Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–20.
ISSN-0037-8712
doi:10.5269/bspm.64565

Entropy solutions for nonlinear parabolic problems involving the generalized $p(x)$-Laplace operator and L^1 data

Mohamed Badr BENBOUBKER* and Urbain TRAORE

ABSTRACT: In this paper we prove the existence of an entropy solution to nonlinear parabolic equations with nonhomogeneous Neumann boundary conditions and initial data in L^1. By a time discretization technique we analyze the existence, the uniqueness and the stability questions. The functional setting involves Lebesgue and Sobolev spaces with variable exponents.

Key Words: Nonlinear Parabolic problem, variable exponents, entropy solution, Neumann-type boundary conditions, semi-discretization.

Contents

1. Introduction

Let Ω be a bounded open domain of $\mathbb{R}^d$, $(d \geq 3)$ with Lipschitz boundary $\partial\Omega$, T is a fixed positive number, in this paper we study the existence and the uniqueness of entropy solution for the following nonlinear parabolic problem

$$(P) \begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}\left(\Phi\left(\nabla u - \Theta\left(u\right)\right)\right) + \left|u\right|^{p(x)-2} u + \alpha\left(u\right) = f \text{ in } Q_T =]0, T[\times\Omega, \\ \Phi\left(\nabla u - \Theta\left(u\right)\right).\eta + \gamma\left(u\right) = g \text{ on } \sum_T =]0, T[\times\partial\Omega, \\ u(0, .) = u_0 \text{ in } \Omega \end{cases}$$

where α, γ, Θ are continuous functions defined on $\mathbb{R}$ and verify some assumptions which will be given later, η denotes the unit vector normal to $\partial\Omega$ and

$$\Phi\left(\xi\right) = \left|\xi\right|^{p(x)-2} \xi, \quad \forall \xi \in \mathbb{R}^d.$$

The usual weak formulations of parabolic problems in the case where the initial data are in L^1 do not allow to ensure the existence and uniqueness of the solutions. To overcome these difficulties, new formulations and types of solutions have been developed: the so-called SOLA (Solution Obtained as the Limit of Approximations) solutions introduced by A. Dallaglio, the renormalized solutions by R. Diperna and P.-L. Lions and the entropy solutions formulated by Ph. Bénilan and *al* in [3] (for more details see [15], [9] and the references therein). In this paper, we will focus on the entropic formulation.

We recall that the elliptic version of the problem (P) has been already studied by Azroul et al. (cf. [5]). They introduced the notion of entropy solutions and established an existence result for L^1 data and later Jamea and El Hachimi proved in [17] the uniqueness of this entropy solution.

* Corresponding author.
2010 *Mathematics Subject Classification:* 35K55, 35K61, 35J60, 35Dxx.
Submitted August 01, 2022. Published September 24, 2025

The main purpose of this paper is to extend the results in [5,17] to the case of parabolic equations and to generalize the results in [15] to the Sobolev spaces with variable exponents.
The study of differential equations with variable exponents is a very active field (see [2,6,7,20,21,22]). In these papers, the authors consider a Leray-Lions type operator, which permit them to exploit the growth condition, the coerciveness condition and the monotonicity condition of the operator to achieve their work. Unfortunately, in this work, due to the term Θ in the operator, we don't have such Leray-Lions conditions and we can't use the same techniques as in these papers to study the problem.
To overcome this difficulty we apply a time discretization of given continuous problem by the Euler forward scheme and we study the existence, the uniqueness and the stability questions. Let's recall that this method has been used in the literature for the study of some nonlinear parabolic problems, we refer for example to [8,11,14,15] for some details. This method is usually used to prove existence of solutions as well as to compute numerical approximations. Our motivations for the study of the problem (P) comes from the applications in filtration of a fluid in a partially saturated porous medium and flow through a porous medium in a turbulent regime (see [15]).

The rest of the paper is organized as follows : after some preliminary results in Section 2, we introduce the Euler forward scheme associated with the problem (P) in Section 3. We analyse the stability of the discretized problem and we study the existence of an entropy solution to the parabolic problem (P), in Section 4.

2. Preliminaries

In this paper, we assume that

$$\begin{cases} p(.) : \overline{\Omega} & \rightarrow & \mathbb{R} \quad \text{is a continuous function such that} \\ 1 < p_- & \leq & p_+ < +\infty, \end{cases} \tag{2.1}$$

where $p_- := ess \inf_{x\in\Omega} p(x)$ and $p_+ := ess \sup_{x\in\Omega} p(x)$.
We denote the Lebesgue space with variable exponent $L^{p(.)}(\Omega)$ (see [10]) as the set of all measurable function $u : \Omega \to \mathbb{R}$ for which the convex modular

$$\rho_{p(.)}(u) := \int_\Omega |u|^{p(x)}\, dx$$

is finite.
If the exponent is bounded, i.e., if $p_+ < +\infty$, then the expression

$$\|u\|_{p(.)} := \inf\left\{\lambda > 0 : \rho_{p(.)}(u/\lambda) \leq 1\right\}$$

defines a norm in $L^{p(.)}(\Omega)$, called the Luxemburg norm.
The space $\left(L^{p(.)}(\Omega), \|.\|_{p(.)}\right)$ is a separable Banach space. Moreover, if $1 < p_- \leq p_+ < +\infty$, then $L^{p(.)}(\Omega)$ is uniformly convex, hence reflexive and its dual space is isomorphic to $L^{p'(.)}(\Omega)$, where $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$.
Finally, we have the Hölder's type inequality

$$\left|\int_\Omega uv dx\right| \leq \left(\frac{1}{p_-} + \frac{1}{p_+}\right) \|u\|_{p(x)} \|v\|_{p'(x)}, \tag{2.2}$$

for all $u \in L^{p(.)}(\Omega)$ and $v \in L^{p'(.)}(\Omega)$.
Let

$$W^{1,p(.)}(\Omega) := \left\{u \in L^{p(.)}(\Omega) : |\nabla u| \in L^{p(.)}(\Omega)\right\},$$

which is Banach space equipped with the following norm

$$\|u\|_{1,p(.)} := \|u\|_{p(.)} + \|\nabla u\|_{p(.)}.$$

The space $\left(W^{1,p(.)}(\Omega), \|.\|_{1,p(.)}\right)$ is a separable and reflexive Banach space.
An important role in manipulating the generalized Lebesgue and Sobolev spaces is played by the modular $\rho_{p(x)}$ of the space $L^{p(.)}(\Omega)$. We have the following result.

Proposition 2.1 *(see [12,25]) If $u_n, u \in L^{p(.)}(\Omega)$ and $p_+ < \infty$, the following properties hold true:*
(i) $\|u\|_{p(.)} > 1 \Rightarrow \|u\|_{p(.)}^{p-} < \rho_{p(.)}(u) < \|u\|_{p(.)}^{p+}$;
(ii) $\|u\|_{p(.)} < 1 \Rightarrow \|u\|_{p(.)}^{p+} < \rho_{p(.)}(u) < \|u\|_{p(.)}^{p-}$;
(iii) $\|u\|_{p(.)} < 1$ $(respectively = 1; > 1) \Leftrightarrow \rho_{p(.)}(u) < 1$ $(respectively = 1; > 1)$;
(iv) $\|u_n\|_{p(.)} \to 0$ $(respectively \to +\infty) \Leftrightarrow \rho_{p(.)}(u_n) < 1$ $(respectively \to +\infty)$;
(v) $\rho_{p(.)}\left(u/\|u\|_{p(.)}\right) = 1.$

For a measurable function $u : \Omega \to \mathbb{R}$ we introduce the following notation:

$$\rho_{1,p(.)}(u) = \int_\Omega |u|^{p(x)}\, dx + \int_\Omega |\nabla u|^{p(x)}\, dx.$$

Proposition 2.2 *(see [23,24]) If $u \in W^{1,p(.)}(\Omega)$, the following properties hold true:*
(i) $\|u\|_{1,p(.)} > 1 \Rightarrow \|u\|_{1,p(.)}^{p-} < \rho_{1,p(.)}(u) < \|u\|_{1,p(.)}^{p+}$;
(ii) $\|u\|_{1,p(.)} < 1 \Rightarrow \|u\|_{1,p(.)}^{p+} < \rho_{1,p(.)}(u) < \|u\|_{1,p(.)}^{p-}$;
(iii) $\|u\|_{1,p(.)} < 1$ $(respectively = 1; > 1) \Leftrightarrow \rho_{1,p(.)}(u) < 1$ $(respectively = 1; > 1)$.

Put

$$p^\partial(x) := (p(x))^\partial = \begin{cases} \dfrac{(N-1)\,p(x)}{N-p(x)}, & \text{if } \ p(x) < N \\ \infty, & \text{if } \ p(x) \geq N. \end{cases}$$

Proposition 2.3 *(see [24]) Let $p \in C(\bar{\Omega})$ and $p_- > 1$. If $q \in C(\partial\Omega)$ satisfies the condition*

$$1 < q(x) < p^\partial(x) \quad \forall x \in \partial\Omega,$$

then, there is a compact embedding $W^{1,p(.)}(\Omega) \hookrightarrow L^{q(.)}(\partial\Omega)$.
In particular, there is a compact embedding $W^{1,p(.)}(\Omega) \hookrightarrow L^{p(.)}(\partial\Omega)$.

Let us introduce the following notation: given two bounded measurable functions $p(x), q(x) : \Omega \to \mathbb{R}$, we write

$$q(x) \ll p(x) \ \text{ if } \ ess \inf_{x\in\Omega}(p(x) - q(x)) > 0.$$

For the next section, we need the following lemmas.

