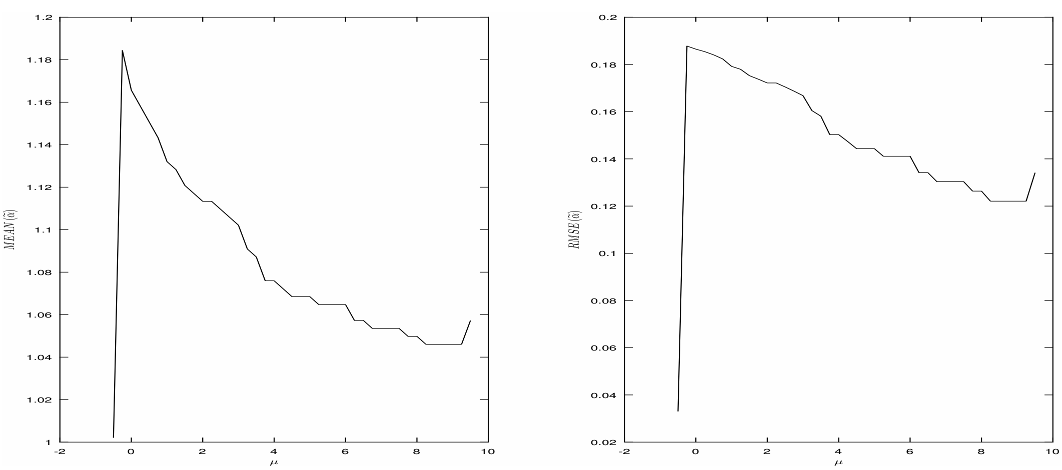

Figure 4: Mean($\widetilde{\alpha}$) and $RMSE\,(\widetilde{\alpha})$ against μ for the ROU process with one-sided barrier, $\alpha = 1.0$, $\beta = 1$, $b^L = 0.5$.

4.2. ROU with two-sided barriers

The results of the simulation of the second model are reported in Table 4 below

Length	500	1000	1500
Mean($\tilde{\alpha}$)	1.0177	1.0274	1.0212
($RMSE$)	(0.1905)	(0.1468)	0.1134)
Mean($\tilde{\mu}$)	0.1168	0.0928	0.0955
($RMSE$)	(0.0977)	(0.0741)	(0.0608)
Mean($\tilde{\beta}$)	1.0605	1.0007	1.0128
($RMSE$)	(0.6045)	(0.3255)	(0.3157)

design(1): $\alpha = 1$, $\mu = 0.1$ and $\beta = 1.0$, $b^L = -5, b^U = 2$.

Length	500	1000	1500
Mean($\tilde{\alpha}$)	0.5666	0.5596	0.5548
($RMSE$)	(0.1441)	(0.1064)	(0.0790)
Mean($\tilde{\mu}$)	0.1282	0.0770	0.0664
($RMSE$)	(0.1067)	(0.0616)	(0.0538)
Mean($\tilde{\beta}$)	1.0575	0.9065	0.9289
($RMSE$)	(1.0390)	(0.3858)	(0.3383)

design(2): $\alpha = 0.5$, $\mu = 0.05$ and $\beta = 1.0$, $b^L = -5, b^U = 2$.

Table 4: The results of the simulation by the least squares estimator for *ROU* with two-sided barriers.

The asymptotic distribution of estimated density is shown in Figure 4, followed by the box plot summary of the descriptive statistics of each estimate in Figure 5.

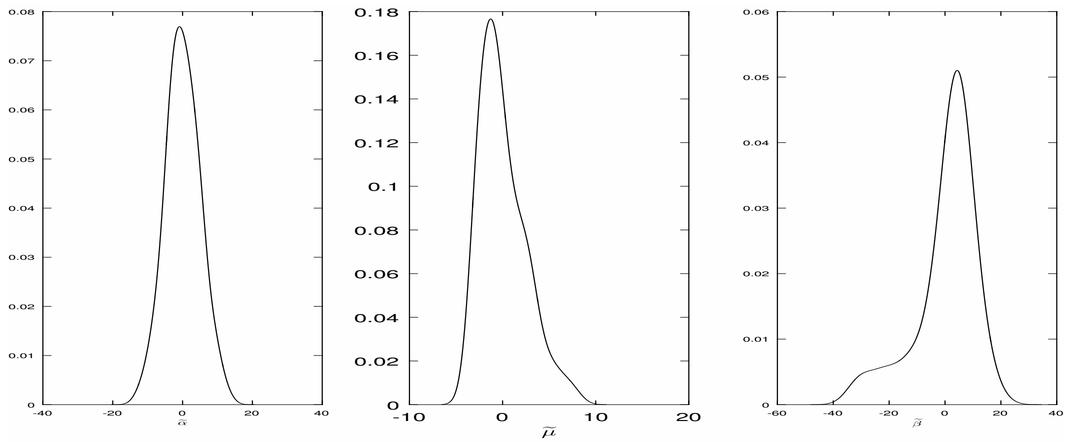

Figure 5: The distribution of *LSE* for *ROU* with two-sided barriers.

