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Existence and Ulam-Hyers Stability Results for a Class of Fractional Integro-Differential Equations Involving Nonlocal Fractional Integro-Differential Boundary Conditions

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ABSTRACT: In this paper, we investigate the existence and uniqueness of solutions for a class of fractional integro-differential boundary value problems involving both Riemann–Liouville and Caputo fractional derivatives, and supplemented with multi-point and nonlocal Riemann-Liouville fractional integral and Caputo fractional derivative boundary conditions. Our results are based on some known tools of fixed point theory. We also study the Ulam–Hyers stability for the proposed fractional problems. Finally, some illustrative examples are included to verify the validity of our results.

Key Words: Ulam-Hyers stability, Riemann-Liouville fractional integral, Caputo fractional derivative, fractional integro-differential equations, existence, fixed point theorem, nonlocal boundary conditions.

Contents

1. Introduction

Fractional differential equations have recently proved to be valuable tools in many fields, such as viscoelasticity, engineering, physics, chemistry, mechanics, and economics, see [30], [36], [34], [37], [25]. In the recent years, there has been a significant development in ordinary and partial differential equations involving fractional derivatives, see the monographs of Kilbas et al. [25], Miller and Ross [32], [38], and the papers [9,27,46,47,26,23,8] and the cited references therein. Integral boundary conditions received much attention due to its applicability in various fields such as population dynamics, blood flow models, chemical engineering, cellular systems, underground water flow, heat transmission, plasma physics, thermoelasticity, etc.

Nowadays, there is a huge increase in investigation of nonlocal conditions for their many successful applications in various physical phenomena and engineering such as thermodynamics, elasticity and wave propagation. Further details can be found in the work by Byszewski [12,13].

Recently, boundary value problems of fractional differential equations involving both Riemann-Liouville fractional integral and Caputo fractional derivative in boundary conditions have received much interest and attention of several researchers. Many authors have studied the existence of solutions of the fractional boundary value problems using various boundary conditions and by different approaches. We refer the readers to the papers [1,2,3,4,24,16,17,18,28,40,41,42,43,44,19,20,21,22].

Very recently, Agarwal et al. [1] studied the following fractional order boundary value problem

$$^{C}D^{q}x(t) = f(t, x(t)),\ 1 < q \leq 2,\ t \in [0, 1],$$

supplemented by boundary conditions, of the form

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$$x(0) = \delta x(\sigma),\ a\ {}^C D^p x(\zeta_1) + b\ {}^C D^p x(\zeta_2) = \sum_{i=1}^{m-2} \alpha_i x(\beta_i),\ 0 < p < 1.$$

Together with the above fractional differential equation they also investigated the boundary conditions

$$x(0) = \delta_1 \int_0^\sigma x(s)ds,\ a\ {}^C D^p x(\zeta_1) + b\ {}^C D^p x(\zeta_2) = \sum_{i=1}^{m-2} \alpha_i x(\beta_i),\ 0 < p < 1,$$

where ${}^C D^q, {}^C D^p$ denote the Caputo fractional derivatives of orders q, p and $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given continuous function and $\delta, \delta_1, a, b, \alpha_i \in \mathbb{R}$, with $0 < \sigma < \zeta_1 < \beta_1 < \beta_2 < ... < \beta_{m-2} < \zeta_2 < 1$.
The existence and uniqueness results were proved via some well known tools of the fixed point theory.

In [4], the authors studied the existence and uniqueness of solutions to the fractional differential equation with four-point nonlocal Riemann-Liouville fractional integral boundary conditions of different order given by

$${}^C D^q x(t) = f(t, x(t)),\ 1 < q \leq 2,\ t \in [0,1],$$

$$x(0) = a \int_0^\eta \frac{(\eta - s)^{\beta-1}}{\Gamma(\beta)} x(s)ds,\ 0 < \beta \leq 1,$$

$$x(1) = b \int_0^\sigma \frac{(\sigma - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds,\ 0 < \alpha \leq 1,$$

where ${}^C D^q$ is the Caputo fractional derivative of order q, f is a given continuous function, and a, b, η, σ are real constants with $0 < \eta, \sigma < 1$. The main results are based on standard tools of fixed point theory and Leray-Schauder nonlinear alternative.

Bashir et al. in [3] discussed the existence and uniqueness of solutions of a new class of fractional boundary value problems

$${}^C D^q x(t) = f(t, x(t)),\ t \in [0,1],\ q \in (1,2],$$

$$x(0) = 0,\ x(\xi) = a \int_\eta^1 x(s)ds,$$

where ${}^C D^q$ denotes the Caputo fractional derivative of order q, f is a given continuous function, and a is a positive real constant, $\xi \in (0,1)$ with $\xi < \eta < 1$. The existence results are obtained with the aid of some classical fixed point theorems.

In [41], Sudsutad and Tariboon studied the existence and uniqueness of solutions for a boundary value problem of fractional order differential equation with three-point fractional integral boundary conditions given by

$${}^C D^q x(t) = f(t, x(t)),\ t \in [0,1],\ q \in (1,2],$$

$$x(0) = 0,\ x(1) = \alpha \int_0^\eta \frac{(\eta - s)^{p-1}}{\Gamma(p)} x(s)ds,\ 0 < \eta < 1,\ p > 0,$$

where ${}^C D^q$ denotes the Caputo fractional derivative of order q, $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function and $\alpha \in \mathbb{R}$ is such that $\alpha \neq\ \Gamma(p+2)/\eta^{p+1}$.

Tariboon et al. [43] have also studied the following fractional boundary value problem with three-point nonlocal Riemann-Liouville integral boundary conditions

$$D^{\alpha}x(t) = f(t, x(t)),\ 0 < t < T,\ \alpha \in (1, 2],$$

$$x(\eta) = 0,\ I^{\nu}x(T) = \int_0^T \frac{(T-s)^{\nu-1}}{\Gamma(\nu)} x(s)ds = 0,$$

where D^{α} denotes the Riemann-Liouville fractional derivative of order $\alpha > 0$, $\eta \in (0, T)$ is a given constant. The existence and uniqueness results were proved via the Banach contraction principle, the Banach's fixed point theorem and Hölder's inequality, the Krasnoselskii fixed point theorem and the Leray-Schauder nonlinear alternative.

More recently, in [35], both Riemann-Liouville and Caputo fractional derivatives were considered in the boundary values problem

$$^{RL}D^{q}\left(^{C}D^{r}x(t)\right) = f(t, x(t)),\ t \in (0, T),$$

$$x'(\xi) = \lambda^{C}D^{\nu}x(\eta),\ u(T) = \mu[I^{p}x](\zeta),$$

where $^{RL}D^{q}$ denotes the Riemann–Liouville fractional derivative of order q, $0 < q < 1$, $^{C}D^{r}$, $^{C}D^{\nu}$ denote the Caputo fractional derivatives of orders r and ν respectively, $0 < r < 1$, $0 < \nu < q + r$, I^{p} is the Riemann–Liouville fractional integral of order $p > 0$, $f : [0, T]\times\mathbb{R} \to \mathbb{R}$ and $\lambda, \mu \in \mathbb{R}$, $\xi, \eta, \zeta \in (0, T)$. Some existence and uniqueness results are established via Banach contraction mapping principle, Krasnoselskii fixed-point theorem and nonlinear alternative of Leray–Schauder type.

Inspired and motivated by the recent paper [35], in this work, we introduce a new class of boundary value problems of integro-fractional differential equations supplemented with nonlocal Riemann-Liouville fractional integral and Caputo fractional derivative boundary conditions. In precise terms, we consider the following nonlocal problem:

$$^{RL}D^{q}\left(^{C}D^{r}x(t) + \lambda\right) = f(t, x(t), (Hx)(t)),\ t \in [0, T], \tag{1.1}$$

$$x'(\xi) = \sum_{i=1}^{m} \alpha_i {}^{C}D^{\nu}x(\eta_i),\ x(T) = \sum_{i=1}^{m} \beta_i [I^{p}x](\sigma_i), \tag{1.2}$$

where $^{c}D^{\mu}$ is the Caputo fractional derivative of order $\mu \in \{r, \nu\}$ such that $0 < q \le 1$, $0 < r \le 1$ $0 < \nu \le 1 < q + r$, I^{p} is the Riemann-Liouville fractional integral of order $p > 0$, $\xi, \eta_i, \sigma_i \in (0, T)$ and λ, α_i, β_i, $i = 1, 2$ are appropriate real constants. $f : [0, T] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a given continuous function, and

$$(Hx)(t) = \int_0^t \psi(t, s)x(s)ds,$$

where $\psi : [0, T] \times [0, T] \to [0, \infty)$ is a continuous function.

Integro-differential equations appear in a variety of different areas of applied mathematics and physics, often as approximation to partial differential equations. Some of the applications are unsteady aerodynamics and aero elastic phenomena, viscoelasticity, fluid dynamics, theory of population dynamics, etc. For more details, see [5,29,31,33] and the references therein.

Many of the physical systems can be described by integral boundary conditions, for example in population dynamics, chemical engineering, thermoelasticity, blood flow problems, thermoelasticity, underground water flow, and so forth. A detailed description of the integral boundary conditions can be found, for instance, in [6]. For more details of boundary value problems with nonlocal and integral boundary conditions, see [7,10,11,14,45] and references therein.

The main contribution of this paper is the extension of some results found in [35] to a more general type of integro-fractional differential equations.

The paper is organized as follows. In Section 2, we present some backgrounds on fractional calculus and the fixed point theorems. Also, we give an useful auxiliary lemma. Section 3 deals with the existence result for problem (1.1)-(1.2) which is obtained via the Banach contraction principle, Schaefer's fixed point Theorem, and Krasnoselskii's fixed point Theorem. Besides, we discuss the Ulam–Hyers stability criteria for the main fractional problem. Finally, some examples are given to illustrate the viability of the main results

2. Preliminaries

In this section, we recall some basic definitions of fractional calculus and an auxiliary lemma to define the solution for the problem (1.1)-(1.2) is presented.