Lemma 2.1 *(see [1]) Let $\xi, \eta \in \mathbb{R}^N$ and let $1 < p < \infty$. We have*

$$\frac{1}{p}|\xi|^p - \frac{1}{p}|\eta|^p \leq |\xi|^{p-2}\xi.(\xi - \eta).$$

Lemma 2.2 *(see [16]) Let $(v_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions in Ω. If v_n converges in measure to v and is uniformly bounded in $L^{p(x)}(\Omega)$ for some $1 \ll p(x) \in L^\infty(\Omega)$, then v_n strongly converges to v in $L^1(\Omega)$.*

For a measurable set U in $\mathbb{R}^d$, meas(U) denotes its measure, C_i and C will denote various positive constants. For a Banach space X and $a < b$, $L^q(a, b; X)$ is the space of measurable functions $u : [a, b] \to X$ such that

$$\left(\int_a^b \|u\|_X^q\, dt\right)^{\frac{1}{q}} := \|u\|_{L^q(a,b;X)} < \infty. \tag{2.3}$$

For a given constant $k>0$ we define the cut-off function $T_k:\mathbb{R}\to\mathbb{R}$ by

$$T_k(s):=\begin{cases} s & \text{if} \quad |s|\le k\\ k\text{sign}(s) & \text{if} \quad |s|>k\end{cases}$$

with

$$\text{sign}(s):=\begin{cases} 1 & \text{if} \quad s>0\\ 0 & \text{if} \quad s=0\\ -1 & \text{if} \quad s<0.\end{cases}$$

Let $J_k:\mathbb{R}\to\mathbb{R}^+$ be defined by

$$J_k(x)=\int_0^x T_k(s)ds$$

(J_k is a primitive of T_k). We have (see [13])

$$\left\langle \frac{\partial v}{\partial t}, T_k(s)\right\rangle = \frac{d}{dt}\left(\int_\Omega J_k(v)dx\right) \text{ in } L^1(]0,T[),$$

which implies that

$$\int_0^t \left\langle \frac{\partial v}{\partial t}, T_k(s)\right\rangle = \int_\Omega J(v(t))dx - \int_\Omega J(v(0))dx.$$

For all $u\in W^{1,p(.)}(\Omega)$ we denote by $\tau(u)$ the trace of u on $\partial\Omega$ in the usual sense.
In the sequel, we will identify at the boundary, u and $\tau(u)$.
Set

$$\mathcal{T}^{1,p(.)}(\Omega)=\left\{u:\Omega\to\mathbb{R}, \text{ measurable such that } T_k(u)\in W^{1,p(x)}(\Omega), \text{ for any } k>0\right\}.$$

Proposition 2.4 *(see [9]) Let $u\in\mathcal{T}^{1,p(.)}(\Omega)$. Then there exists a unique measurable function $v:\Omega\to\mathbb{R}^N$ such that $\nabla T_k(u)=v\chi_{\{|u|<k\}}$, for all $k>0$. The function v is denoted by ∇u. Moreover, if $u\in W^{1,p(.)}(\Omega)$ then $v\in\left(L^{p(.)}(\Omega)\right)^N$ and $v=\nabla u$ in the usual sense.*

We denote by $\mathcal{T}_{tr}^{1,p(.)}(\Omega)$ (cf [3,4,18,19]) the set of functions $u\in\mathcal{T}^{1,p(.)}(\Omega)$ such that there exists a sequence $(u_n)_{n\in\mathbb{N}}\subset W^{1,p(.)}(\Omega)$ satisfying the following conditions:

i) $u_n\to u$ a.e. in Ω.

ii) $\nabla T_k(u_n)\to\nabla T_k(u)$ in $\left(L^1(\Omega)\right)^N$ for any $k>0$.

iii) There exists a measurable function v on $\partial\Omega$, such that $u_n\to v$ a.e. on $\partial\Omega$.

The function v is the trace of u in the generalized sense introduced in [3,4]. In the sequel, the trace of $u\in\mathcal{T}_{tr}^{1,p(.)}(\Omega)$ on $\partial\Omega$ will be denoted by $tr(u)$. If $u\in W^{1,p(.)}(\Omega)$, $tr(u)$ coincides with $\tau(u)$ in the usual sense. Moreover $u\in\mathcal{T}_{tr}^{1,p(.)}(\Omega)$ and for every $k>0$, $\tau(T_k(u))=T_k(tr(u))$ and if $\varphi\in W^{1,p(.)}(\Omega)\cap L^\infty(\Omega)$ then $(u-\varphi)\in\mathcal{T}_{tr}^{1,p(.)}(\Omega)$ and $tr(u-\varphi)=tr(u)-tr(\varphi)$.

3. The semi-discrete problem

In this section, we study the Euler forward scheme associated with the problem (P). We make the following hypotheses :

(H_1) α and γ are continuous functions defined on $\mathbb{R}$ such that there exists two positive real numbers M_1, M_2 with $|\alpha(x)|\le M_1$, $|\gamma(x)|\le M_2$, $\alpha(x).x\ge 0$, $\gamma(x).x\ge 0$ for all $x\in\mathbb{R}$ and $\alpha(0)=\gamma(0)=0$.

(H_2) $f\in L^1(Q_T)$, $g\in L^1(\sum_T)$ and $u_0\in L^1(\Omega)$.

(H_3) $\Theta:\mathbb{R}\to\mathbb{R}^N$ is a continuous function such that $\Theta(0)=0$ and $|\Theta(x)-\Theta(y)|\le C_0|x-y|$ for all $x,y\in\mathbb{R}$, C_0 is a positive constant such that

$$C_0 < \min\left((\frac{p_-}{2})^{\frac{1}{p_-}}, (\frac{p_-}{2})^{\frac{1}{p_+}} \right).$$

The Euler forward scheme associated with the problem (P) is given by :

$$(P_n)\begin{cases} U^n - \tau \mathrm{div}\left(\Phi\left(\nabla U^n - \Theta\left(U^n\right)\right)\right) + \tau \left|U^n\right|^{p(x)-2} U^n + \tau\alpha\left(U^n\right) = \tau f_n + U^{n-1} \text{ in } \Omega \\ \Phi\left(\nabla U^n - \Theta\left(U^n\right)\right).\eta + \gamma\left(U^n\right) = g_n \text{ on } \partial\Omega, \\ U^0 = u_0 \text{ in } \Omega \end{cases}$$

where $N\tau = T$, $0 < \tau < 1$, $1 \leq n \leq N$ and

$$\begin{aligned} f_n(.) &= \tfrac{1}{\tau}\textstyle\int_{(n-1)\tau}^{n\tau} f(s,.)ds \quad \text{in } \Omega, \\ g_n(.) &= \tfrac{1}{\tau}\textstyle\int_{(n-1)\tau}^{n\tau} g(s,.)ds \quad \text{on } \partial\Omega. \end{aligned}$$

Definition 3.1 *An entropy solution to the discretized problems (P_n) is a sequence $(U^n)_{0\leq n\leq N}$ such that $U^0 = u_0$ and U^n is defined by induction as an entropy solution to the problem*

$$\begin{cases} U^n - \tau\, div\left(\Phi\left(\nabla U^n - \Theta\left(U^n\right)\right)\right) + \tau \left|U^n\right|^{p(x)-2} U^n + \tau\alpha\left(U^n\right) = \tau f_n + U^{n-1} \text{ in } \Omega \\ \Phi\left(\nabla U^n - \Theta\left(U^n\right)\right).\eta + \gamma\left(U^n\right) = g_n \text{ on } \partial\Omega \end{cases}$$

i.e. $U^n \in \mathcal{T}_{tr}^{1,p(.)}(\Omega)$, $\left|U^n\right|^{p(.)-2} U^n \in L^1(\Omega)$, $\alpha\left(U^n\right) \in L^1(\Omega)$, $\gamma\left(U^n\right) \in L^1(\partial\Omega)$ *and*

$$\begin{aligned} &\tau \int_\Omega \Phi(\nabla U^n - \Theta(U^n))\nabla T_k(U^n - \varphi)dx + \tau \int_\Omega |U^n|^{p(x)-2} u T_k(u-\varphi)dx \\ &+ \int_\Omega (\tau\alpha(U^n) + U^n)T_k(U^n - \varphi)dx + \tau \int_{\partial\Omega} \gamma(U^n)T_k(U^n - \varphi)d\sigma \\ &\leq \int_\Omega (\tau f_n + U^{n-1})T_k(U^n - \varphi)dx + \tau \int_{\partial\Omega} g_n T_k(U^n - \varphi)d\sigma, \end{aligned} \tag{3.1}$$

for any $\varphi \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$ and every $k > 0$.

We have the following result.

Lemma 3.1 *Let hypotheses $(H1) - (H3)$ be satisfied. If $(U^n)_{0\leq n\leq N}$ is an entropy solution of problems (P_n), then $U^n \in L^1(\Omega)$ for all $n = 1, \cdots, N$.*

Proof. For $n = 1$, we take $\varphi = 0$ in (3.1), to get,

$$\begin{aligned} &\tau \int_\Omega \Phi(\nabla U^1 - \Theta(U^1))\nabla T_k(U^1)dx + \tau \int_\Omega |U^1|^{p(x)-2} U^1 T_k(U^1)dx \\ &+ \int_\Omega (\tau\alpha(U^1) + U^1)T_k(U^1)dx + \tau \int_{\partial\Omega} \gamma(U^1)T_k(U^1)d\sigma \\ &\leq \int_\Omega (\tau f_1 + u_0)T_k(U^1)dx + \tau \int_{\partial\Omega} g_1 T_k(U^1)d\sigma, \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\tau \int_\Omega \Phi(\nabla T_k(U^1) - \Theta(T_k(U^1)))\nabla T_k(U^1)dx + \tau \int_\Omega \frac{1}{p(x)}|\Theta(T_k(U^1))|^{p(x)}dx + \tau \int_\Omega |U^1|^{p(x)-2} U^1 T_k(U^1)dx \\ &+ \int_\Omega (\tau\alpha(U^1) + U^1)T_k(U^1)dx + \tau \int_{\partial\Omega} \gamma(U^1)T_k(U^1)d\sigma \\ &\leq \int_\Omega (\tau f_1 + u_0)T_k(U^1)dx + \tau \int_{\partial\Omega} g_1 T_k(U^1)d\sigma + \tau \int_\Omega \frac{1}{p(x)}|\Theta(T_k(U^1))|^{p(x)}dx. \end{aligned} \tag{3.2}$$

In Lemma 2.1, we take $\xi = (\nabla T_k(U^1) - \Theta(T_k(U^1)))$ and $\eta = -\Theta(T_k(U^1))$ to obtain

$$\begin{aligned} & |\nabla T_k(U^1) - \Theta(T_k(U^1)|^{p(x)-2}(\nabla T_k(U^1) - \Theta(T_k(U^1)).\nabla T_k(T_k(U^1)) \\ & +\frac{1}{p(x)}|\Theta(T_k(U^1))|^{p(x)} \\ \geq \ & \frac{1}{p(x)}|\nabla T_k(U^1) - \Theta(T_k(U^1))|^{p(x)}. \end{aligned}$$