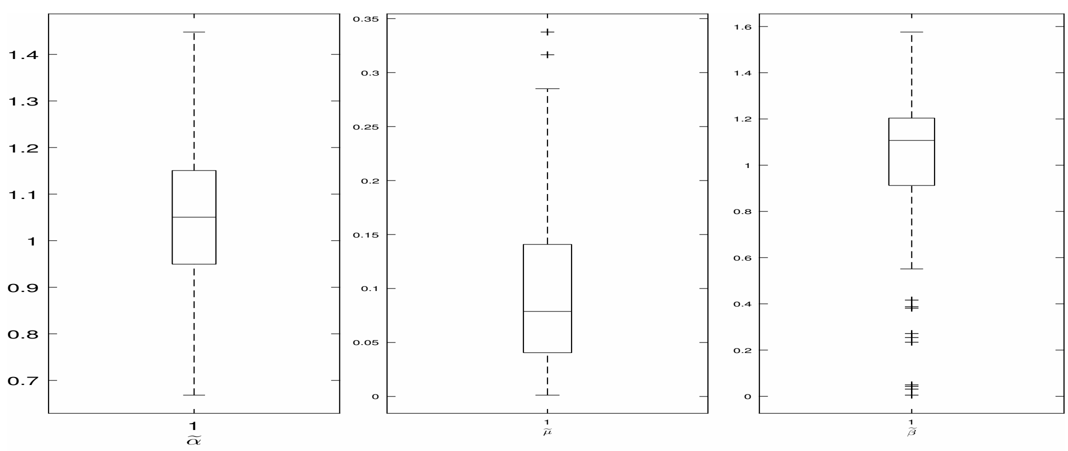

Figure 6: The box plot of *LSE* for *ROU* with two-sided barriers.

As in the first model, the sensitivity of the estimator $\tilde{\alpha}$ for α with respect to changes in μ exists. We remark that the influence of the change of μ on the values of the mean of $\tilde{\alpha}$ is reported in Table 5. For example, in the interval $[0, 0.1]$, if $\mu \in (0, 0.09]$, the empirical RMSE (root mean squared error) of the estimator increases as μ increases. However, if $\mu > 0.09$, we have the inverse where RMSE is maximized at 0.09 as shown in Figure 6.

μ	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1
Mean($\tilde{\alpha}$)	0.5940	0.5957	0.5975	0.5994	0.6012	0.6032	0.6052	0.6072	0.6093	0.5854
$RMSE(\tilde{\alpha})$	0.1199	0.1202	0.1205	0.1208	0.1210	0.1213	0.1217	0.1221	0.1225	0.1106

Table 5: Mean ($\tilde{\alpha}$) and $RMSE$ ($\tilde{\alpha}$) against μ for the ROU process with two-sided barriers, with $\alpha = 0.5$, $\beta = 1$, and $b^L = -0.5$, $b^U = 2$.

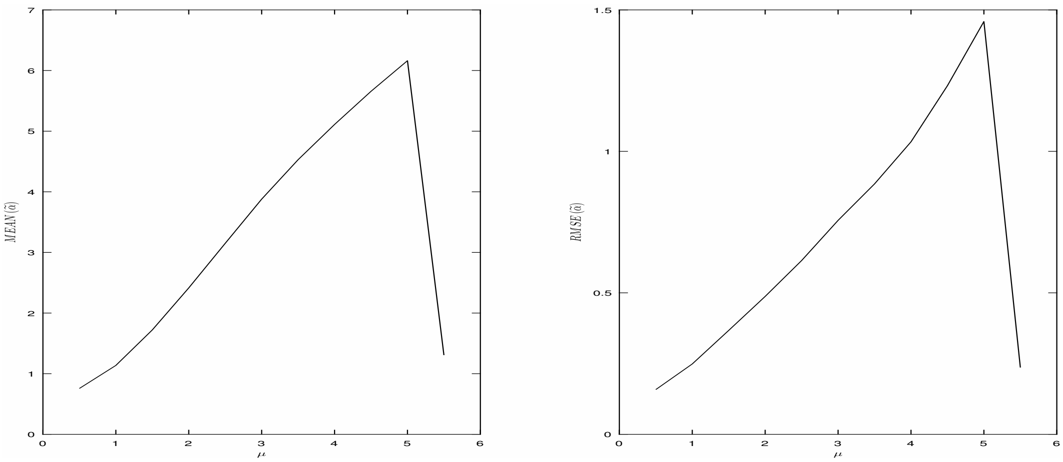

Figure 7: Mean($\widetilde{\alpha}$) and $RMSE\,(\widetilde{\alpha})$ against μ for the ROU process with two-sided barriers, $\alpha = 0.5$, $\beta = 1$, $b^L = -0.5$ and $b^U = 2$.