Definition 2.1. *The Riemann-Liouville fractional integral of order q for a continuous function f is defined as*

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{f(s)}{(t-s)^{1-q}} ds, \ \ q > 0,$$

provided the integral exists, where $\Gamma(.)$ is the gamma function, which is defined by $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$.

Lemma 2.2. *Let $q, p > 0$, then $I^q I^p f(t) = I^{q+p} f(t)$.*

Definition 2.3. *For at least n-times continuously differentiable function $f : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as*

$$^cD^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{q+1-n}} ds, \ \ n-1 < q < n, \ \ n = [q] + 1,$$

where $[q]$ denotes the integer part of the real number q, provided the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.4. *The Riemann–Liouville fractional derivative of order q for a function $f : (0, \infty) \to \mathbb{R}$ is defined by*

$$^{RL}D^q f(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-q-1} f(s) ds, \ \ q > 0, \ \ n = [q] + 1,$$

provided the right-hand side is pointwise defined on $(0, \infty)$.

Lemma 2.5 ([38], [25])**.** *If $\beta > \alpha > 0$ and $x \in L_1[0, 1]$, then*

(i) $^cD^\alpha I^\beta x(t) = I^{\beta-\alpha} x(t)$, holds almost everywhere on $[0, 1]$ and it is valid at any point $t \in [0, 1]$ if $x \in C[0, 1]$; $^cD^\alpha I^\alpha x(t) = x(t)$, for all $t \in [0, 1]$.

(ii) $^cD^\alpha t^{\lambda-1} = \frac{\Gamma(\lambda)}{\Gamma(\lambda-\alpha)} t^{\lambda-\alpha-1}$, $\lambda > [\alpha]$ and $^cD^\alpha t^{\lambda-1} = 0$, $\lambda < [\alpha]$.

(iii) $I^\alpha t^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} t^{\beta+\alpha-1}$.

Lemma 2.6 ([25])**.** *For $q > 0$, the general solution of the fractional differential equation $^cD^q x(t) = 0$ is given by*

$$x(t) = c_0 + c_1 t + ... + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0, 1, ..., n-1$ $(n = [q] + 1)$.

According to Lemma 2.6, it follows that

$$I^q \ {}^cD^q x(t) = x(t) + c_0 + c_1 t + ... + c_{n-1} t^{n-1},$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, ..., n-1$ $(n = [q] + 1)$.

Lemma 2.7 ([25]). *Let $q>0$. Then, for $x\in C(0,T)\cap L(0,T)$, the following formula holds:*

$$I^q\big({}^{RL}D^q x\big)(t)=x(t)+c_1t^{q-1}+c_2t^{q-2}+...+c_nt^{q-n},$$

where $c_i\in\mathbb{R}$, $i=1,...,n$ $(n=[q]+1)$.

Next, we define Ulam-Hyers and generalized Ulam-Hyers stability of the problem (1.1)-(1.2).

Definition 2.8. *The fractional differential equation (1.1)-(1.2) is Ulam-Hyers stable if there exists $c_f>0$ such that for each $\epsilon>0$ and for each solution $y\in C([0,T],\mathbb{R})$ of the inequality*

$$\Big|{}^{RL}D^q\big({}^CD^r y(t)+\lambda\big)-f(t,y(t),(Hy)(t))\Big|\le\epsilon,\ \ t\in[0,T], \tag{2.1}$$

there exists a solution x of problem (1.1)-(1.2) with

$$\big|y(t)-x(t)\big|\le c_f\epsilon,\ \ t\in[0,T].$$

Definition 2.9. *The fractional differential equation (1.1)-(1.2) is called generalized Ulam-Hyers stable if there exists $\varphi_f\in C(\mathbb{R}_+,\mathbb{R}_+)$ with $\varphi_f(0)=0$ such that for each $\epsilon>0$ and for each solution $y\in C([0,T],\mathbb{R})$ of the inequality (2.1), there exists a solution x of problem (1.1)-(1.2) for which*

$$\big|y(t)-x(t)\big|\le\varphi_f(\epsilon),\ \ t\in[0,T].$$

Remark 2.10. *A function $y\in C([0,T],\mathbb{R})$ is a solution of (2.1) if and only if there exists a function $z\in C([0,T],\mathbb{R})$ (which depends on y) such that*

(i) $|z(t)|\le\epsilon,\ t\in[0,T]$,

(ii) ${}^{RL}D^q\big({}^CD^r y(t)+\lambda\big)=f(t,y(t),(Hy)(t))+z(t),\ t\in[0,T]$.

Lemma 2.11. *Let $\Delta_1\neq0$ and $\Delta_2\neq0$. Then, for any $h\in C([0,T],\mathbb{R})$, the linear fractional boundary value problem*

$${}^{RL}D^q\big({}^CD^r x(t)+\lambda\big)=h(t),\ \ t\in[0,T], \tag{2.2}$$

$$x'(\xi)=\sum_{i=1}^m\alpha_i{}^CD^\nu x(\eta_i),\ \ x(T)=\sum_{i=1}^m\beta_i[I^px](\sigma_i), \tag{2.3}$$

has an integral solution given by

$$\begin{aligned} x(t) &= I^{q+r}h(t)+\frac{1}{\Delta_1}\left(\sum_{i=1}^m\alpha_iI^{q+r-\nu}h(\eta_i)-I^{q+r-1}h(\xi)\right)\left(\frac{\Gamma(q)}{\Gamma(q+r)}t^{q+r-1}+\frac{\Delta_3}{\Delta_2}\right)\\ &\quad+\frac{1}{\Delta_2}\left[\sum_{i=1}^m\beta_iI^{q+r+p}h(\sigma_i)-I^{q+r}h(T)\right]+\lambda\left[\frac{1}{\Delta_1}\left(\frac{\Gamma(q)}{\Gamma(q+r)}t^{q+r-1}+\frac{\Delta_3}{\Delta_2}\right)\right.\\ &\quad\times\left(\frac{\xi^{r-1}}{\Gamma(r)}-\frac{1}{\Gamma(r-\nu+1)}\sum_{i=1}^m\alpha_i\eta_i^{r-\nu}\right)+\frac{1}{\Delta_2}\left(\frac{T^r}{\Gamma(r+1)}-\frac{1}{\Gamma(p+r+1)}\sum_{i=1}^m\beta_i\sigma_i^{p+r}\right)\\ &\quad\left.-\frac{t^r}{\Gamma(r+1)}\right], \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} \Delta_1&=\frac{\Gamma(q)}{\Gamma(q+r-1)}\xi^{q+r-2}-\frac{\Gamma(q)}{\Gamma(q+r-\nu)}\sum_{i=1}^m\alpha_i\eta_i^{q+r-\nu-1},\\ \Delta_2&=1-\frac{1}{\Gamma(p+1)}\sum_{i=1}^m\beta_i\sigma_i^p,\\ \Delta_3&=\frac{\Gamma(q)}{\Gamma(q+r+p)}\sum_{i=1}^m\beta_i\sigma_i^{q+r+p-1}-\frac{\Gamma(q)}{\Gamma(q+r)}T^{q+r-1}. \end{aligned} \tag{2.5}$$

Proof. Applying the Riemann–Liouville fractional integral of order q to both sides of equation in (2.2), and in view of Lemma 2.7, we get

$$^{C}D^{r}x(t)+\lambda=I^{q}h(t)+c_{1}t^{q-1}, \tag{2.6}$$

where $c_1\in\mathbb{R}$ is arbitrary constant. Now, in view of Lemmas 2.5-2.6, by taking the Riemann–Liouville fractional integral of order r to both sides of equation in (2.6), we get

$$x(t)=I^{q+r}h(t)+c_{1}\frac{\Gamma(q)}{\Gamma(q+r)}t^{q+r-1}+c_{2}-\lambda\frac{t^{r}}{\Gamma(r+1)}. \tag{2.7}$$

From (2.7), we have

$$x'(t)=I^{q+r-1}h(t)+c_{1}\frac{\Gamma(q)}{\Gamma(q+r-1)}t^{q+r-2}-\lambda\frac{t^{r-1}}{\Gamma(r)}, \tag{2.8}$$

$$^{C}D^{\nu}x(t)=I^{q+r-\nu}h(t)+c_{1}\frac{\Gamma(q)}{\Gamma(q+r-\nu)}t^{q+r-\nu-1}-\lambda\frac{t^{r-\nu}}{\Gamma(r-\nu+1)}, \tag{2.9}$$

$$I^{p}x(t)=I^{q+r+p}h(t)+c_{1}\frac{\Gamma(q)}{\Gamma(q+r+p)}t^{q+r+p-1}+c_{2}\frac{t^{p}}{\Gamma(p+1)}-\lambda\frac{t^{p+r}}{\Gamma(p+r+1)}. \tag{2.10}$$

From (2.8) and (2.9), the boundary condition $x'(\xi)=\sum_{i=1}^{m}\alpha_{i}{}^{C}D^{\nu}x(\eta_{i})$ implies that

$$\begin{aligned} I^{q+r-1}h(\xi)+c_{1}\frac{\Gamma(q)}{\Gamma(q+r-1)}\xi^{q+r-2}-\lambda\frac{\xi^{r-1}}{\Gamma(r)} &= \sum_{i=1}^{m}\alpha_{i}\Bigg(I^{q+r-\nu}h(\eta_{i})+c_{1}\frac{\Gamma(q)}{\Gamma(q+r-\nu)}\eta_{i}^{q+r-\nu-1} \\ &\quad -\lambda\frac{\eta_{i}^{r-\nu}}{\Gamma(r-\nu+1)}\Bigg), \end{aligned}$$

which, on solving yields

$$c_{1} = \frac{1}{\Delta_{1}}\left(\sum_{i=1}^{m}\alpha_{i}I^{q+r-\nu}h(\eta_{i})-I^{q+r-1}h(\xi)+\lambda\left(\frac{\xi^{r-1}}{\Gamma(r)}-\sum_{i=1}^{m}\alpha_{i}\frac{\eta_{i}^{r-\nu}}{\Gamma(r-\nu+1)}\right)\right).$$