This implies that

$$\tau \int_\Omega \Phi(\nabla T_k(U^1) - \Theta(T_k(U^1)))\nabla T_k(U^1)dx + \tau \int_\Omega \frac{1}{p(x)}|\Theta(T_k(U^1))|^{p(x)}dx \geq 0.$$

Thanks to assumption $(H1)$, we have

$$\tau(\int_\Omega \tau\alpha(U^1)T_k(U^1)dx + \int_{\partial\Omega} \gamma(U^1)T_k(U^1)d\sigma) \geq 0.$$

Using hypothesis $(H3)$ and the fact that $\tau < 1$, $p(x) > 1$ for all $x \in \Omega$ we get

$$\tau \int_\Omega \frac{1}{p(x)}|\Theta(T_k(U^1))|^{p(x)}dx \leq \int_\Omega (C_0k)^{p_+}dx \leq (C_0k)^{p_+}\text{meas}(\Omega).$$

Therefore, since the third term in the inequality (3.2) is nonnegative, it follows that

$$\int_\Omega U^1T_k(U^1)dx \leq k\tau[\|f_1\|_1 + \|g_1\|_{L^1(\partial\Omega)}] + k\,\|u_0\|_1 + (C_0k)^{p_+}\text{meas}(\Omega).$$

Since

$$\sum_{n=1}^{N} \tau[\|f_n\|_1 + \|g_n\|_{L^1(\partial\Omega)}] \leq \|f\|_1 + \|g\|_{L^1(\partial\Omega)}\,,$$

it follows that

$$\int_\Omega U^1T_k(U^1)dx \leq k(\|f\|_1 + \|g\|_{L^1(\partial\Omega)} + \|u_0\|_1) + (C_0k)^{p_+}\text{meas}(\Omega). \tag{3.3}$$

We have

$$\lim_{k\to 0} U^1\frac{T_k(U^1)}{k}dx = |U^1|.$$

Then dividing (3.3) by k and letting $k \to 0$; we deduce by Fatou's lemma that

$$\left\|U^1\right\|_1 \leq C_1, \tag{3.4}$$

where C_1 is a constant depending on f, g, and u_0□

Theorem 3.1 *Let hypotheses $(H1)-(H3)$ be satisfied. Then for all $N \in \mathbb{N}$, the problems (P_n) have an entropy solution $U^n \in \mathcal{T}^{1,p(.)}_{tr}(\Omega) \cap L^1(\Omega)$ for all $n = 1, \cdots, N$.*

Proof. The problem (P_1) can be rewrite in the following form

$$-\tau\text{div}\left(\Phi\left(\nabla u - \Theta\left(u\right)\right)\right) + \tau\left|u\right|^{p(x)-2}u + \overline{\alpha}(u) = F_1 \text{ in } \Omega$$

$$\Phi(\nabla u - \Theta(u)).\eta + \gamma(u) = G_1 \text{ on } \partial\Omega$$

with

$$\overline{\alpha}(s) := \alpha(s) + s, \quad F_1 := \tau f + u_0, \quad G_1 := g_1.$$

From the assumption (H_2), we have $(F_1, G_1) \in L^1(\Omega) \times L^1(\partial\Omega)$, and using $(H1)$, we obtain that $\overline{\alpha}$ is continuous, $\overline{\alpha}(0) = 0$ and $\overline{\alpha}(s)s \geq 0$ for all $s \in \mathbb{R}$.
Hence, using [5, Theorem 3.1], we get the existence of an entropy solution to the problem (P_1); $U^1 \in \mathcal{T}_{tr}^{1,p(.)}(\Omega)$, and $\alpha\left(U^1\right) \in L^1(\Omega)$, $\gamma\left(U^1\right) \in L^1(\partial\Omega)$.

Thanks to Lemma 3.1, by induction, we deduce in the same manner that for $n = 2, \cdots, N$, the problem

$$\begin{aligned} & u - \operatorname{div}\left(\Phi\left(\nabla u - \Theta\left(u\right)\right)\right) + \tau\left|u\right|^{p(x)-2} u + \tau\alpha\left(u\right) = \tau f^n + U^{n-1} \text{ in } \Omega \\ & \Phi\left(\nabla u - \Theta\left(u\right)\right).\eta + \gamma(u) = g^n \text{ on } \partial\Omega, \end{aligned}$$

has an entropy solution $U^n \in \mathcal{T}_{tr}^{1,p(.)}(\Omega) \cap L^1(\Omega)$, $\alpha\left(U^n\right) \in L^1(\Omega)$, $\gamma\left(U^n\right) \in L^1(\partial\Omega)$. Moreover if $1 < p_- < p_+ \leq 2$ from [17, Theorem 1] this entropy solution is unique.

4. Stability

This section is devoted to the a priori estimates for the discrete entropy solution $(U^n)_{1\leq n\leq N}$. These result are essentials for the study of the convergence of the Euler forward scheme.

Theorem 4.1 *Let hypotheses $(H1)-(H3)$ be satisfied. Then there exist a positives constants $C(u_0, f, g)$ and $C(u_0, p_+, f, g)$ depending on the data but not on N such that for all $n = 1, \cdots, N$, we have*

1. $\left\|U^n\right\|_1 \leq C(u_0, f, g)$,
2. $\tau \sum_{i=1}^n \left\|\alpha(U^i)\right\|_1 + \tau \sum_{i=1}^n \left\|\gamma(U^i)\right\|_1 + \tau \sum_{i=1}^n \left\|\left|U^i\right|^{p(x)-2} U^i\right\|_1 \leq C(u_0, f, g)$,
3. $\sum_{i=1}^n \left\|U^i - U^{i-1}\right\|_1 \leq C(u_0, f, g)$,
4. $\tau \sum_{i=1}^n \rho_{1,p(x)}(T_k(U^i)) \leq kC(u_0, p_+, f, g)$.

Proof. 1 and 2. For $\varphi = 0$ as a test function in (3.1), we have

$$\begin{aligned} & \frac{\tau}{k}\left(\int_\Omega \Phi(\nabla T_k(U^i) - \Theta(T_k(U^i)))\nabla T_k(U^i)dx + \int_\Omega \frac{1}{p(x)}|\Theta(T_k(U^i))|^{p(x)}dx\right) \\ & +\tau\int_\Omega |U^i|^{p(x)-2}\frac{U^i T_k(U^i)}{k}dx + \int_\Omega U^i\frac{T_k(U^i)}{k}dx + \int_\Omega \tau\alpha(U^i)\frac{T_k(U^i)}{k}dx + \tau\int_{\partial\Omega}\gamma(U^i)\frac{T_k(U^i}{k}d\sigma \\ & \leq \tau(\|f_i\|_1 + \|g_i\|_1) + \left\|U^{i-1}\right\|_1 + \tau\int_\Omega \frac{1}{kp(x)}|\Theta(T_k(U^i))|^{p(x)}dx. \end{aligned}$$

Since

$$\int_\Omega \Phi(\nabla T_k(U^i) - \Theta(T_k(U^i)))\nabla T_k(U^i)dx + \int_\Omega \frac{1}{p(x)}|\Theta(T_k(U^i))|^{p(x)}dx \geq 0,$$

it follows that

$$\begin{aligned} & \tau\int_\Omega |U^i|^{p(x)-2}\frac{U^i T_k(U^i)}{k}dx + \int_\Omega U^i\frac{T_k(U^i)}{k}dx + \int_\Omega \tau\alpha(U^i)\frac{T_k(U^i)}{k}dx + \tau\int_{\partial\Omega}\gamma(U^i)\frac{T_k(U^i}{k}\sigma \\ & \leq \tau(\|f_i\|_1 + \|g_i\|_1) + \left\|U^{i-1}\right\|_1 + k^{p_+-1}C_0^{p_+}\text{meas}(\Omega). \end{aligned}$$

Then letting $k \to 0$ and using Fatou's lemma, we deduce that

$$\begin{aligned} & \tau\left\||U^i|^{p(x)-1}\right\|_1 + \left\|U^i\right\|_1 + \tau\left\|\alpha(U^i)\right\|_1 + \tau\left\|\gamma(U^i)\right\|_1 \\ & \qquad \leq \tau(\|f_i\|_1 + \|g_i\|_1) + \left\|U^{i-1}\right\|_1. \end{aligned} \tag{4.1}$$

Now, we sum (4.1) from $i=1$ to n to obtain

$$\|U^n\|_1+\tau\sum_{i=1}^{n}\left\|\alpha(U^i)\right\|_1+\tau\sum_{i=1}^{n}\left\|\gamma(U^i)\right\|_1+\tau\sum_{i=1}^{n}\left\||U^i|^{p(x)-1}\right\| \leq \|f\|_1+\|g\|_1+\|u_0\|_1 \tag{4.2}$$

which gives the inequalities 1 and 2.

3 : For $k\geq 1$, we take $\varphi=T_h(U^i-sign(U^i-U^{i-1}))$, $(h>1)$ as a test function in (3.1), then letting $h\to\infty$, we obtain,

$$\begin{aligned}&\tau\lim_{h\to\infty}\mathcal{I}(k,h)+\left\|U^i-U^{i-1}\right\|_1 \\ &\leq \tau\left(\|f_i\|_1+\|g_i\|_{L^1(\partial\Omega)}+\left\|\alpha(U^i)\right\|_1+\left\|\gamma(U^i)\right\|_1+\left\||U^i|^{p(x)-1}\right\|_1\right),\end{aligned} \tag{4.3}$$

where

$$\begin{aligned}\mathcal{I}(k,h) &:= \int_\Omega \Phi(\nabla U^i-\Theta(U^i))\nabla T_k(U^i-T_h(U^i-sign(U^i-U^{i-1})))dx \\ &= \int_{\Omega_{k,h}\cap\overline{\Omega(k)}}\Phi(\nabla U^i-\Theta(U^i))\nabla U^i dx\end{aligned}$$

and

$$\begin{aligned}\Omega_{k,h} &:= \left\{|U^i-T_h(U^i-sign(U^i-U^{i-1}))|\leq k\right\}, \\ \overline{\Omega(k)} &= \left\{|U^i-sign(U^i-U^{i-1})|>h\right\}.\end{aligned}$$

Since

$$\Omega_{k,h}\cap\overline{\Omega(k)}\subset\left\{k-1\leq|U^i|\leq k+h\right\},$$

as in the proof of [1, Lemma 3.6], one show that

$$\lim_{h\to\infty}\mathcal{I}(k,h)=0.$$

Then, it follows that

$$\left\|U^i-U^{i-1}\right\|_1\leq k\tau\left(\|f_i\|_1+\left\|g_{L^1(\partial\Omega)}\right\|_1+\left\|\alpha(U^i)\right\|_1+\left\|\gamma(U^i)\right\|_1+\left\||U^i|^{p(x)-1}\right\|_1\right). \tag{4.4}$$

Then, summing (4.4) from $i=1$ to n and by the stability result 2, we obtain the stability result 3.