5. Conclusion

In our work, we have considered the reflected Ornstein-Uhlenbeck (ROU) processes in both one-sided barrier and two-sided barriers with multiple parameters case. We conclude that the application of the least squares estimation (LSE) for the reflected Ornstein-Uhlenbeck (ROU) processes based on continuous observations provides us with new formulas for the estimators of the parameters. The computed moments of the ROU processes are used to obtain an explicit expression of the estimator for volatility. Then, we proved the strong consistency and the asymptotic normality of the LSE. A simulation-based parameter estimation scheme was developed for both models. The finite sample study showed that the estimation algorithm is effective, which confirms our proposed method.

References

1. Ata, B and Harrison, J.M and Shepp, L.A., *Drift rate control of a Brownian processing system* , Annals of Applied Probability 15, 1145-1160, (2005).

2. Yang, X and Ren, G and Wang, Y and Bo, L and Li, D., *Modeling the exchange rate in a target zone by a reflected Ornstein-Uhlenbeck process*, SSRN Electronic Journal, (2016).

3. Bo, L and Tang, D and Wang, Y and Yang, X., *On the conditional default probability in a regulated market: a structural approach*, Quantitative Finance 11, 1695-1702, (2011).

4. Bo, L and Wang, Y and Yang, X., *Some integral functionals of reflected SDEs and their applications in finance* , Quantitative Finance 11 , 343-348, (2010).

5. Bo, L and Wang, Y and Yang, X and Zhang, G., *Maximum likelihood estimation for reflected Ornstein-Uhlenbeck processes*, Journal of Statistical Planning and Inference 141, 588-596, (2011).

6. Bo, L and Zhang, L and Wang, Y., *On the first passage times of reflected O-U processes with two-sided barriers*, Queueing Systems 54, 313-316, (2006).

7. Glasserman, P., *Monte Carlo Methods in Financial Engineering*, Springer, New York, (2004).

8. Harrison, M., *Brownian Motion and Stochastic Flow Systems*, John Wiley Sons, NewYork, (1986).

9. Linetsky, V., *On the transition densities for reflected diffusion*, Adv. in Appl. Probab. 37, 435-460, (2005).

10. Hu, Y and Lee, C and Lee, M and Song, J., *Parameter estimation for reflected Ornstein-Uhlenbeck processes with discrete observations*, Statistical Inference for Stochastic, 18(3), 279-291, (2015).

11. Kallenberg, O., *Fandtation of Modern Probability*, Springer-Verlagn, New York, (1997).

12. Krugman, P.R., *Target zones and exchange rate dynamics*, Quarterly Journal of Economics 106, 669-682, (1991).

13. Lee, C and Bishwal, J.P.N and Lee, M.H., *Sequential maximum likelihood estimation for reflected Ornstein-Uhlenbeck processes*, J. Statist. Plann. Inference 142, 1234-1242, (2012).

14. Lépingle, D., *Euler scheme for reflected stochastic differential equations*, Mathematics and Computers in Simulation 38,119-126, (1995).

15. Lions, P and Sznitman, A., *Stochastic differential equations with reflecting boundary conditions*, Comm. Pure Appl. Math. 17, 511-537, (1984).

16. Ricciardi, L.M., *Stochastic population models II. Diffusion models*, Lecture Note at the International School on Mathematical Ecology, (1985).

17. Ricciardi, L M and Sacerdote, L., *On the probability densities of an Ornstein-Uhlenbeck process with a reflecting boundary*, Journal of Applied Probability 24, 355-369, (1987).

18. Prakasa Rao, B.L.S., *Statistical Inference for Diffusion Type Processes*, Arnold (Co-published by Oxford University Press), New York, , (1999).

19. Valdivieso, L and Schoutens, W and Tuerlinckx, F., *Maximum likelihood estimation in processes of Ornstein-Uhlenbeck type*, Statistical Inference for Stochastic Processes 12, 1-19, (2009).

20. Ward, A and Glynn, P., *A diffusion approximation for Markovian queue with reneging*, Queueing Systems 43,103-128, (2003).

21. Ward, A and Glynn, P., *Properties of the reflected Ornstein-Uhlenbeck process*, Queueing Systems 44, 109-123, (2003).

22. Whitt, W., *Stochastic Process Limits. Springer Series in Operations Research*, Springer-Verlag, New York, (2002).

23. Zhu, C., *Some limiting results of reflected Ornstein-Uhlenbeck processes with two-sided barriers*, Bull. Korean Math. Soc. Vol 54(2), 573-581, (2016).

Fateh Merahi,
Department of Statistics and Data Science,
Faculty of Mathematics and Computer Science,
Batna 2 University, Batna, Algeria.
E-mail address: f.merahi@univ-batna2.dz

and

Abdelouahab Bibi,
Department of Mathematics,
Larbi Ben M'hidi University, Oum El Bouaghi, Algeria.
E-mail address: abd.bibi@gmail.com