Using the boundary condition $x(T)=\sum_{i=1}^{m}\beta_{i}[I^{p}x](\sigma_{i})$, we find that

$$\begin{aligned} c_{2} &= \frac{1}{\Delta_{2}}\left(\sum_{i=1}^{m}\beta_{i}I^{q+r+p}h(\sigma_{i})-I^{q+r}h(T)+\lambda\left(\frac{T^{r}}{\Gamma(r+1)}-\sum_{i=1}^{m}\beta_{i}\frac{\sigma_{i}^{p+r}}{\Gamma(p+r+1)}\right)\right) \\ &\quad +\frac{c_{1}}{\Delta_{2}}\left(\sum_{i=1}^{m}\beta_{i}\frac{\Gamma(q)}{\Gamma(q+r+p)}\sigma_{i}^{q+r+p-1}-\frac{\Gamma(q)}{\Gamma(q+r)}T^{q+r-1}\right). \end{aligned}$$

Inserting the value of c_1 and c_2 in (2.7), we obtain

$$\begin{aligned}
x(t) &= I^{q+r}h(t) + c_1\frac{\Gamma(q)}{\Gamma(q+r)}t^{q+r-1} + c_2 - \lambda\frac{t^r}{\Gamma(r+1)} \\
&= I^{q+r}h(t) + c_1\frac{\Gamma(q)}{\Gamma(q+r)}t^{q+r-1} + \frac{c_1}{\Delta_2}\left(\sum_{i=1}^{m}\beta_i\frac{\Gamma(q)}{\Gamma(q+r+p)}\sigma_i^{q+r+p-1} - \frac{\Gamma(q)}{\Gamma(q+r)}T^{q+r-1}\right) \\
&\quad + \frac{1}{\Delta_2}\left(\sum_{i=1}^{m}\beta_i I^{q+r+p}h(\sigma_i) - I^{q+r}h(T) + \lambda\left(\frac{T^r}{\Gamma(r+1)} - \sum_{i=1}^{m}\beta_i\frac{\sigma_i^{p+r}}{\Gamma(p+r+1)}\right)\right) \\
&\quad -\lambda\frac{t^r}{\Gamma(r+1)} \\
&= I^{q+r}h(t) + \frac{1}{\Delta_1}\left(\sum_{i=1}^{m}\alpha_i I^{q+r-\nu}h(\eta_i) - I^{q+r-1}h(\xi)\right)\left(\frac{\Gamma(q)}{\Gamma(q+r)}t^{q+r-1} + \frac{\Delta_3}{\Delta_2}\right) \\
&\quad + \frac{1}{\Delta_2}\left[\sum_{i=1}^{m}\beta_i I^{q+r+p}h(\sigma_i) - I^{q+r}h(T)\right] + \lambda\left[\frac{1}{\Delta_1}\left(\frac{\Gamma(q)}{\Gamma(q+r)}t^{q+r-1} + \frac{\Delta_3}{\Delta_2}\right)\right. \\
&\quad \times\left(\frac{\xi^{r-1}}{\Gamma(r)} - \frac{1}{\Gamma(r-\nu+1)}\sum_{i=1}^{m}\alpha_i\eta_i^{r-\nu}\right) + \frac{1}{\Delta_2}\left(\frac{T^r}{\Gamma(r+1)} - \frac{1}{\Gamma(p+r+1)}\sum_{i=1}^{m}\beta_i\sigma_i^{p+r}\right) \\
&\quad \left. -\frac{t^r}{\Gamma(r+1)}\right].
\end{aligned}$$

The converse of the Lemma follows by direct computation. This completes the proof. □

3. Existence results

In this section, we establish sufficient conditions for the existence of solutions to the fractional integro-differential boundary value problem (1.1)-(1.2) using certain fixed point theorems.

Let $\mathcal{C}$ be the Banach space of all continuous functions from $[0,T]$ into $\mathbb{R}$ equipped with the norm: $\|x\| = \sup\{|x(t)|, t \in [0,T]\}$. We define the operator $\mathcal{A} : \mathcal{C} \to \mathcal{C}$ by

$$\begin{aligned}
(\mathcal{A}x)(t) &= I^{q+r}f(t,x(t),(Hx)(t)) + \frac{1}{\Delta_1}\left(\sum_{i=1}^{m}\alpha_i I^{q+r-\nu}f(\eta_i,x(\eta_i),(Hx)(\eta_i))\right. \\
&\quad \left. -I^{q+r-1}f(\xi,x(\xi),(Hx)(\xi))\right)\left(\frac{\Gamma(q)}{\Gamma(q+r)}t^{q+r-1} + \frac{\Delta_3}{\Delta_2}\right) \\
&\quad + \frac{1}{\Delta_2}\left[\sum_{i=1}^{m}\beta_i I^{q+r+p}f(\sigma_i,x(\sigma_i),(Hx)(\sigma_i)) - I^{q+r}f(T,x(T),(Hx)(T))\right] \\
&\quad +\lambda\left[\frac{1}{\Delta_1}\left(\frac{\Gamma(q)}{\Gamma(q+r)}t^{q+r-1} + \frac{\Delta_3}{\Delta_2}\right)\left(\frac{\xi^{r-1}}{\Gamma(r)} - \frac{1}{\Gamma(r-\nu+1)}\sum_{i=1}^{m}\alpha_i\eta_i^{r-\nu}\right)\right. \\
&\quad \left. + \frac{1}{\Delta_2}\left(\frac{T^r}{\Gamma(r+1)} - \frac{1}{\Gamma(p+r+1)}\sum_{i=1}^{m}\beta_i\sigma_i^{p+r}\right) - \frac{t^r}{\Gamma(r+1)}\right]. \qquad (3.1)
\end{aligned}$$

Obviously, the fixed points of the operator $\mathcal{A}$ are the solutions of the fractional integro-differential boundary value problem (1.1)-(1.2).

In order to prove our main results, the following well known fixed point theorems are needed.

Theorem 3.1 ([15]). *Let (X,d) be a complete metric space and $T : X \to X$ be a contraction. Then T has a unique fixed point in X.*

Theorem 3.2 ([39]). *Let X be a Banach space. Assume that $F : X \to X$ is a completely continuous operator and the set $V = \{x \in X : x = \gamma Fx,\ 0 < \gamma < 1\}$ is bounded. Then F has a fixed point in X.*

Theorem 3.3 ([39]). *Let E be a closed convex, bounded and nonempty subset of a Banach space X. Let A, B be the operators such that*

(1) $Ax + By \in E$, for any $x, y \in E$;

(2) A is a completely continuous operator;

(3) B is a contraction operator.

Then there exists at least one fixed point $z \in E$ such that $z = Az + Bz$.

In the following, for computational convenience, we set

$$\begin{aligned}
\Omega &= \frac{T^{q+r}}{\Gamma(q+r+1)} + \frac{1}{|\Delta_1|}\left(\sum_{i=1}^{m}|\alpha_i|\frac{\eta_i^{q+r-\nu}}{\Gamma(q+r-\nu+1)} + \frac{\xi^{q+r-1}}{\Gamma(q+r)}\right)\left(\frac{\Gamma(q)}{\Gamma(q+r)}T^{q+r-1} + \left|\frac{\Delta_3}{\Delta_2}\right|\right)\\
&\quad + \frac{1}{|\Delta_2|}\left(\sum_{i=1}^{m}|\beta_i|\frac{\sigma_i^{q+r+p}}{\Gamma(q+r+p+1)} + \frac{T^{q+r}}{\Gamma(q+r+1)}\right),\\
\Lambda &= |\lambda|\left[\frac{1}{|\Delta_1|}\left(\frac{\Gamma(q)}{\Gamma(q+r)}T^{q+r-1} + \left|\frac{\Delta_3}{\Delta_2}\right|\right)\left(\frac{\xi^{r-1}}{\Gamma(r)} + \frac{1}{\Gamma(r-\nu+1)}\sum_{i=1}^{m}|\alpha_i|\eta_i^{r-\nu}\right) + \frac{1}{|\Delta_2|}\right.\\
&\quad \left.\times\left(\frac{T^{r}}{\Gamma(r+1)} + \frac{1}{\Gamma(p+r+1)}\sum_{i=1}^{m}|\beta_i|\sigma_i^{p+r}\right) + \frac{T^{r}}{\Gamma(r+1)}\right]. \qquad (3.2)
\end{aligned}$$

Now, we are in a position to present the main results of this paper. The first one existence result is based on Banach's contraction mapping principle (Theorem 3.1).

Theorem 3.4. *Assume that $\psi : [0,T] \times [0,T] \to [0,\infty)$ is continuous, and let $f : [0,T] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a continuous function with $f(t,0,0) \not\equiv 0$ on $[0,T]$ and satisfying the Lipschitz condition:*
(H_1) $|f(t,x_1,y_1) - f(t,x_2,y_2)| \le L_1|x_1 - x_2| + L_2|y_1 - y_2|,\ \ L_1, L_2 > 0,$ *for all* $t \in [0,T],\ x_i, y_i \in \mathbb{R},\ i = 1,2.$
Then the problem (1.1)-(1.2) has a unique solution on $[0,T]$ provided that

$$\left(L_1 + \overline{\psi}L_2T\right)\Omega < 1,$$

where $\overline{\psi} = \max\{\psi(t,s) : (t,s) \in [0,T] \times [0,T]\}$, Ω is given by (3.2).