4. We take $\varphi=0$ as a test function in (3.1) to get

$$\begin{aligned}&\tau\Big(\int_\Omega|\nabla T_k(U^i)-\Theta(T_k(U^i))|^{p(x)-2}(\nabla T_k(U^i)-\Theta(T_k(U^i)))\nabla T_k(U^i)dx+\int_\Omega|T_k(U^i)|^{p(x)}dx\Big) \\ \leq\ & k\tau(\|f_i\|_1+\|g_i\|_{L^1(\partial\Omega)}+\left\|\alpha(U^i)\right\|_1+\left\|\gamma(U^i)\right\|_1)+k\left\|U^i-U^{i-1}\right\|_1.\end{aligned}$$

Using Lemma 2.1 and the inequality

$$(a+b)^p\leq 2^{p-1}(a^p+b^p)\quad\forall a,b\in\mathbb{R}^+,\ 1\leq p<\infty \tag{4.5}$$

we obtain respectively

$$\begin{aligned}&|\nabla T_k(U^i)-\Theta(T_k(U^i))|^{p(x)-2}(\nabla T_k(U^i)-\Theta(T_k(U^i)))\nabla T_k(T_k(U^i)) \\ \geq\ & \frac{1}{p(x)}|\nabla T_k(U^i)-\Theta(T_k(U^i))|^{p(x)}-\frac{1}{p(x)}|\Theta(T_k(U^i)|^{p(x)}\end{aligned}$$

and

$$|\nabla T_k(U^i)|^{p(x)}\leq 2^{p(x)-1}(|\nabla T_k(U^i)-\Theta(T_k(U^i))|^{p(x)}+|\Theta(T_k(U^i))|^{p(x)}).$$

Then, using the assumption $(H3)$ it follows that

$$\begin{aligned} & |\nabla T_k(U^i) - \Theta(T_k(U^i))|^{p(x)-2}(\nabla T_k(U^i) - \Theta(T_k(U^i)))\nabla T_k(U^i) + |T_k(U^i|^{p(x)} \qquad (4.6)\\ \geq\ & \frac{1}{2^{p(x)-1}}\frac{1}{p(x)}|\nabla T_k(U^i)|^{p(x)} + |T_k(U^i|^{p(x)} - \frac{2}{p(x)}|\Theta(T_k(U^i)|^{p(x)}\\ \geq\ & \frac{1}{2^{p_+-1}}\frac{1}{p_+}|\nabla T_k(U^i)|^{p(x)} + (1-\frac{1}{p_-}C_0^{p(x)})|T_k(U^i|^{p(x)}. \qquad (4.7)\end{aligned}$$

So, the choice of C_0 in $(H3)$ gives the existence of a positive constant C such that

$$\begin{aligned} & \int_\Omega (|\nabla T_k(U^i) - \Theta(T_k(U^i))|^{p(x)-2}(\nabla U^i - \Theta(U^i))\nabla T_k(U^i)dx + |T_k(U^i|^{p(x)})dx\\ \geq\ & \min\left\{\frac{1}{2^{p_+-1}}\frac{1}{p_+}, C\right\}(\int_\Omega |\nabla T_k(U^i)|^{p(x)}dx + \int_\Omega |T_k(U^i)|^{p(x)}dx)\\ \geq\ & \min\left\{\frac{1}{2^{p_+-1}}\frac{1}{p_+}, C\right\}\rho_{1,p(.)}(T_k(U^i)).\end{aligned}$$

Therefore,

$$\begin{aligned}\tau\rho_{1,p(.)}(T_k(U^i)) \leq\ & \frac{1}{\min\left\{\frac{1}{p_+2^{p_+-1}}, C\right\}}\Big[k\tau(\|f_i\|_1 + \|g_i\|_{L^1(\partial\Omega)} + k\left\|\alpha(U^i)\right\|_1 + \left\|\gamma(U^i)\right\|_1)\\ & +k\left\|U^i - U^{i-1}\right\|_1\Big]. \qquad (4.8)\end{aligned}$$

Now, summing (4.8) from $i = 1$ to n and using the stability results 1, 2, 3, we get

$$\begin{aligned}\tau\sum_{i=1}^n \rho_{1,p(.)}(T_k(U^i)) \leq\ & \frac{1}{\min\left\{\frac{1}{p_+2^{p_+-1}}, C\right\}}\Bigg[k(\|f\|_1 + \|g\|_{L^1(\partial\Omega)} + \tau\sum_{i=1}^n\left\|\alpha(U^i)\right\|_1 + \tau\sum_{i=1}^n\left\|\gamma(U^i)\right\|_1)\\ & + k\sum_{i=1}^n\left\|U^i - U^{i-1}\right\|_1\Bigg]\\ \leq\ & kC(f, g, u_0, p_+)\square\end{aligned}$$

5. Convergence and existence result.

In this section, we prove the existence of an entropy solution of problem (P). First of all, we introduce the appropriated spaces for the entropy formulation of the nonlinear parabolic problem (P).

We define the space:

$$V = \left\{v \in L^{p_-}(0, T; W^{1,p(\cdot)}(\Omega)) : \nabla v \in (L^{p(\cdot)}(Q_T))^d\right\},$$

and

$$\begin{aligned}\mathcal{T}^{1,p(\cdot)}(Q_T) =\ & \Big\{u : \Omega\times(0, T];\text{measurable} \mid T_k(u) \in L^{p_-}(0, T; W^{1,p(\cdot)}(\Omega))\\ & \text{with } \nabla T_k(u) \in (L^{p(\cdot)}(Q_T))^d \ \text{ for every } k > 0\Big\}.\end{aligned}$$

Definition 5.1 *A measurable function u is called an entropy solution to problem (P) if $u \in \mathcal{T}^{1,p(\cdot)}(Q_T)\cap C(0, T; L^1(\Omega))$ and for every $k > 0$,*

$$\begin{aligned}&\int_0^t\int_\Omega \Phi(\nabla u - \Theta(u))\nabla T_k(u-\varphi) + \int_0^t\int_\Omega \alpha(u)T_k(u-\varphi) + \int_0^t\int_{\partial\Omega}\gamma(u)T_k(u-\varphi)\\ &\leq -\int_0^t\left\langle\frac{\partial\varphi}{\partial s}, T_k(u-\varphi)\right\rangle + \int_\Omega J_k(u(0)-\varphi(0)) - \int_\Omega J_k(u(t)-\varphi(t))\\ &+\int_0^t\int_\Omega fT_k(u-\varphi) + \int_0^t\int_{\partial\Omega} gT_k(u-\varphi)\end{aligned}$$

for all $\phi \in L^\infty(Q)\cap V\cap W^{1,1}(0, T; L^1(\Omega))$ and $t \in [0, T]$.

Our main result is the following.

Theorem 5.1 *Let hypotheses $(H1)-(H3)$ be satisfied. Then the nonlinear parabolic problem (P) has an entropy solution.*

Proof : The proof is divided into two steps
Step 1 The Rothe function : We introduce a piecewise linear extension:

$$\begin{cases} u^N(0) & := & u_0, \\ u^N(t) & := & U^{n-1} + (U^n - U^{n-1})\frac{t-t^{n-1}}{\tau} \end{cases} \tag{5.1}$$

for all $t \in]t^{n-1}, t^n]$, $n = 1, \cdots, N$, in Ω and a piecewise constant function

$$\begin{cases} \overline{u}^N(0) & := & u_0, \\ \overline{u}^N(t) & := & U^n,\ \forall t \in]t^{n-1}, t^n],\ n = 1, \cdots, N,\ \text{ in } \Omega, \end{cases} \tag{5.2}$$

where $t^n := n\tau$ and $(U^n)_{1\leq n\leq N}$ is an entropy solution of (P_n).

From the Theorem 4.1, we deduce the existence of a constant $C(T, u_0, f, g)$ not depending on N such that for all $N \in \mathbb{N}$, we have

$$\begin{aligned} &\|\overline{u}^N - u^N\|_{L^1(Q_T)} \leq \frac{1}{N}C(T, u_0, f, g), \\ &\|u^N\|_{L^1(Q_T)} \leq C(T, u_0, f, g), \\ &\|\overline{u}^N\|_{L^1(Q_T)} \leq C(T, u_0, f, g), \\ &\left\||\overline{u}^N|^{p(x)-2}\overline{u}^N\right\|_{L^1(Q_T)} \leq C(T, u_0, f, g), \\ &\left\|\frac{\partial u^N}{\partial t}\right\|_{L^1(Q_T)} \leq C(T, u_0, f, g), \\ &\|\alpha(\overline{u}^N)\|_{L^1(Q_T)} \leq C(T, u_0, f, g), \\ &\|\gamma(\overline{u}^N)\|_{L^1(Q_T)} \leq C(T, u_0, f, g). \end{aligned} \tag{5.3}$$

Moreover combining Proposition 2.1 and Young inequality, we get

$$\begin{aligned} \int_0^T \|T_k(\overline{u}^N)\|_{1,p(x)}^{p_-} dt &\leq \int_0^T \max\left\{\rho_{1,p(x)}(T_k(\overline{u}^N)); \rho_{1,p(x)}(T_k(\overline{u}^N))^{\frac{p_-}{p_+}}\right\} dt \\ &\leq \int_0^T \rho_{1,p(x)}(T_k(\overline{u}^N))dt + \int_0^T \rho_{1,p(x)}(T_k(\overline{u}^N))^{\frac{p_-}{p_+}} dt \\ &\leq \int_0^T \rho_{1,p(x)}(T_k(\overline{u}^N))dt + \int_0^T \left(\frac{p_-}{p_+}\rho_{1,p(x)}(T_k(\overline{u}^N)) + (1 - \frac{p_-}{p_+})\right) dt \\ &\leq \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \rho_{1,p(x)}(T_k(U^n))dt + \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \rho_{1,p(x)}(T_k(U^n))dt + \frac{p_+ - p_-}{p_+}T \\ &\leq 2\sum_{n=1}^N \tau\rho_{1,p(x)}(T_k(U^n)) + T. \end{aligned} \tag{5.4}$$

Consequently from stability result 4 it follows that

$$\|T_k(\overline{u}^N)\|_{L^{p_-}(0,T;W^{1,p(\cdot)}(\Omega))} \leq kC(T, p_+, u_0, f, g) \tag{5.5}$$

where $C(T, p_+, u_0, f, g)$ is a positive constant depending only the data.