Proof. Setting $\sup\{|f(t,0,0)|, t \in [0,T]\} = M < \infty$ and define $B_\rho = \{x \in \mathcal{C} : \|x\| \le \rho\}$, where

$$\rho \ge \frac{M\Omega + \Lambda}{1 - \Omega\left(L_1 + \overline{\psi}L_2T\right)},$$

where Ω, Λ are given by (3.2). As a first step, we show that $\mathcal{A}(B_\rho) \subset B_\rho$. From (H_1), for $x \in B_\rho$, and $t \in [0,T]$, we get

$$\begin{aligned}
|f(t,x(t),(Hx)(t))| &\le |f(t,x(t),(Hx)(t)) - f(t,0,0)| + |f(t,0,0)|\\
&\le L_1|x(t)| + L_2|(Hx)(t)| + M\\
&\le \left(L_1 + \overline{\psi}L_2T\right)\|x\| + M\\
&\le \left(L_1 + \overline{\psi}L_2T\right)\rho + M.
\end{aligned} \qquad (3.3)$$

Using (3.1) and (3.3), we obtain

$$
\begin{aligned}
\|\mathcal{A}x\| &\leq \sup_{t\in[0,T]} \Bigg\{ I^{q+r}|f(t,x(t),(Hx)(t))| + \frac{1}{|\Delta_1|}\Bigg(\sum_{i=1}^{m} |\alpha_i| I^{q+r-\nu}|f(\eta_i,x(\eta_i),(Hx)(\eta_i))| \\
&\quad + I^{q+r-1}|f(\xi,x(\xi),(Hx)(\xi))|\Bigg)\Bigg(\frac{\Gamma(q)}{\Gamma(q+r)}t^{q+r-1} + \left|\frac{\Delta_3}{\Delta_2}\right|\Bigg) \\
&\quad + \frac{1}{|\Delta_2|}\left[\sum_{i=1}^{m}|\beta_i| I^{q+r+p}|f(\sigma_i,x(\sigma_i),(Hx)(\sigma_i))| + I^{q+r}|f(T,x(T),(Hx)(T))|\right]\Bigg\} \\
&\quad + |\lambda|\Bigg[\frac{1}{|\Delta_1|}\Bigg(\frac{\Gamma(q)}{\Gamma(q+r)}T^{q+r-1} + \left|\frac{\Delta_3}{\Delta_2}\right|\Bigg)\Bigg(\frac{\xi^{r-1}}{\Gamma(r)} + \frac{1}{\Gamma(r-\nu+1)}\sum_{i=1}^{m}|\alpha_i|\eta_i^{r-\nu}\Bigg) \\
&\quad + \frac{1}{|\Delta_2|}\Bigg(\frac{T^r}{\Gamma(r+1)} + \frac{1}{\Gamma(p+r+1)}\sum_{i=1}^{m}|\beta_i|\sigma_i^{p+r}\Bigg) + \frac{T^r}{\Gamma(r+1)}\Bigg] \\
&\leq \Big((L_1+\overline{\psi}L_2T)\rho + M\Big)\sup_{t\in[0,1]}\Bigg\{\int_0^t \frac{(t-s)^{q+r-1}}{\Gamma(q+r)}ds + \frac{1}{|\Delta_1|}\Bigg(\sum_{i=1}^{m}|\alpha_i|\int_0^{\eta_i}\frac{(\eta_i-s)^{q+r-\nu-1}}{\Gamma(q+r-\nu)}ds \\
&\quad + \int_0^{\xi}\frac{(\xi-s)^{q+r-2}}{\Gamma(q+r-1)}ds\Bigg)\Bigg(\frac{\Gamma(q)}{\Gamma(q+r)}t^{q+r-1} + \left|\frac{\Delta_3}{\Delta_2}\right|\Bigg) \\
&\quad + \frac{1}{|\Delta_2|}\left[\sum_{i=1}^{m}|\beta_i|\int_0^{\sigma_i}\frac{(\sigma_i-s)^{q+r+p-1}}{\Gamma(q+r+p)}ds + \int_0^T\frac{(T-s)^{q+r-1}}{\Gamma(q+r)}ds\right]\Bigg\} \\
&\quad + |\lambda|\Bigg[\frac{1}{|\Delta_1|}\Bigg(\frac{\Gamma(q)}{\Gamma(q+r)}T^{q+r-1} + \left|\frac{\Delta_3}{\Delta_2}\right|\Bigg)\Bigg(\frac{\xi^{r-1}}{\Gamma(r)} + \frac{1}{\Gamma(r-\nu+1)}\sum_{i=1}^{m}|\alpha_i|\eta_i^{r-\nu}\Bigg) \\
&\quad + \frac{1}{|\Delta_2|}\Bigg(\frac{T^r}{\Gamma(r+1)} + \frac{1}{\Gamma(p+r+1)}\sum_{i=1}^{m}|\beta_i|\sigma_i^{p+r}\Bigg) + \frac{T^r}{\Gamma(r+1)}\Bigg] \\
&\leq \Big((L_1+\overline{\psi}L_2T)\rho + M\Big)\Bigg\{\frac{T^{q+r}}{\Gamma(q+r+1)} + \frac{1}{|\Delta_1|}\Bigg(\sum_{i=1}^{m}|\alpha_i|\frac{\eta_i^{q+r-\nu}}{\Gamma(q+r-\nu+1)} + \frac{\xi^{q+r-1}}{\Gamma(q+r)}\Bigg) \\
&\quad \times\Bigg(\frac{\Gamma(q)}{\Gamma(q+r)}T^{q+r-1} + \left|\frac{\Delta_3}{\Delta_2}\right|\Bigg) + \frac{1}{|\Delta_2|}\Bigg(\sum_{i=1}^{m}|\beta_i|\frac{\sigma_i^{q+r+p}}{\Gamma(q+r+p+1)} + \frac{T^{q+r}}{\Gamma(q+r+1)}\Bigg)\Bigg\} \\
&\quad + |\lambda|\Bigg[\frac{1}{|\Delta_1|}\Bigg(\frac{\Gamma(q)}{\Gamma(q+r)}T^{q+r-1} + \left|\frac{\Delta_3}{\Delta_2}\right|\Bigg)\Bigg(\frac{\xi^{r-1}}{\Gamma(r)} + \frac{1}{\Gamma(r-\nu+1)}\sum_{i=1}^{m}|\alpha_i|\eta_i^{r-\nu}\Bigg) \\
&\quad + \frac{1}{|\Delta_2|}\Bigg(\frac{T^r}{\Gamma(r+1)} + \frac{1}{\Gamma(p+r+1)}\sum_{i=1}^{m}|\beta_i|\sigma_i^{p+r}\Bigg) + \frac{T^r}{\Gamma(r+1)}\Bigg] \\
&= \Big((L_1+\overline{\psi}L_2T)\rho + M\Big)\Omega + \Lambda \leq \rho.
\end{aligned}
$$

Thus, $\mathcal{A}(B_\rho) \subset B_\rho$. Now, for $x, y \in \mathcal{C}$, and for each $t \in [0,T]$, we have

$$
\begin{aligned}
\|(\mathcal{A}x) - (\mathcal{A}y)\| &\leq \sup_{t\in[0,1]}\Bigg\{\int_0^t \frac{(t-s)^{q+r-1}}{\Gamma(q+r)}\big|f(s,x(s),(Hx)(s)) - f(s,y(s),(Hy)(s))\big|ds \\
&\quad + \frac{1}{|\Delta_1|}\Bigg(\sum_{i=1}^{m}|\alpha_i|\int_0^{\eta_i}\frac{(\eta_i-s)^{q+r-\nu-1}}{\Gamma(q+r-\nu)}\big|f(s,x(s),(Hx)(s)) - f(s,y(s),(Hy)(s))\big|ds \\
&\quad + \int_0^{\xi}\frac{(\xi-s)^{q+r-2}}{\Gamma(q+r-1)}\big|f(s,x(s),(Hx)(s)) - f(s,y(s),(Hy)(s))\big|ds\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&\times\left(\frac{\Gamma(q)}{\Gamma(q+r)}t^{q+r-1}+\left|\frac{\Delta_3}{\Delta_2}\right|\right)\\
&+\frac{1}{|\Delta_2|}\left[\sum_{i=1}^{m}|\beta_i|\int_0^{\sigma_i}\frac{(\sigma_i-s)^{q+r+p-1}}{\Gamma(q+r+p)}\big|f(s,x(s),(Hx)(s))-f(s,y(s),(Hy)(s))\big|ds\right.\\
&\left.\left.+\int_0^T\frac{(T-s)^{q+r-1}}{\Gamma(q+r)}\big|f(s,x(s),(Hx)(s))-f(s,y(s),(Hy)(s))\big|ds\right]\right\}\\
\leq\ &\big(L_1+\overline{\psi}L_2T\big)\left\{\frac{T^{q+r}}{\Gamma(q+r+1)}+\frac{1}{|\Delta_1|}\left(\sum_{i=1}^{m}|\alpha_i|\frac{\eta_i^{q+r-\nu}}{\Gamma(q+r-\nu+1)}+\frac{\xi^{q+r-1}}{\Gamma(q+r)}\right)\right.\\
&\left.\times\left(\frac{\Gamma(q)}{\Gamma(q+r)}T^{q+r-1}+\left|\frac{\Delta_3}{\Delta_2}\right|\right)+\frac{1}{|\Delta_2|}\left(\sum_{i=1}^{m}|\beta_i|\frac{\sigma_i^{q+r+p}}{\Gamma(q+r+p+1)}+\frac{T^{q+r}}{\Gamma(q+r+1)}\right)\right\}\\
&\times\|x-y\|\\
\leq\ &\big(L_1+\overline{\psi}L_2T\big)\Omega\|x-y\|.
\end{aligned}
$$

According to the condition $\big(L_1+\overline{\psi}L_2T\big)\Omega<1$, it follows that the operator $\mathcal{A}$ is a contraction. Therefore, by Theorem 3.1 (Banach's contraction principle), there exists a unique fixed point in B_ρ for the operator $\mathcal{A}$ which is a unique solution for the problem (1.1)-(1.2). This completes the proof. □

Corollary 3.5. *Assume that f and ψ are as in Theorem 3.4. Then the boundary value problem*

$$x''(t)=f(t,x(t),(Hx)(t)),\ \ t\in[0,T] \tag{3.4}$$

$$x'(\xi)=\sum_{i=1}^{m}\alpha_i x'(\eta_i),\ \ x(T)=\sum_{i=1}^{m}\beta_i\int_0^{\sigma_i}x(s)ds, \tag{3.5}$$

has a unique solution x on $[0,T]$ provided that

$$\big(L_1+\overline{\psi}L_2T\big)\Omega<1,$$

where $\overline{\psi}=\max\{\psi(t,s):(t,s)\in[0,T]\times[0,T]\}$, Ω is given by

$$\Omega=\frac{T^2}{2}+\frac{\left(\xi+\sum_{i=1}^{m}|\alpha_i|\eta_i\right)\left(T+\left|\frac{\frac{1}{2}\sum_{i=1}^{m}\beta_i\sigma_i^2-T}{1-\sum_{i=1}^{m}\beta_i\sigma_i}\right|\right)}{\left|1-\sum_{i=1}^{m}\alpha_i\right|}+\frac{\frac{1}{6}\sum_{i=1}^{m}|\beta_i|\sigma_i^3+\frac{T^2}{2}}{\left|1-\sum_{i=1}^{m}\beta_i\sigma_i\right|}.$$

Proof. It follows from Theorem 3.4 by choosing $q=r=\nu=p=1$, and $\lambda=0$. □

Now, we establish another existence result for the fractional integro-differential boundary value problem (1.1)-(1.2) by applying Schaefer's fixed point Theorem 3.2.

Theorem 3.6. *Let $f:[0,T]\times\mathbb{R}^2\longrightarrow\mathbb{R}$ be a continuous function. Assume that*
(H_2) There exists a constant $L>0$ such that $|f(t,x,y)|\leq L$, for all $t\in[0,T]$, $x,y\in\mathbb{R}$.
Then the problem (1.1)-(1.2) has at least one solution on $[0,T]$.

Proof. We prove that the operator $\mathcal{A}$ defined by (3.1) has a fixed point by utilizing Schaefer's fixed point theorem. The proof consists of several steps. Firstly, we show that the operator $\mathcal{A}$ is continuous.

Let x_n be a sequence such that $x_n \to x$ in $\mathcal{C}$. Then for each $t \in [0,T]$, we have

$$\begin{aligned}
|(\mathcal{A}x_n)(t) - (\mathcal{A}x)(t)| \quad \leq \quad & \Bigg\{ \int_0^t \frac{(t-s)^{q+r-1}}{\Gamma(q+r)} ds + \frac{1}{|\Delta_1|} \Bigg(\sum_{i=1}^m |\alpha_i| \int_0^{\eta_i} \frac{(\eta_i - s)^{q+r-\nu-1}}{\Gamma(q+r-\nu)} ds \\
& + \int_0^{\xi} \frac{(\xi - s)^{q+r-2}}{\Gamma(q+r-1)} ds \Bigg) \Bigg(\frac{\Gamma(q)}{\Gamma(q+r)} T^{q+r-1} + \left| \frac{\Delta_3}{\Delta_2} \right| \Bigg) \\
& + \frac{1}{|\Delta_2|} \Bigg[\sum_{i=1}^m |\beta_i| \int_0^{\sigma_i} \frac{(\sigma_i - s)^{q+r+p-1}}{\Gamma(q+r+p)} ds + \int_0^T \frac{(T-s)^{q+r-1}}{\Gamma(q+r)} ds \Bigg] \Bigg\} \\
& \times \big\| f(.,x_n(.),(Hx_n)(.)) - f(.,x(.),(Hx)(.)) \big\| \\
\leq \quad & \Omega \big\| f(.,x_n(.),(Hx_n)(.)) - f(.,x(.),(Hx)(.)) \big\|.
\end{aligned}$$

Since f is continuous, then $\|\mathcal{A}x_n - \mathcal{A}x\| \to 0$ as $n \to \infty$. Therefore $\mathcal{A}$ is continuous.

Now, it will be shown that $\mathcal{A}$ maps bounded sets into bounded sets in $\mathcal{C}$. For $\rho > 0$, let $B_\rho = \{x \in \mathcal{C} : \|x\| \leq \rho\}$ be bounded set in $\mathcal{C}$. In view of the condition (H_2), it is easy to establish that $\|\mathcal{A}x\| \leq L\Omega + \Lambda$, $x \in B_\rho$.

Thus $\mathcal{A}$ is uniformly bounded on B_ρ. Moreover, for $t_1, t_2 \in [0,T]$ with $t_1 < t_2$ and $x \in B_\rho$, we get the following estimates

$$\begin{aligned}
|(\mathcal{A}x)(t_2) - (\mathcal{A}x)(t_1)| = & \Bigg| \int_0^{t_2} \frac{(t_2-s)^{q+r-1}}{\Gamma(q+r)} f(s,x(s),(Hx)(s)) ds \\
& - \int_0^{t_1} \frac{(t_1-s)^{q+r-1}}{\Gamma(q+r)} f(s,x(s),(Hx)(s)) ds + \frac{\Gamma(q)}{\Delta_1 \Gamma(q+r)} (t_2^{q+r-1} - t_1^{q+r-1}) \\
& \times \Bigg(\sum_{i=1}^m \alpha_i \int_0^{\eta_i} \frac{(\eta_i - s)^{q+r-\nu-1}}{\Gamma(q+r-\nu)} f(s,x(s),(Hx)(s)) ds \\
& - \int_0^{\xi} \frac{(\xi-s)^{q+r-2}}{\Gamma(q+r-1)} f(s,x(s),(Hx)(s)) ds \Bigg) + \lambda \Bigg(\frac{\Gamma(q)}{\Delta_1 \Gamma(q+r)} \Big(t_2^{q+r-1} - t_1^{q+r-1} \Big) \\
& \times \Bigg(\frac{\xi^{r-1}}{\Gamma(r)} - \frac{1}{\Gamma(r-\nu+1)} \sum_{i=1}^m \alpha_i \eta_i^{r-\nu} \Bigg) - \frac{t_2^r - t_1^r}{\Gamma(r+1)} \Bigg) \Bigg| \\
\leq & L \Bigg\{ \frac{1}{\Gamma(q+r+1)} \big[(t_2^{q+r} - t_1^{q+r}) + 2(t_2 - t_1)^{q+r} \big] + \frac{\Gamma(q)}{|\Delta_1| \Gamma(q+r)} \\
& \times \Big(t_2^{q+r-1} - t_1^{q+r-1} \Big) \Bigg(\sum_{i=1}^m \frac{|\alpha_i| \eta_i^{q+r-\nu}}{\Gamma(q+r-\nu+1)} + \frac{\xi^{q+r-1}}{\Gamma(q+r)} + |\lambda| \\
& \times \Bigg(\frac{\xi^{r-1}}{\Gamma(r)} + \sum_{i=1}^m \frac{|\alpha_i| \eta_i^{r-\nu}}{\Gamma(r-\nu+1)} \Bigg) \Bigg) + \frac{|\lambda|}{\Gamma(r+1)} (t_2^r - t_1^r) \Bigg\}.
\end{aligned}$$

As $t_2 \to t_1$, the right-hand side tends to zero independently of $x \in B_\rho$. Thus, by the Arzelá-Ascoli theorem, the operator $\mathcal{A}$ is completely continuous.

Next, we need to show that the set $\mathcal{V} = \{x \in \mathcal{C} : x = \gamma \mathcal{A}x,\ 0 < \gamma < 1\}$ is bounded.

Let $x \in \mathcal{V}$ and $t \in [0,T]$. Then

$$
\begin{aligned}
x(t) &= \gamma\Bigg\{ I^{q+r} f(t,x(t),(Hx)(t)) + \frac{1}{\Delta_1}\left(\sum_{i=1}^{m} \alpha_i I^{q+r-\nu} f(\eta_i, x(\eta_i), (Hx)(\eta_i))\right. \\
&\quad \left. - I^{q+r-1} f(\xi, x(\xi), (Hx)(\xi))\right)\left(\frac{\Gamma(q)}{\Gamma(q+r)} t^{q+r-1} + \frac{\Delta_3}{\Delta_2}\right) \\
&\quad + \frac{1}{\Delta_2}\left[\sum_{i=1}^{m} \beta_i I^{q+r+p} f(\sigma_i, x(\sigma_i), (Hx)(\sigma_i)) - I^{q+r} f(T, x(T), (Hx)(T))\right] \\
&\quad + \lambda\Bigg[\frac{1}{\Delta_1}\left(\frac{\Gamma(q)}{\Gamma(q+r)} t^{q+r-1} + \frac{\Delta_3}{\Delta_2}\right)\left(\frac{\xi^{r-1}}{\Gamma(r)} - \frac{1}{\Gamma(r-\nu+1)}\sum_{i=1}^{m} \alpha_i \eta_i^{r-\nu}\right) \\
&\quad + \frac{1}{\Delta_2}\left(\frac{T^r}{\Gamma(r+1)} - \frac{1}{\Gamma(p+r+1)}\sum_{i=1}^{m} \beta_i \sigma_i^{p+r}\right) - \frac{t^r}{\Gamma(r+1)}\Bigg]\Bigg\},
\end{aligned}
$$

which implies using $\gamma < 1$ that

$$
\|x\| = \sup_{t\in[0,1]} \{|\gamma(\mathcal{A}x)(t)|\} \le L\Omega + \Lambda.
$$

Therefore, $\mathcal{V}$ is bounded. By Schaefer's fixed point Theorem 3.2, we conclude that the operator $\mathcal{A}$ has a fixed point which is a solution of the fractional order boundary value problem (1.1)-(1.2). This completes the proof. □

Our next result on existence is based on Krasnoselskii's fixed point Theorem 3.3.

Theorem 3.7. *Let $f : [0,T] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a continuous function satisfying the assumption (H_1). Moreover, it is assumed that*
(H_3) $|f(t,x(t),(Hx)(t))| \le \sigma(t)$, $\forall (t,x) \in [0,T] \times \mathbb{R}$, where $\sigma \in C([0,T],\mathbb{R}^+)$.
Then the boundary value problem (1.1)-(1.2) has at least one solution on $[0,T]$ if

$$
(L_1 + \overline{\psi} L_2 T)\left(\Omega - \frac{T^{q+r}}{\Gamma(q+r+1)}\right) < 1,
$$

where Ω is given by (3.2), and $\overline{\psi} = \max_{(t,s)\in[0,T]\times[0,T]} \psi(t,s)$.