Lemma 5.1 *Let hypotheses $(H1)-(H3)$ be satisfied. Then the sequence $(\overline{u}^N)_{N\in\mathbb{N}}$ converges in measure and a.e. in Q_T.*

Proof Let ε, r, k be positive numbers. For $N, M\in\mathbb{N}$, we have the inclusion

$$\begin{aligned}\left\{|\overline{u}^N-\overline{u}^M|>r\right\} &\subset \left\{|\overline{u}^N|>k\right\}\cup\left\{|\overline{u}^M|>k\right\}\\ &\cup \left\{|\overline{u}^N|\leq k,\, |\overline{u}^M|\leq k, |\overline{u}^N-\overline{u}^M|>r\right\}.\end{aligned}$$

Firstly, we have

$$\text{meas}\left\{|\overline{u}^N|>k\right\}\leq \frac{1}{k}\left\|\overline{u}^N\right\|_{L^1(Q_T)}\leq \frac{1}{k}C(T,u_0,f,g).$$

Similarly, we have

$$\text{meas}\left\{|\overline{u}^M|>k\right\}\leq \frac{1}{k}\left\|\overline{u}^N\right\|_{L^1(Q_T)}\leq \frac{1}{k}C(T,u_0,f,g).$$

Therefore, for k large enough, we have

$$\text{meas}(\left\{|\overline{u}^M|>k\right\}\cup\left\{|\overline{u}^M|>k\right\})\leq\frac{\varepsilon}{2}. \tag{5.6}$$

Secondly, by Proposition 2.1, we have

$$\left\|T_k(\overline{u}^N)\right\|_{L^{p(\cdot)}(Q_T)}\leq \max\left\{\left(\int_0^T\int_\Omega |T_k(\overline{u}^N|^{p(x)}dxdt\right)^{\frac{1}{p_-}};\left(\int_0^T\int_\Omega |T_k(\overline{u}^N|^{p(x)}dxdt\right)^{\frac{1}{p_+}}\right\}$$

and also, we have

$$\begin{aligned}\int_0^T\int_\Omega |T_k(\overline{u}^N|^{p(x)}dxdt &\leq \int_0^T\rho_{1,p(x)}(T_k(\overline{u}^N))dt\\ &\leq \sum_{n=1}^N\int_{t_n}^{t_{n-1}}\rho_{1,p(x)}(T_k(U^n))dt\\ &\leq \sum_{n=1}^N\tau\rho_{1,p(x)}(T_k(U^n)).\end{aligned}$$

Therefore, using the stability result 4, it follows

$$\left\|T_k(\overline{u}^N)\right\|_{L^{p(\cdot)}(Q_T)}\leq\left(k^{\frac{1}{p_-}}+k^{\frac{1}{p_+}}\right)\max\left\{C(u_0,p_+,f,g)^{\frac{1}{p_+}},C(u_0,p_+,f,g)^{\frac{1}{p_+}}\right\}. \tag{5.7}$$

Hence the sequences $(T_k(\overline{u}^N))_{N\in\mathbb{N}}$ is bounded in $L^{p(\cdot)}(Q_T)$. Then, there exists a subsequence, still denoted by $(T_k(\overline{u}^N))_{N\in\mathbb{N}}$, that is a Cauchy sequence in $L^{p(\cdot)}(Q_T)$) and in measure. Thus, there exists $N_0\in\mathbb{N}$ such that for all $N, M\geq N_0$, we have

$$\text{meas}(\left\{|\overline{u}^N|\leq k,\, |\overline{u}^M|\leq k, |\overline{u}^N-\overline{u}^M|>r\right\})<\frac{\varepsilon}{2}. \tag{5.8}$$

Then, by (5.6) and (5.8), $(\overline{u}^N)_{N\in\mathbb{N}}$ converges in measure. Therefore there exists an element $u\in\mathcal{M}(Q_T)$ (the set of measure on Q_T) such that

$$\overline{u}^N\to u\ \text{ a.e in}\, Q_T\square$$

Using the same method as above in the proof of (5.7), one show that

$$\left\|\nabla\left(T_k(\overline{u}^N)\right)\right\|_{L^{p(\cdot)}(Q_T)}\leq\left(k^{\frac{1}{p_-}}+k^{\frac{1}{p_+}}\right)\max\left\{C(u_0,p_+,f,g)^{\frac{1}{p_+}},C(u_0,p_+,f,g)^{\frac{1}{p_+}}\right\}, \tag{5.9}$$

which implies that the sequence $(\nabla T_k(\overline{u}^N))_{N\in\mathbb{N}}$ is uniformly bounded in $(L^{p(\cdot)}(Q_T))^d$. Hence there exists a subsequence, still denoted by $\left(\nabla T_k(\overline{u}^N)\right)_{N\in\mathbb{N}}$

$$\left(\nabla T_k(\overline{u}^N)\right)_{N\in\mathbb{N}}\quad\text{converges to an element }\ V\text{ in }L^{p(\cdot)}(Q_T).$$

Since

$$T_k(\overline{u}^N) \quad \text{converges to } \ T_k(u) \text{ in } L^{p(\cdot)}(Q_T)$$

it follows that

$$\nabla T_k(\overline{u}^N) \quad \text{converges to } \ \nabla T_k(u) \text{ in } (L^{p(\cdot)}(Q_T))^d.$$

and by (5.5) we conclude that

$$T_k(u) \in L^{p_-}(0,T;W^{1,p(\cdot)}(\Omega)) \ \text{ for all } \ k>0.$$

Lemma 5.2 *$(\overline{u}^N)_{N\in\mathbb{N}}$ converges a.e. in $\sum_T$.*

Proof. We know that the trace operator is compact from $W^{1,1}(\Omega)$ into $L^1(\partial\Omega)$, then there exists a constant C such that

$$\int_0^T \left\|T_k(\overline{u}^N(t)) - T_k(u(t))\right\|_{L^1(\partial\Omega)} dt \leq C\int_0^T \left\|T_k(\overline{u}^N(t)) - T_k(u(t))\right\|_{W^{1,1}(\Omega)} dt.$$

Therefore,

$$T_k(\overline{u}^N(t)) \to T_k(u) \,\text{in}\, L^1(\sum_T) \,\text{and a.e. on}\, \sum_T.$$

So, there exists $A\subset \sum_T$ such that $T_k(\overline{u}^N(t))$ converges to $T_k(u(t))$ on $\sum_T \setminus A$ with meas$(A)=0$. For every $k>0$, we set

$$A_k = \left\{(t,x)\in \sum_T : |T_k(u(t))| < k\right\}, \quad \text{and } B = \sum_T \setminus \bigcup_{k=1}^{\infty} A_k.$$

We have, by Hölder's inequality

$$\begin{aligned}
\text{meas}\,(B) &= \frac{1}{k}\int_B |T_k\,(u)|\,d\sigma \\
&\leq \frac{1}{k}\int_0^T \|T_k(u)\|_{L^1(\partial\Omega)}\,dt \\
&\leq \frac{1}{k}\int_0^T \|T_k\,(u)\|_{W^{1,1}(\Omega)}\,dt \\
&\leq \frac{1}{k}\int_0^T\int_\Omega (|T_k\,(u)\,| + |\nabla T_k\,(u)\,|) \\
&\leq \frac{1}{k}\left(\frac{1}{p_-}+\frac{1}{p_+}\right)\|1\|_{L^{p'(\cdot)}(Q_T)}\left(\|T_k\,(u)\|_{L^{p(\cdot)}(Q_T)} + \|\nabla T_k\,(u)\|_{(L^{p(\cdot)}(Q_T))^d}\right).
\end{aligned} \tag{5.10}$$

Thanks to (5.7) and (5.9), for all $k>0$, we have

$$\begin{aligned}
\left\|T_k\left(\overline{u}^N\right)\right\|_{L^{p(\cdot)}(Q)} + \left\|\nabla T_k\left(\overline{u}^N\right)\right\|_{(L^{p(\cdot)}(Q))^d} \leq\ & 2\left(k^{\frac{1}{p_-}} + k^{\frac{1}{p_+}}\right) \\
&\times\max\left\{C(u_0,p_+,f,g)^{\frac{1}{p_+}}, C(u_0,p_+,f,g)^{\frac{1}{p_+}}\right\}
\end{aligned} \tag{5.11}$$

We now use the Fatou's lemma in (5.11) to get

$$\begin{aligned}
\|T_k\,(u)\|_{L^{p(\cdot)}(Q)} + \|\nabla T_k\,(u)\|_{(L^{p(\cdot)}(Q))^d} \leq\ & 2\left(k^{\frac{1}{p_-}} + k^{\frac{1}{p_+}}\right) \\
&\times\max\left\{C(u_0,p_+,f,g)^{\frac{1}{p_+}}, C(u_0,p_+,f,g)^{\frac{1}{p_+}}\right\},
\end{aligned}$$

and (5.10) becomes

$$\text{meas}\,(B) \leq 2\left(\frac{1}{k^{1-\frac{1}{p_-}}} + \frac{1}{k^{1-\frac{1}{p_+}}}\right)\max\left\{C(u_0,p_+,f,g)^{\frac{1}{p_+}}, C(u_0,p_+,f,g)^{\frac{1}{p_+}}\right\}. \tag{5.12}$$

Therefore, we get by letting $k \to \infty$ in (5.12) that meas$(B) = 0$.
Let us now define on $\partial\Omega$, the function v by

$$v(t,x) = T_k(u(t))(x) \,\text{if}\, (x,t) \in A_k.$$

We take $(x,t) \in \sum_T \backslash (A \cup B)$; then there exists $k > 0$ such that $(x,t) \in A_k$ and we have

$$\overline{u}^N(t,x) - v(t,x) = (\overline{u}^N(t,x) - T_k(\overline{u}^N(t))(x)) + (T_k(\overline{u}^N(t))(x) - T_k(u(t))(x)).$$

Since $(x,t) \in A_k$, we have $\left|T_k(\overline{u}^N(t))(x)\right| < k$ from which we deduce that $T_k(\overline{u}^N(t))(x) = \overline{u}^N(t,x)$.
Therefore,

$$\overline{u}^N(t,x) - v(t,x) = (T_k(\overline{u}^N(t))(x) - T_k(u(t))(x)) \to 0, \text{ as } N \to \infty.$$