Proof. If we denote $B_\rho = \{x \in \mathcal{C} : \|x\| \le \rho\}$, where $\rho \ge \Omega\|\sigma\| + \Lambda$ with $\|\sigma\| = \sup_{t\in[0,T]} |\sigma(t)|$, and Ω is given by (3.2). Then B_ρ is a bounded closed convex subset of $\mathcal{C}$.

For $t \in [0,T]$, we define two operators on B_ρ as

$$
\begin{aligned}
(\mathcal{A}_1 x)(t) &= \int_0^t \frac{(t-s)^{q+r-1}}{\Gamma(q+r)} f(s,x(s),(Hx)(s))ds, \\
(\mathcal{A}_2 x)(t) &= \frac{1}{\Delta_1}\left(\frac{\Gamma(q)}{\Gamma(q+r)} t^{q+r-1} + \frac{\Delta_3}{\Delta_2}\right) \\
&\quad \times \left(\sum_{i=1}^{m} \alpha_i I^{q+r-\nu} f(\eta_i, x(\eta_i), (Hx)(\eta_i)) - I^{q+r-1} f(\xi, x(\xi), (Hx)(\xi))\right) \\
&\quad + \lambda\Bigg[\frac{1}{\Delta_1}\left(\frac{\Gamma(q)}{\Gamma(q+r)} t^{q+r-1} + \frac{\Delta_3}{\Delta_2}\right)\left(\frac{\xi^{r-1}}{\Gamma(r)} - \frac{1}{\Gamma(r-\nu+1)}\sum_{i=1}^{m} \alpha_i \eta_i^{r-\nu}\right) \\
&\quad + \frac{1}{\Delta_2}\left(\frac{T^r}{\Gamma(r+1)} - \frac{1}{\Gamma(p+r+1)}\sum_{i=1}^{m} \beta_i \sigma_i^{p+r}\right) - \frac{t^r}{\Gamma(r+1)}\Bigg] \\
&\quad + \frac{1}{\Delta_2}\left[\sum_{i=1}^{m} \beta_i I^{q+r+p} f(\sigma_i, x(\sigma_i), (Hx)(\sigma_i)) - I^{q+r} f(T, x(T), (Hx)(T))\right].
\end{aligned}
$$

For $x,y\in B_\rho$, we find that $\|\mathcal{A}_1x+\mathcal{A}_2y\|\le \Omega\|\sigma\|+\Lambda\le\rho$, which implies that $\mathcal{A}_1x+\mathcal{A}_2y\in B_\rho$.

Using (H_1) and (3.2), for $x,y\in\mathcal{C}$, we obtain

$$\begin{aligned}
\|(\mathcal{A}_2x)-(\mathcal{A}_2y)\| &\le \sup_{t\in[0,1]}\Bigg\{\frac{1}{|\Delta_1|}\Bigg(\int_0^\xi \frac{(\xi-s)^{q+r-2}}{\Gamma(q+r-1)}|ds+\sum_{i=1}^m|\alpha_i|\int_0^{\eta_i}\frac{(\eta_i-s)^{q+r-\nu-1}}{\Gamma(q+r-\nu)}ds\Bigg)\\
&\quad\times\Bigg(\frac{\Gamma(q)}{\Gamma(q+r)}t^{q+r-1}+\left|\frac{\Delta_3}{\Delta_2}\right|\Bigg)+\frac{1}{|\Delta_2|}\Bigg[\sum_{i=1}^m|\beta_i|\int_0^{\sigma_i}\frac{(\sigma_i-s)^{q+r+p-1}}{\Gamma(q+r+p)}ds\\
&\quad+\int_0^T\frac{(T-s)^{q+r-1}}{\Gamma(q+r)}ds\Bigg]\Bigg\}\big(L_1+\overline{\psi}L_2T\big)\|x-y\|\\
&\le \big(L_1+\overline{\psi}L_2T\big)\Bigg\{\frac{1}{|\Delta_1|}\Bigg(\sum_{i=1}^m|\alpha_i|\frac{\eta_i^{q+r-\nu}}{\Gamma(q+r-\nu+1)}+\frac{\xi^{q+r-1}}{\Gamma(q+r)}\Bigg)\Bigg(\frac{\Gamma(q)}{\Gamma(q+r)}T^{q+r-1}\\
&\quad+\left|\frac{\Delta_3}{\Delta_2}\right|\Bigg)+\frac{1}{|\Delta_2|}\Bigg(\sum_{i=1}^m|\beta_i|\frac{\sigma_i^{q+r+p}}{\Gamma(q+r+p+1)}+\frac{T^{q+r}}{\Gamma(q+r+1)}\Bigg)\Bigg\}\|x-y\|\\
&\le \big(L_1+\overline{\psi}L_2T\big)\Bigg(\Omega-\frac{T^{q+r}}{\Gamma(q+r+1)}\Bigg)\|x-y\|,
\end{aligned}$$

which shows that the operator $\mathcal{A}_2$ is a contraction since $\big(L_1+\overline{\psi}L_2T\big)\Big(\Omega-\frac{T^{q+r}}{\Gamma(q+r+1)}\Big)<1$.

For $x\in B_\rho$, we have

$$\|\mathcal{A}_1x\|\le\sup_{t\in[0,1]}\left\{\int_0^t\frac{(t-s)^{q+r-1}}{\Gamma(q+r)}|f(s,x(s),(Hx)(s))|ds\right\}\le\frac{\|\sigma\|T^{q+r}}{\Gamma(q+r+1)}.$$

Therefore, $\mathcal{A}_1$ is uniformly bounded on B_ρ. Now, we prove the compactness of the operator $\mathcal{A}_1$.

Let $t_1,t_2\in[0,T]$ with $t_1<t_2$ and $x\in B_\rho$. Then, we obtain

$$|(\mathcal{A}_1x)(t_2)-(\mathcal{A}_1x)(t_1)| \le \frac{\tilde{f}}{\Gamma(q+r+1)}\Big[(t_2^{q+r}-t_1^{q+r})+2(t_2-t_1)^{q+r}\Big],$$

where $\sup_{(t,x,Hx)\in[0,T]\times B_\rho\times B_{\widetilde{\rho}}}|f(t,x,Hx)|=\tilde{f}$, with $\widetilde{\rho}=\rho\overline{\psi}T$. Obviously, the right-hand side of the above inequality tends to zero independently of $x\in B_\rho$ as $t_2\to t_1$. So $\mathcal{A}_1$ is relatively compact on B_ρ. Hence, by the Arzelá-Ascoli theorem, $\mathcal{A}_1$ is compact on B_ρ. Continuity of f implies that the operator $\mathcal{A}_1$ is continuous. Therefore, $\mathcal{A}_1$ is completely continuous. Thus all the hypothesis of Theorem 3.3 are satisfied and consequently the problem (1.1)-(1.2) has at least one solution on $[0,T]$. This completes the proof. □

Theorem 3.8. *Let $f:[0,T]\times\mathbb{R}^2\longrightarrow\mathbb{R}$ be a continuous function and assume that the assumption (H_1) holds with $\big(L_1+\overline{\psi}L_2T\big)\Omega<1$. Then the boundary value problem (1.1)-(1.2) is Ulam-Hyers stable on $[0,T]$ and consequently generalized Ulam-Hyers stable.*

Proof. Let $\epsilon>0$ and $y\in C([0,T],\mathbb{R})$ be the solution of inequality (2.1), and let $x\in C([0,T],\mathbb{R})$ be the unique solution of the following problem

$$\begin{cases} {}^{RL}D^q\big({}^CD^rx(t)+\lambda\big)=f(t,x(t),(Hx)(t)),\ t\in[0,T],\\ x'(\xi)=\sum_{i=1}^m\alpha_i{}^CD^\nu x(\eta_i),\ x(T)=\sum_{i=1}^m\beta_i[I^px](\sigma_i).\end{cases}$$

Since y is a solution of (2.1), we have by Remark 2.10

$${}^{RL}D^q\big({}^CD^ry(t)+\lambda\big)=f(t,y(t),(Hy)(t))+z(t),\ t\in[0,T],$$

and

$$\begin{aligned} y(t) &= I^{q+r}f(t,y(t),(Hy)(t))+\frac{1}{\Delta_1}\left(\sum_{i=1}^{m}\alpha_i I^{q+r-\nu}f(\eta_i,y(\eta_i),(Hy)(\eta_i))\right.\\ &\left.-I^{q+r-1}f(\xi,y(\xi),(Hy)(\xi))\right)\left(\frac{\Gamma(q)}{\Gamma(q+r)}t^{q+r-1}+\frac{\Delta_3}{\Delta_2}\right)\\ &+\frac{1}{\Delta_2}\left[\sum_{i=1}^{m}\beta_i I^{q+r+p}f(\sigma_i,y(\sigma_i),(Hy)(\sigma_i))-I^{q+r}f(T,y(T),(Hy)(T))\right]\\ &+I^{q+r}z(t)+\frac{1}{\Delta_1}\left(\sum_{i=1}^{m}\alpha_i I^{q+r-\nu}z(\eta_i)-I^{q+r-1}z(\xi)\right)\left(\frac{\Gamma(q)}{\Gamma(q+r)}t^{q+r-1}+\frac{\Delta_3}{\Delta_2}\right)\\ &+\frac{1}{\Delta_2}\left[\sum_{i=1}^{m}\beta_i I^{q+r+p}z(\sigma_i)-I^{q+r}z(T)\right]\\ &+\lambda\left[\frac{1}{\Delta_1}\left(\frac{\Gamma(q)}{\Gamma(q+r)}t^{q+r-1}+\frac{\Delta_3}{\Delta_2}\right)\left(\frac{\xi^{r-1}}{\Gamma(r)}-\frac{1}{\Gamma(r-\nu+1)}\sum_{i=1}^{m}\alpha_i\eta_i^{r-\nu}\right)\right.\\ &\left.+\frac{1}{\Delta_2}\left(\frac{T^r}{\Gamma(r+1)}-\frac{1}{\Gamma(p+r+1)}\sum_{i=1}^{m}\beta_i\sigma_i^{p+r}\right)-\frac{t^r}{\Gamma(r+1)}\right]. \end{aligned}$$

Then y is a solution of the following inequality

$$\begin{aligned} &\left|y(t)-I^{q+r}f(t,y(t),(Hy)(t))-\frac{1}{\Delta_1}\left(\sum_{i=1}^{m}\alpha_i I^{q+r-\nu}f(\eta_i,y(\eta_i),(Hy)(\eta_i))\right.\right.\\ &\left.-I^{q+r-1}f(\xi,y(\xi),(Hy)(\xi))\right)\left(\frac{\Gamma(q)}{\Gamma(q+r)}t^{q+r-1}+\frac{\Delta_3}{\Delta_2}\right)\\ &-\frac{1}{\Delta_2}\left[\sum_{i=1}^{m}\beta_i I^{q+r+p}f(\sigma_i,y(\sigma_i),(Hy)(\sigma_i))-I^{q+r}f(T,y(T),(Hy)(T))\right]\\ &-\lambda\left[\frac{1}{\Delta_1}\left(\frac{\Gamma(q)}{\Gamma(q+r)}t^{q+r-1}+\frac{\Delta_3}{\Delta_2}\right)\left(\frac{\xi^{r-1}}{\Gamma(r)}-\frac{1}{\Gamma(r-\nu+1)}\sum_{i=1}^{m}\alpha_i\eta_i^{r-\nu}\right)\right.\\ &\left.\left.+\frac{1}{\Delta_2}\left(\frac{T^r}{\Gamma(r+1)}-\frac{1}{\Gamma(p+r+1)}\sum_{i=1}^{m}\beta_i\sigma_i^{p+r}\right)-\frac{t^r}{\Gamma(r+1)}\right]\right|\\ =&\left|I^{q+r}z(t)+\frac{1}{\Delta_1}\left(\sum_{i=1}^{m}\alpha_i I^{q+r-\nu}z(\eta_i)-I^{q+r-1}z(\xi)\right)\left(\frac{\Gamma(q)}{\Gamma(q+r)}t^{q+r-1}+\frac{\Delta_3}{\Delta_2}\right)\right.\\ &\left.+\frac{1}{\Delta_2}\left[\sum_{i=1}^{m}\beta_i I^{q+r+p}z(\sigma_i)-I^{q+r}z(T)\right]\right|\\ \leq&\ \epsilon\left[\frac{T^{q+r}}{\Gamma(q+r+1)}+\frac{1}{|\Delta_1|}\left(\sum_{i=1}^{m}|\alpha_i|\frac{\eta_i^{q+r-\nu}}{\Gamma(q+r-\nu+1)}+\frac{\xi^{q+r-1}}{\Gamma(q+r)}\right)\left(\frac{\Gamma(q)}{\Gamma(q+r)}T^{q+r-1}+\left|\frac{\Delta_3}{\Delta_2}\right|\right)\right.\\ &\left.+\frac{1}{|\Delta_2|}\left(\sum_{i=1}^{m}|\beta_i|\frac{\sigma_i^{q+r+p}}{\Gamma(q+r+p+1)}+\frac{T^{q+r}}{\Gamma(q+r+1)}\right)\right]\\ \leq&\ \epsilon\Omega. \end{aligned}$$

Thus, for each $t \in [0, T]$, we have

$$
\begin{aligned}
|y(t)-x(t)| &= \Bigg| y(t) - I^{q+r} f(t, x(t), (Hx)(t)) - \frac{1}{\Delta_1}\Bigg(\sum_{i=1}^{m} \alpha_i I^{q+r-\nu} f(\eta_i, x(\eta_i), (Hx)(\eta_i)) \\
&\quad - I^{q+r-1} f(\xi, x(\xi), (Hx)(\xi)) \Bigg) \Bigg(\frac{\Gamma(q)}{\Gamma(q+r)} t^{q+r-1} + \frac{\Delta_3}{\Delta_2} \Bigg) \\
&\quad - \frac{1}{\Delta_2} \Bigg[\sum_{i=1}^{m} \beta_i I^{q+r+p} f(\sigma_i, x(\sigma_i), (Hx)(\sigma_i)) - I^{q+r} f(T, x(T), (Hx)(T)) \Bigg] \\
&\quad - \lambda \Bigg[\frac{1}{\Delta_1} \Bigg(\frac{\Gamma(q)}{\Gamma(q+r)} t^{q+r-1} + \frac{\Delta_3}{\Delta_2} \Bigg) \Bigg(\frac{\xi^{r-1}}{\Gamma(r)} - \frac{1}{\Gamma(r-\nu+1)} \sum_{i=1}^{m} \alpha_i \eta_i^{r-\nu} \Bigg) \\
&\quad + \frac{1}{\Delta_2} \Bigg(\frac{T^r}{\Gamma(r+1)} - \frac{1}{\Gamma(p+r+1)} \sum_{i=1}^{m} \beta_i \sigma_i^{p+r} \Bigg) - \frac{t^r}{\Gamma(r+1)} \Bigg] \Bigg| \\
&= \Bigg| y(t) - I^{q+r} f(t, y(t), (Hy)(t)) - \frac{1}{\Delta_1}\Bigg(\sum_{i=1}^{m} \alpha_i I^{q+r-\nu} f(\eta_i, y(\eta_i), (Hy)(\eta_i)) \\
&\quad - I^{q+r-1} f(\xi, y(\xi), (Hy)(\xi)) \Bigg) \Bigg(\frac{\Gamma(q)}{\Gamma(q+r)} t^{q+r-1} + \frac{\Delta_3}{\Delta_2} \Bigg) \\
&\quad - \frac{1}{\Delta_2} \Bigg[\sum_{i=1}^{m} \beta_i I^{q+r+p} f(\sigma_i, y(\sigma_i), (Hy)(\sigma_i)) - I^{q+r} f(T, y(T), (Hy)(T)) \Bigg] \\
&\quad + I^{q+r} f(t, y(t), (Hy)(t)) + \frac{1}{\Delta_1}\Bigg(\sum_{i=1}^{m} \alpha_i I^{q+r-\nu} f(\eta_i, y(\eta_i), (Hy)(\eta_i)) \\
&\quad - I^{q+r-1} f(\xi, y(\xi), (Hy)(\xi)) \Bigg) \Bigg(\frac{\Gamma(q)}{\Gamma(q+r)} t^{q+r-1} + \frac{\Delta_3}{\Delta_2} \Bigg) \\
&\quad + \frac{1}{\Delta_2} \Bigg[\sum_{i=1}^{m} \beta_i I^{q+r+p} f(\sigma_i, y(\sigma_i), (Hy)(\sigma_i)) - I^{q+r} f(T, y(T), (Hy)(T)) \Bigg] \\
&\quad - I^{q+r} f(t, x(t), (Hx)(t)) - \frac{1}{\Delta_1}\Bigg(\sum_{i=1}^{m} \alpha_i I^{q+r-\nu} f(\eta_i, x(\eta_i), (Hx)(\eta_i)) \\
&\quad - I^{q+r-1} f(\xi, x(\xi), (Hx)(\xi)) \Bigg) \Bigg(\frac{\Gamma(q)}{\Gamma(q+r)} t^{q+r-1} + \frac{\Delta_3}{\Delta_2} \Bigg) \\
&\quad - \frac{1}{\Delta_2} \Bigg[\sum_{i=1}^{m} \beta_i I^{q+r+p} f(\sigma_i, x(\sigma_i), (Hx)(\sigma_i)) - I^{q+r} f(T, x(T), (Hx)(T)) \Bigg] \\
&\quad - \lambda \Bigg[\frac{1}{\Delta_1} \Bigg(\frac{\Gamma(q)}{\Gamma(q+r)} t^{q+r-1} + \frac{\Delta_3}{\Delta_2} \Bigg) \Bigg(\frac{\xi^{r-1}}{\Gamma(r)} - \frac{1}{\Gamma(r-\nu+1)} \sum_{i=1}^{m} \alpha_i \eta_i^{r-\nu} \Bigg) \\
&\quad + \frac{1}{\Delta_2} \Bigg(\frac{T^r}{\Gamma(r+1)} - \frac{1}{\Gamma(p+r+1)} \sum_{i=1}^{m} \beta_i \sigma_i^{p+r} \Bigg) - \frac{t^r}{\Gamma(r+1)} \Bigg] \Bigg| \\
&\le \epsilon\Omega + \big(L_1 + \overline{\psi} L_2 T\big)\Omega \|y - x\|.
\end{aligned}
$$

Then, it follows that

$$\|y - x\| \le \epsilon\Omega + \big(L_1 + \overline{\psi} L_2 T\big)\Omega \|y - x\|,$$

which yields that

$$\|y-x\| \leq \frac{\epsilon\Omega}{1-\left(L_1+\overline{\psi}L_2T\right)\Omega}.$$

If we set $c_f = \frac{\Omega}{1-\left(L_1+\overline{\psi}L_2T\right)\Omega}$, then the problem (1.1)-(1.2) is Ulam-Hyers stable. Moreover, by choosing $\varphi_f(\epsilon) = \frac{\Omega\epsilon}{1-\left(L_1+\overline{\psi}L_2T\right)\Omega}$ with $\varphi_f(0) = 0$, the generalized Ulam-Hyers stability condition is also satisfied. $\square$

4. Examples

In this section, we present some numerical examples where our results can be applied.