This means that $\left(\overline{u}^N\right)$ converges to v a.e. on $\sum_T$ □

Lemma 5.3 *The sequence $(\overline{u}^N)_{N\in\mathbb{N}}$ converges to u in $C(0,T;L^1(\Omega))$.*

Proof. Let $(t^n = n\tau_N)_{n=1}^N$ and $(t^m = m\tau_M)_{n=1}^M$ be two partitions of the interval $[0,T]$ and let $(u^N(t), \overline{u}^N(t))$, $(u^M(t); \overline{u}^M(t))$ be the semi-discrete solutions defined by (5.1), (5.2) and corresponding to the respective partitions. Let $\varphi \in L^\infty(\Omega) \cap V \cap W^{1,1}(0,T;L^1(\Omega))$. We rewrite 3.1 in the forms

$$\begin{aligned}
&\int_0^t \left\langle \frac{\partial u^N}{\partial s}, T_k(\overline{u}^N - \varphi)\right\rangle ds + \int_0^t\int_\Omega \Phi(\nabla\overline{u}^N - \Theta(\overline{u}^N)).\nabla T_k(\overline{u}^N - \varphi)dxds \\
&+ \int_0^t\int_\Omega |\overline{u}^N|^{p(x)-2}\overline{u}^N T_k(\overline{u}^N - \varphi)dxds + \int_0^t\int_\Omega \alpha(\overline{u}^N)T_k(\overline{u}^N - \varphi)dxds \\
&+ \int_0^t\int_{\partial\Omega} \gamma(\overline{u}^N)T_k(\overline{u}^N - \varphi)d\sigma ds \\
&\le \int_0^t\int_\Omega f_N T_k(\overline{u}^N - \varphi)dxds + \int_0^t\int_{\partial\Omega} g_N T_k(\overline{u}^N - \varphi)dxds
\end{aligned} \tag{5.13}$$

and

$$\begin{aligned}
&\int_0^t \left\langle \frac{\partial u^M}{\partial s}, T_k(\overline{u}^M - \varphi)\right\rangle ds + \int_0^t\int_\Omega \Phi(\nabla\overline{u}^M - \Theta(\overline{u}^M)).\nabla T_k(\overline{u}^M - \varphi)dxds \\
&+ \int_0^t\int_\Omega |\overline{u}^M|^{p(x)-2}\overline{u}^M T_k(\overline{u}^M - \varphi)dxds + \int_0^t\int_\Omega \alpha(\overline{u}^M)T_k(\overline{u}^M - \varphi)dxds \\
&+ \int_0^t\int_{\partial\Omega} \gamma(\overline{u}^M)T_k(\overline{u}^M - \varphi)d\sigma ds \\
&\le \int_0^t\int_\Omega f_M T_k(\overline{u}^M - \varphi)dxds + \int_0^t\int_{\partial\Omega} g_M T_k(\overline{u}^M - \varphi)dxds,
\end{aligned} \tag{5.14}$$

where

$$f_N(t,x) = f_n(x), \quad g_N(t,x) = g_n(x) \quad \forall t \in]t^{n-1}, t^n],$$

$$f_M(t,x) = f_m(x), \quad g_M(t,x) = g_m(x) \quad \forall t \in]t^{m-1}, t^m].$$

Let $h > 1$, in the inequality (5.13) we take $\varphi = T_h(\overline{u}^M)$ and in inequality (5.14) we take $\varphi = T_h(\overline{u}^N)$.

Summing both inequalities, we get, for $k=1$,

$$\begin{aligned}
&\int_0^t \left\langle \frac{\partial(u^N-u^M)}{\partial s}, T_1(u^N-u^M)\right\rangle ds + I_{N,M}(h) + \int_0^t\int_\Omega |\overline{u}^N|^{p(x)-2}\overline{u}^N T_1(\overline{u}^N - T_h(\overline{u}^M))dxds \\
&+\int_0^t\int_\Omega |\overline{u}^M|^{p(x)-2}\overline{u}^M T_1(\overline{u}^M - T_h(\overline{u}^N))dxds + \int_0^t\int_\Omega \alpha(\overline{u}^N)T_1(\overline{u}^N - T_h(\overline{u}^M))dxds \\
&+\int_0^t\int_\Omega \alpha(\overline{u}^M)T_1(\overline{u}^M - T_h(\overline{u}^N))dxds \\
&+\int_0^t\int_{\partial\Omega} [\gamma(\overline{u}^N)T_1(\overline{u}^N - T_h(\overline{u}^M))d + \gamma(\overline{u}^M)T_1(\overline{u}^M - T_h(\overline{u}^N))]d\sigma ds \\
\leq\; & \int_0^t \left\langle \frac{\partial(u^N-u^M)}{\partial s}, T_1(u^N-u^M)\right\rangle - \left\langle \frac{\partial u^N}{\partial s}, T_1(\overline{u}^N - T_h(\overline{u}^M))\right\rangle ds \qquad (5.15)\\
&-\int_0^t \left\langle \frac{\partial u^M}{\partial s}, T_1(\overline{u}^M - T_h(\overline{u}^N))\right\rangle ds \\
&+\int_0^t\int_\Omega [f_N T_1(\overline{u}^N - T_h(\overline{u}^M)) + f_M T_1(\overline{u}^M - T_h(\overline{u}^N))]dxds \\
&+\int_0^t\int_{\partial\Omega} [g_N T_1(\overline{u}^N - T_h(\overline{u}^M)) + g_M T_1(\overline{u}^M - T_h(\overline{u}^N))]d\sigma
\end{aligned}$$

where

$$\begin{aligned}
I_{N,M}(h) &= \int_0^t\int_\Omega \Phi(\nabla\overline{u}^N - \Theta(\overline{u}^N)).\nabla T_1(\overline{u}^N - T_h(\overline{u}^M))dxds \\
&\quad + \int_0^t\int_\Omega \Phi(\nabla\overline{u}^M - \Theta(\overline{u}^M)).\nabla T_1(\overline{u}^M - T_h(\overline{u}^N))dxds.
\end{aligned}$$

We have

$$\begin{aligned}
\left|\int_0^t \left\langle \frac{\partial(u^N-u^M)}{\partial s}, T_1(u^N-u^M)\right\rangle ds\right| &\leq \left\|\frac{\partial(u^N-u^M)}{\partial s}\right\|_{L^1(Q_T)} \left\|T_1(u^N-u^M)\right\|_{L^\infty(Q_T)} \\
&\leq 2C(T,f,g,u_0)\left\|T_1(u^N-u^M)\right\|_{L^\infty(Q_T)}.
\end{aligned}$$

Since

$$\lim_{N,M\to\infty}\left\|T_1(u^N-u^M)\right\|_{L^\infty(Q_T)} = 0,$$

it follows that

$$\lim_{h\to\infty}\lim_{N,M\to\infty}\int_0^t \left\langle \frac{\partial(u^N-u^M)}{\partial s}, T_1(u^N-u^M)\right\rangle ds = 0. \qquad (5.16)$$

Similarly, we show that

$$\lim_{h\to\infty}\lim_{N,M\to\infty}\left(\int_0^t \left\langle \frac{\partial u^N}{\partial s}, T_1(\overline{u}^N - T_h(\overline{u}^M))\right\rangle + \left\langle \frac{\partial u^M}{\partial s}, T_1(\overline{u}^M - T_h(\overline{u}^N))\right\rangle ds\right) = 0,$$

$$\lim_{h\to\infty}\lim_{N,M\to\infty}\int_0^t\int_\Omega [f_N T_1(\overline{u}^N - T_h(\overline{u}^M)) + f_M T_1(\overline{u}^M - T_h(\overline{u}^N))]dxds = 0,$$

$$\lim_{h\to\infty}\lim_{N,M\to\infty}\int_0^t\int_{\partial\Omega} [g_N T_1(\overline{u}^N - T_h(\overline{u}^M)) + g_M T_1(\overline{u}^M - T_h(\overline{u}^N))]d\sigma = 0,$$

and

$$\lim_{h\to\infty}\lim_{N,M\to\infty}\int_0^t\int_\Omega \alpha(\overline{u}^N)T_1(\overline{u}^N - T_h(\overline{u}^M))dxds + \int_0^t\int_\Omega \alpha(\overline{u}^M)T_1(\overline{u}^M - T_h(\overline{u}^N))dxds = 0,$$

$$\lim_{h\to\infty}\lim_{N,M\to\infty}\int_0^t\int_{\partial\Omega}[\gamma(\overline{u}^N)T_1(\overline{u}^N-T_h(\overline{u}^M))d+\gamma(\overline{u}^M)T_1(\overline{u}^M-T_h(\overline{u}^N))]d\sigma ds=0.$$

By the Fatou's lemma one show that $|\overline{u}^N|^{p(.)-2}\overline{u}^N$ converges to $|u|^{p(.)-2}u$ in $L^1(Q_T)$, then by the generalized Lebesgue convergence theorem, we have

$$\lim_{N\to\infty}\int_0^t\int_\Omega|\overline{u}^N|^{p(x)-2}\overline{u}^NT_1(\overline{u}^N-T_h(\overline{u}^M))dxds=\int_0^t\int_\Omega|u|^{p(x)-2}uT_1(u-T_h(\overline{u}^M))dxds$$

and by the Lebesgue convergence theorem we obtain respectively

$$\lim_{M\to\infty}\int_0^t\int_\Omega|u|^{p(x)-2}uT_1(u-T_h(\overline{u}^M))dxds=\int_0^t\int_\Omega|u|^{p(x)-2}uT_1(u-T_h(u))dxds$$

and

$$\lim_{h\to\infty}\int_0^t\int_\Omega|u|^{p(x)-2}uT_1(u-T_h(u))dxds=0$$

that is,

$$\lim_{h\to\infty}\lim_{N,M\to\infty}\int_0^t\int_\Omega|\overline{u}^N|^{p(x)-2}\overline{u}^NT_1(\overline{u}^N-T_h(\overline{u}^M))dxds=0.$$

Similarly, we get

$$\lim_{h\to\infty}\lim_{N,M\to\infty}\int_0^t\int_\Omega|\overline{u}^M|^{p(x)-2}\overline{u}^MT_1(\overline{u}^M-T_h(\overline{u}^N))dxds=0.$$

Then, letting N, $M\to\infty$ and $h\to\infty$, in (5.15)we get

$$\lim_{h\to\infty}\lim_{N,M\to\infty}\int_0^t\left\langle\frac{\partial(u^N-u^M)}{\partial s},T_1(u^N-u^M)\right\rangle ds+\lim_{h\to\infty}\lim_{N,M\to\infty}I_{N,M}(h)\leq 0. \tag{5.17}$$

Since

$$\left\langle\frac{\partial v}{\partial t},T_k(v)\right\rangle=\frac{d}{dt}\int_\Omega J_k(v)\quad\text{in } L^1(]0,T[),$$

inequality (5.17), becomes

$$\lim_{N,M\to\infty}\int_\Omega J_1(u^N(t)-u^M(t))dx+\lim_{h\to\infty}\lim_{N,M\to\infty}I_{N,M}(h)\leq 0. \tag{5.18}$$

Now, we show that

$$\lim_{h\to\infty}\lim_{N,M\to\infty}I_{N,M}(h)\geq 0.$$

We consider the decomposition

$$I_{N,M}(h)=\sum_{i=1}^4L_i(h),$$

where

$$L_i(h)=\int_0^t\int_{\Omega_i(h)}\Phi(\nabla\overline{u}^N-\Theta(\overline{u}^N)).\nabla T_1(\overline{u}^N-T_h(\overline{u}^M))dxds$$
$$+\int_0^t\int_{\Omega_i(h)}\Phi(\nabla\overline{u}^M-\Theta(\overline{u}^M)).\nabla T_1(\overline{u}^M-T_h(\overline{u}^N))dxds$$

and

$$\Omega_1(h)=\left\{|\overline{u}^N|\leq h,\ |\overline{u}^M|\leq h\right\}\qquad\Omega_2(h)=\left\{|\overline{u}^N|\leq h,\ |\overline{u}^M|>h\right\}$$
$$\Omega_3(h)=\left\{|\overline{u}^N|>h,\ |\overline{u}^M|\leq h\right\}\qquad\Omega_4(h)=\left\{|\overline{u}^N|>h,\ |\overline{u}^M|>h\right\}.$$

On the one hand, we have

$$L_1(h) = \int_0^t \int_{\Omega_1^1(h)} [\Phi(\nabla \overline{u}^N - \Theta(\overline{u}^N)) - \Phi(\nabla \overline{u}^M - \Theta(\overline{u}^M))].\nabla(\overline{u}^N - \overline{u}^M) dxds$$

$$= \int_0^t \int_{\Omega_1^1(h)} [\Phi(\nabla \overline{u}^N - \Theta(\overline{u}^N)) - \Phi(\nabla \overline{u}^M - \Theta(\overline{u}^M))].\Psi_\Theta(\overline{u}^N, \overline{u}^M) dxds$$

$$+ \int_0^t \int_{\Omega_1^1(h)} [\Phi(\nabla \overline{u}^N - \Theta(\overline{u}^N)) - \Phi(\nabla \overline{u}^M - \Theta(\overline{u}^M))].(\Theta(\overline{u}^N) - \Theta(\overline{u}^M)) dxds$$

$$\geq \int_0^t \int_{\Omega_1^1(h)} [\Phi(\nabla T_h(\overline{u}^N) - \Theta(T_h(\overline{u}^N))) - \Phi(\nabla T_h(\overline{u}^M) - \Theta(T_h(\overline{u}^M)))].\Lambda_\Theta^h(\overline{u}^N, \overline{u}^M) dxds,$$

where

$$\Psi_\Theta(\overline{u}^N, \overline{u}^M) = \nabla \overline{u}^N - \Theta(\overline{u}^N) - (\nabla \overline{u}^M - \Theta(\overline{u}^M)),$$

and

$$\Lambda_\Theta^h(\overline{u}^N, \overline{u}^M) = \Theta(T_h(\overline{u}^N)) - \Theta(T_h(\overline{u}^M)),$$
$$\Omega_1^1(h) = \left\{ |\overline{u}^N| \leq h, \, |\overline{u}^M| \leq h, |\overline{u}^N - \overline{u}^M| \leq 1 \right\}.$$

Since

$$\Theta(T_h(\overline{u}^N) - \Theta(T_h(\overline{u}^M) \to 0 \quad \text{strongly in } (L^{p(x)}(Q_T))^d$$

and $\Phi(\nabla T_h(\overline{u}^N) - \Theta(T_h(\overline{u}^N))) - \Phi(\nabla T_h(\overline{u}^M) - \Theta(T_h(\overline{u}^M)))$ converges weakly in $(L^{p'(x)}(Q_T))^d$, it follows that the integral

$$\int_0^t \int_{\Omega_1^1(h)} [\Phi(\nabla T_h(\overline{u}^N) - \Theta(T_h(\overline{u}^N))) - \Phi(\nabla T_h(\overline{u}^M) - \Theta(T_h(\overline{u}^M)))].\Lambda_\Theta^h(\overline{u}^N, \overline{u}^M) dxds$$

tends to zero. Therefore

$$\lim_{h\to\infty} \lim_{N,M\to\infty} L_1(h) \geq 0.$$

On the other hand by the hypothesis $(H3)$, we have

$$\begin{aligned} L_2(h) &= \int_0^t \int_{\Omega_2^1(h)} \Phi(\nabla \overline{u}^N - \Theta(\overline{u}^N)).\nabla \overline{u}^N dxds \\ &\quad + \int_0^t \int_{\Omega_2^2(h)} \Phi(\nabla \overline{u}^M - \Theta(\overline{u}^M)).\nabla(\overline{u}^M - \overline{u}^N) dxds \\ &\geq -\int_0^t \int_{\Omega_2^2(h)} \Phi(\nabla \overline{u}^M - \Theta(\overline{u}^M)).\nabla \overline{u}^N dxds, \end{aligned}$$

where

$$\Omega_2^1(h) = \left\{ |\overline{u}^N| \leq h, \, |\overline{u}^M| > h, |\overline{u}^N - h\text{sign}(\overline{u}^M)| \leq 1 \right\}, \qquad \Omega_2^2(h) = \left\{ |\overline{u}^N| \leq h, \, |\overline{u}^M| > h, |\overline{u}^N - \overline{u}^M| \leq 1 \right\}$$

We take $\varphi = T_h(\overline{u}^N)$ in (5.13), to get

$$\lim_{h\to\infty} \lim_{N\to\infty} \int_0^t \int_{\left\{h\leq|\overline{u}^N|\leq h+k\right\}} \Phi(\nabla \overline{u}^N - \Theta(\overline{u}^N)).\nabla \overline{u}^N dxdt = 0.$$

This implies

$$\lim_{h\to\infty} \lim_{N\to\infty} \int_0^t \int_{\left\{h\leq|\overline{u}^N|\leq h+k\right\}} |\nabla \overline{u}^N - \Theta(\overline{u}^N)|^{p(x)} dxdt = 0, \quad k > 0, \tag{5.19}$$

and

$$\lim_{h\to\infty}\lim_{N\to\infty}\int_0^t\int_{\{h\le|\overline{u}^N|\le h+k\}}|\nabla\overline{u}^N|^{p(x)}dxdt=0,\quad k>0. \tag{5.20}$$

Now by Young inequality, we have

$$\begin{aligned}
&\left|\int_0^t\int_{\Omega_2^2(h)}\Phi(\nabla\overline{u}^M-\Theta(\overline{u}^M)).\nabla\overline{u}^N dxdt\right|\\
\le\ &\int_0^t\int_{\Omega_2^2(h)}|\nabla\overline{u}^M-\Theta(\overline{u}^M)|^{p(x)-1}|\nabla\overline{u}^N|dxdt\\
\le\ &\int_0^t\int_{\{h\le|\overline{u}^M|\le h+1\}}\frac{1}{p'(x)}|\nabla\overline{u}^M-\Theta(\overline{u}^M)|^{p(x)}dxdt+\int_0^t\int_{\{h-1\le|\overline{u}^N|\le h\}}\frac{1}{p(x)}|\nabla\overline{u}^M|^{p(x)}dxdt\\
\le\ &\int_0^t\int_{\{h\le|\overline{u}^M|\le h+1\}}\frac{1}{p'_-}|\nabla\overline{u}^M-\Theta(\overline{u}^M)|^{p(x)}dxdt+\int_0^t\int_{\{h-1\le|\overline{u}^N|\le h\}}\frac{1}{p_-}|\nabla\overline{u}^M|^{p(x)}dxdt
\end{aligned}$$

Thus (5.19) and (5.20) give

$$\lim_{N,M\to\infty}\int_0^t\int_{\Omega_2^2(h)}\Phi(\nabla\overline{u}^M-\Theta(\overline{u}^M)).\nabla\overline{u}^N=0,$$

which implies that

$$\lim_{h\to\infty}\lim_{N,M\to\infty}L_2(h)\ge 0.$$

Similarly, we show that

$$\lim_{h\to\infty}\lim_{N,M\to\infty}(L_3(h)+L_4(h))\ge 0.$$

Therefore

$$\lim_{h\to\infty}\lim_{N,M\to\infty}I_{N,M}(h)\ge 0.$$

Thus (5.18), becomes

$$\lim_{N,M\to\infty}\int_\Omega J_1(u^N(t)-u^M(t))dx=0. \tag{5.21}$$

Since

$$\begin{aligned}
&\frac{1}{2}\int_{\{|u^N-u^M|\le 1\}}|u^N(t)-u^M(t)|^2dx+\int_{\{|u^N-u^M|\ge 1\}}|u^N(t)-u^M(t)|dx\\
\le\ &\int_\Omega J_1(u^N(t)-u^M(t))dx;
\end{aligned}$$

we have

$$\int_{\{|u^N-u^M|\ge 1\}}|u^N(t)-u^M(t)|dx \tag{5.22}$$

$$=\int_{\{|u^N-u^M|\le 1\}}|u^N(t)-u^M(t)|dx+\int_{\{|u^N-u^M|\ge 1\}}|u^N(t)-u^M(t)|dx \tag{5.23}$$

$$\le C_\Omega\left(\int_{\{|u^N-u^M|\le 1\}}|u^N(t)-u^M(t)|^2dx\right)^{\frac{1}{2}}+\int_{\{|u^N-u^M|\ge 1\}}|u^N(t)-u^M(t)|dx \tag{5.24}$$

$$\le C_2(\Omega)\left(\int_\Omega J_1(u^N(t)-u^M(t))dx\right)^{\frac{1}{2}}+\int_\Omega J_1(u^N(t)-u^M(t))dx. \tag{5.25}$$

By (5.21) and (5.22) we deduce that $(u^N)_{N\in\mathbb{N}}$ is a Cauchy sequence in $C(0,T;L^1(\Omega))$. Hence $(u^N)_{N\in\mathbb{N}}$ converges to u in $C(0,T;L^1(\Omega))$.

Step 2 : Existence of entropy solution Now, we prove that the limit function u is an entropy solution of the problem (P). Since $u^N(0)=U^0=u_0$ for all $N\in\mathbb{N}$, we have $u(0,.)=u_0$, and inequality (5.13) implies

$$\begin{aligned}
&\int_0^t \left\langle \frac{\partial u^N}{\partial s}, T_k(\overline{u}^N-\varphi)-T_k(u^N-\varphi)\right\rangle ds + \int_0^t\int_\Omega \Phi(\nabla\overline{u}^N-\Theta(\overline{u}^N)).\nabla T_k(\overline{u}^N-\varphi)dxds\\
&+\int_0^t\int_\Omega |\overline{u}^N|^{p(x)-2}\overline{u}^N T_k(\overline{u}^N-\varphi)dxds+\int_0^t\int_\Omega \alpha(\overline{u}^N)T_k(\overline{u}^N-\varphi)dxds\\
&+\int_0^t\int_{\partial\Omega}\gamma(\overline{u}^N)T_k(\overline{u}^N-\varphi)d\sigma ds \qquad (5.26)\\
&\le \int_0^t \left\langle \frac{\varphi}{\partial s}, T_k(u^N-\varphi)-T_k(u^N-\varphi)\right\rangle ds+\int_\Omega J_k(u^N(0)-\varphi(0))dx-\int_\Omega J_k(u^N(t)-\varphi(t))dx\\
&+\int_0^t\int_\Omega f_N T_k(\overline{u}^N-\varphi)dxds+\int_0^t\int_{\partial\Omega} g_N T_k(\overline{u}^N-\varphi)dxds.
\end{aligned}$$

Let $\overline{k}=k+\|\varphi\|_\infty$. Then

$$\begin{aligned}
\int_0^t\int_\Omega \Phi(\nabla\overline{u}^N-\Theta(\overline{u}^N)).\nabla T_k(\overline{u}^N-\varphi)dxds &= \int_0^t\int_\Omega \Phi(\nabla T_{\overline{k}}(\overline{u}^N)-\Theta(T_{\overline{k}}(\overline{u}^N))).\nabla T_k(T_{\overline{k}}(\overline{u}^N)-\varphi)dxds\\
&= \int_0^t\int_\Omega [\Phi(\nabla T_{\overline{k}}(\overline{u}^N)-\Theta(T_{\overline{k}}(\overline{u}^N))).\nabla T_{\overline{k}}(\overline{u}^N)\\
&\quad -\Phi(\nabla T_{\overline{k}}(\overline{u}^N)-\Theta(T_{\overline{k}}(\overline{u}^N))).\nabla\varphi]\mathbf{1}_{Q(N,k)},
\end{aligned}$$

where $Q(N,k)=\left\{|T_{\overline{k}}(\overline{u}^N)-\varphi|\le k\right\}$. Thus, the inequality (5.26) becomes

$$\begin{aligned}
&\int_0^t \left\langle \frac{\partial u^N}{\partial s}, T_k(\overline{u}^N-\varphi)-T_k(u^N-\varphi)\right\rangle ds\\
&-\int_0^t\int_\Omega \Phi(\nabla T_{\overline{k}}(\overline{u}^N)-\Theta(T_{\overline{k}}(\overline{u}^N))).\nabla\varphi\mathbf{1}_{Q(N,k)}\\
&+\int_0^t\int_\Omega [\Phi(\nabla T_{\overline{k}}(\overline{u}^N)-\Theta(T_{\overline{k}}(\overline{u}^N))).\nabla T_{\overline{k}}(\overline{u}^N)+\frac{1}{p(x)}|\Theta(T_{\overline{k}}(\overline{u}^N))|^{p(x)}]\mathbf{1}_{Q(N,k)}\\
&+\int_0^t\int_\Omega |\overline{u}^N|^{p(x)-2}\overline{u}^N T_k(\overline{u}^N-\varphi)dxds+\int_0^t\int_\Omega \alpha(\overline{u}^N)T_k(\overline{u}^N-\varphi)dxds\\
&+\int_0^t\int_{\partial\Omega}\gamma(\overline{u}^N)T_k(\overline{u}^N-\varphi)d\sigma ds\\
&\le -\int_0^t \left\langle \frac{\partial\varphi}{\partial s}, T_k(u^N-\varphi)\right\rangle ds+\int_\Omega J_k(u^N(0)-\varphi(0))dx-\int_\Omega J_k(u^N(t)-\varphi(t))dx \qquad (5.27)\\
&+\int_0^t\int_\Omega f_N T_k(\overline{u}^N-\varphi)dxds+\int_0^t\int_{\partial\Omega} g_N T_k(\overline{u}^N-\varphi)dxds+\int_0^t\int_\Omega \frac{1}{p(x)}|\Theta(T_{\overline{k}}(\overline{u}^N))|^{p(x)}dxdt
\end{aligned}$$

On the one hand, we have

$$\Theta(T_{\overline{k}}(\overline{u}^N))\to\Theta(T_{\overline{k}}(u)) \quad \text{strongly in } (L^{p(.)}(Q_T))^d, \qquad (5.28)$$

$$\nabla T_{\overline{k}}(\overline{u}^N)\to\nabla T_{\overline{k}}(u) \quad \text{strongly in } (L^{p(.)}(Q_T))^d. \qquad (5.29)$$

Thus,

$$\Phi(\nabla T_{\overline{k}}(\overline{u}^N)-\Theta(T_{\overline{k}}(\overline{u}^N)))\to\Phi(\nabla T_{\overline{k}}(u)-\Theta(T_{\overline{k}}(u))) \quad \text{converges strongly in } (L^{p'(.)}(Q_T))^d$$

Now, as $\nabla\varphi\mathbf{1}_{Q(N,k)}$ converges in $L^{p(.)}(Q_T)$ to $\nabla\varphi\mathbf{1}_{Q(k)}$, we get

$$\int_0^t\int_\Omega \Phi(\nabla T_{\overline{k}}(\overline{u}^N)-\Theta(T_{\overline{k}}(\overline{u}^N))).\nabla\varphi\mathbf{1}_{Q(N,k)}\to\int_0^t\int_\Omega \Phi(\nabla T_{\overline{k}}(u)-\Theta(T_{\overline{k}}(u))).\nabla\varphi\mathbf{1}_{Q(k)}$$

where $Q(k)=\left\{|T_{\overline{k}}(u)-\varphi|\leq k\right\}$. We know that

$$\left[\Phi(\nabla T_{\overline{k}}(\overline{u}^N)-\Theta(T_{\overline{k}}(\overline{u}^N))).\nabla T_{\overline{k}}(\overline{u}^N)+\frac{1}{p(x)}|\Theta(T_{\overline{k}}(\overline{u}^N))|^{p(x)}\right]\mathbf{1}_{Q(N,k)}\geq 0.$$

Therefore, by (5.28), (5.29) and Fatou's lemma,

$$\begin{aligned}&\int_0^t\int_\Omega\left[\Phi(\nabla T_{\overline{k}}(u)-\Theta(T_{\overline{k}}(u))).\nabla T_{\overline{k}}(u)+\frac{1}{p(x)}|\Theta(T_{\overline{k}}(u))|^{p(x)}\right]\mathbf{1}_{Q(N,k)}dxds\\ \leq\ &\liminf\int_0^t\int_\Omega\left[\Phi(\nabla T_{\overline{k}}(\overline{u}^N)-\Theta(T_{\overline{k}}(\overline{u}^N))).\nabla T_{\overline{k}}(\overline{u}^N)+\frac{1}{p(x)}|\Theta(T_{\overline{k}}(\overline{u}^N))|^{p(x)}\right]\mathbf{1}_{Q(N,k)}dxds\end{aligned}$$

By hypothesis $(H3)$, we have

$$\frac{1}{p(x)}|\Theta(T_{\overline{k}}(\overline{u}^N))|^{p(x)}\leq(C\overline{k})^{p_+}$$

which implies by (5.28) and the dominated convergence theorem that

$$\int_0^t\int_\Omega\frac{1}{p(x)}|\Theta(T_{\overline{k}}(\overline{u}^N))|^{p(x)}dxds\to\int_0^t\int_\Omega\frac{1}{p(x)}|\Theta(T_{\overline{k}}(u))|^{p(x)}dxds.$$

By Lemma 5.3, we deduce that $u^N(t)\to u(t)$ in $L^1(\Omega)$ for all $t\in[0,T]$, which implies that

$$\int_\Omega J_k(u^N(t)-\varphi(t))dx\to\int_\Omega J_k(u(t)-\varphi(t))dx\quad\forall t\in[0,T].\tag{5.30}$$

We follow the method used in the proof of equality (5.16) to show that

$$\lim_{N\to\infty}\int_0^t\left\langle\frac{\partial u^N}{\partial s},T_k(\overline{u}^N-\varphi)-T_k(u^N-\varphi)\right\rangle ds=0.\tag{5.31}$$

Finally, letting $N\to\infty$ in (5.26) and using the above results and also the continuities of α,γ and the facts that

$$\begin{aligned}f_N\to f\quad&\text{in }L^1(Q_T),\\ g_N\to g\quad&\text{in }L^1(\sum_T),\\ T_{\overline{k}}(\overline{u}^N-\varphi)\to T_{\overline{k}}(u-\varphi)\quad&\text{in }L^\infty(Q_T),\\ T_{\overline{k}}(\overline{u}^N-\varphi)\to T_{\overline{k}}(u-\varphi)\quad&\text{in }L^\infty(\sum_T)\end{aligned}$$

we deduce that u is an entropy solution of the nonlinear parabolic problem $(P)\square$

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Mohamed Badr BENBOUBKER,

Higher School of Technology, Sidi Mohamed Ben Abdellah University,

Fez, Morocco.

E-mail address: simo.ben@hotmail.com

and

Urbain TRAORE,

Laboratoire de Mathématiques et Informatique (LAMI), Université Joseph KI-ZERBO

Ouagadougou, Burkina Faso.

E-mail address: urbain.traore@yahoo.fr