Example 4.1. *Consider the following fractional boundary value problem*

$$\begin{cases} {}^{RL}D^{\frac{2}{3}}\left({}^{C}D^{\frac{3}{5}}x(t)+2\right) = \frac{e^{-\cos^2 t}}{(35e^t+1)\sqrt{t+16}}\sin x + \frac{397}{18144}\int_0^t \left(t+\cos\left(\frac{\pi}{2}s\right)\right)x(s)ds + \frac{\sqrt{2}}{2}\frac{t}{t+1},\ t\in[0,3], \\ x'(\frac{1}{3}) = {}^{c}D^{\frac{1}{4}}x(\frac{1}{7}) + 3\ {}^{c}D^{\frac{1}{4}}x(\frac{1}{5}),\ x(3) = 4[I^{\frac{3}{4}}x](\frac{1}{2}) + \frac{1}{6}[I^{\frac{3}{4}}x](\frac{4}{5}). \end{cases} \tag{4.1}$$

Here, $q=\frac{2}{3}$, $r=\frac{3}{5}$, $\nu=\frac{1}{4}$, $p=\frac{3}{4}$, $\lambda=2$, $T=3$, $\xi=\frac{1}{3}$, $\eta_1=\frac{1}{7}$, $\eta_2=\frac{1}{5}$, $\alpha_1=1$, $\alpha_2=3$, $\beta_1=4$, $\beta_2=\frac{1}{6}$, $\sigma_1=\frac{1}{2}$, $\sigma_2=\frac{4}{5}$, *and* $f(t,x,y) = \frac{e^{-\cos^2 t}}{(35e^t+1)\sqrt{t+16}}\sin x + y + \frac{\sqrt{2}}{2}\frac{t}{t+1}$, $\psi(t,s) = \frac{137}{2016}\left(t+\cos\left(\frac{\pi}{2}s\right)\right)$. *With the given values, it is easy to see that* $0<\nu\leq 1<q+r$, $\overline{\psi} = \frac{137}{1008}$, $\Delta_1 \simeq -4.42055$, $\Delta_2 = 1-\frac{2^{\frac{5}{4}}+\frac{\sqrt{2}}{3\times 5^{\frac{3}{4}}}}{\Gamma(\frac{7}{4})} \simeq -1.74127$ *and* $\Delta_3 \simeq 0.826943$.
Clearly, we have $|f(t,x,Hx)-f(t,y,Hy)| \leq \frac{1}{144}|x-y| + \frac{397}{1233}|Hx-Hy|$ *and thus* (H_1) *is satisfied with* $L_1=\frac{1}{144}$, $L_2=\frac{397}{1233}$. *Furthermore, upon computation, we get*

$$\begin{aligned} (L_1+\overline{\psi}L_2T)\Omega &= \frac{209}{1512}\Bigg\{\frac{T^{q+r}}{\Gamma(q+r+1)} + \frac{1}{|\Delta_1|}\left(\sum_{i=1}^{m}|\alpha_i|\frac{\eta_i^{q+r-\nu}}{\Gamma(q+r-\nu+1)} + \frac{\xi^{q+r-1}}{\Gamma(q+r)}\right) \\ &\quad\times\left(\frac{\Gamma(q)}{\Gamma(q+r)}T^{q+r-1} + \left|\frac{\Delta_3}{\Delta_2}\right|\right) \\ &\quad+\frac{1}{|\Delta_2|}\left(\sum_{i=1}^{m}|\beta_i|\frac{\sigma_i^{q+r+p}}{\Gamma(q+r+p+1)} + \frac{T^{q+r}}{\Gamma(q+r+1)}\right)\Bigg\} \\ &\simeq 0.927633 < 1. \end{aligned}$$

Thus, for the given boundary value problem (4.1), all the conditions of Theorem 3.4 are satisfied. So, by Theorem 3.4, there exists a unique solution for the problem (4.1) on $[0,3]$. *In addition, we have* $c_f = \frac{\Omega}{1-\left(L_1+\overline{\psi}L_2T\right)\Omega} \simeq 92.73449 > 0$. *Hence, by Theorem 3.8, problem (4.1) is Ulam–Hyers stable and also generalized Ulam–Hyers stable.*

Example 4.2. *Consider a fractional boundary value problem given by*

$$\begin{cases} {}^{RL}D^{\frac{1}{3}}\left({}^{C}D^{\frac{5}{7}}x(t)-\sqrt{2}\right) = \frac{e^{-2t}[2+\sin(t^2-t)]}{(3+|\cos x|)(4+t^3x^2)^2} + \frac{1}{16}\cos\left(\int_0^t (t^2+s)x(s)ds\right),\ t\in[0,2], \\ x'(\frac{1}{5}) = \frac{1}{4}\ {}^{c}D^{\frac{2}{3}}x(\frac{3}{2}),\ x(2) = \frac{1}{2}[I^{\frac{4}{3}}x](1), \end{cases} \tag{4.2}$$

where, $q=\frac{1}{3}$, $r=\frac{5}{7}$, $\nu=\frac{2}{3}$, $p=\frac{4}{3}$, $\lambda=-\sqrt{2}$, $T=2$, $\alpha=\frac{1}{4}$, $\beta=\frac{1}{2}$, $\eta=\frac{3}{2}$, $\xi=\frac{1}{5}$, $\sigma=1$, *and* $f(t,x,y) = \frac{e^{-2t}[2+\sin(t^2-t)]}{(3+|\cos x|)(4+t^3x^2)^2} + \frac{1}{16}\cos(y)$, $\psi(t,s) = t^2+s$. *By simple calculations, we find that* $0<\nu\leq 1<q+r$ *and*

$$\Delta_1 = \frac{5^{\frac{20}{21}}\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{1}{21}\right)} - \frac{\Gamma\left(\frac{1}{3}\right)}{2^{\frac{29}{21}}3^{\frac{13}{21}}\Gamma\left(\frac{8}{21}\right)} \simeq 0.382722,\ \textit{and}\ \Delta_2 = 1-\frac{1}{2\Gamma\left(\frac{7}{3}\right)} \simeq 0.580058.$$

We easily get $|f(t,x,y)| \leq \frac{1}{8}$. *Hence, all the conditions of Theorem 3.6 are satisfied. Thus, by Theorem 3.6 the fractional order boundary value problem* (4.2) *has at least one solution on* $[0,2]$:

Example 4.3. *As a third example we consider the fractional boundary value problem*

$$\begin{cases} {}^{RL}D^{\frac{1}{2}}\big({}^{C}D^{\frac{3}{4}}x(t)+3\big) = \frac{e^{-t}\cos(t\sqrt{2})}{(1+|x|)(2+e^t)^2\sqrt{t+25}} + \frac{132+7\sqrt{3}}{1560}\sin\left(\int_0^t \frac{1}{4}\big(t+s+t^3-s^3\big)x(s)ds\right) + \frac{t}{t+1}, \ t\in[0,1], \\ x'(\frac{5}{9}) = \frac{3}{2}\ {}^{c}D^{\frac{6}{7}}x(\frac{2}{7}) + 3\ {}^{c}D^{\frac{6}{7}}x(\frac{3}{10}), \ x(1) = [I^{\frac{5}{3}}x](\frac{1}{2}) + 2[I^{\frac{5}{3}}x](\frac{4}{5}), \end{cases} \tag{4.3}$$

where, $q=\frac{1}{2}$, $r=\frac{3}{4}$, $\nu=\frac{6}{7}$, $p=\frac{5}{3}$, $\lambda=3$, $T=1$, $\alpha_1=\frac{3}{2}$, $\alpha_2=3$, $\eta_1=\frac{2}{7}$, $\eta_2=\frac{3}{10}$, $\beta_1=1$, $\beta_2=2$, $\xi=\frac{5}{9}$, $\sigma_1=\frac{1}{2}$, $\sigma_2=\frac{4}{5}$, *and* $f(t,x,y)=\frac{e^{-t}\cos(t\sqrt{2})}{(1+|x|)(2+e^t)^2\sqrt{t+25}}+\frac{132+7\sqrt{3}}{1560}\sin(y)+\frac{t}{t+1}$, $\psi(t,s)=\frac{1}{4}\big(t+s+t^3-s^3\big)$. *With the given values, it is found that*

$$\Delta_1 = \frac{3\sqrt{3\pi}}{5^{\frac{3}{4}}\Gamma\big(\frac{1}{4}\big)} - \frac{\left(\frac{3}{2}\big(\frac{7}{2}\big)^{\frac{17}{28}} + 3^{\frac{11}{28}}10^{\frac{17}{28}}\right)\sqrt{\pi}}{\Gamma\big(\frac{11}{28}\big)} \simeq -6.64589,$$

$$\Delta_2 = 1 - \frac{2^{-\frac{5}{3}} + 2^{\frac{13}{3}}5^{-\frac{5}{3}}}{\Gamma\big(\frac{8}{3}\big)} \simeq -0.125778,$$

$$\Delta_3 = -\frac{\sqrt{\pi}}{\Gamma\big(\frac{5}{4}\big)} + \frac{\left(2^{-\frac{23}{12}} + 2^{\frac{29}{6}}5^{-\frac{23}{12}}\right)\sqrt{\pi}}{\Gamma\big(\frac{35}{12}\big)} \simeq -0.456032.$$

Also we have $\overline{\psi} = \frac{1}{2}\big(1+\frac{1}{3\sqrt{3}}\big)$.

Since $|f(t,x,Hx)-f(t,y,Hy)| \leq \frac{1}{45}|x-y| + \frac{132+7\sqrt{3}}{1560}|Hx-Hy|$, *then* (H_1) *is satisfied with* $L_1=\frac{1}{45}$, $L_2=\frac{132+7\sqrt{3}}{1560}$. *Further,*

$$|f(t,x,y)| = \left|\frac{e^{-t}\cos(t\sqrt{2})}{(1+|x|)(2+e^t)^2\sqrt{t+25}} + \frac{t}{t+1}\right| \leq \frac{1}{5}\left(\frac{e^{-t}}{9} + \frac{1692+7\sqrt{3}}{312}\right).$$

$$\begin{aligned} \big(L_1+\overline{\psi}L_2T\big)\left(\Omega - \frac{T^{q+r}}{\Gamma(q+r+1)}\right) &= \frac{47+5\sqrt{3}}{720}\left\{\frac{1}{|\Delta_1|}\left(\sum_{i=1}^m |\alpha_i|\frac{\eta_i^{q+r-\nu}}{\Gamma(q+r-\nu+1)} + \frac{\xi^{q+r-1}}{\Gamma(q+r)}\right)\right. \\ &\quad \times\left(\frac{\Gamma(q)}{\Gamma(q+r)}T^{q+r-1} + \left|\frac{\Delta_3}{\Delta_2}\right|\right) \\ &\quad \left. + \frac{1}{|\Delta_2|}\left(\sum_{i=1}^m |\beta_i|\frac{\sigma_i^{q+r+p}}{\Gamma(q+r+p+1)} + \frac{T^{q+r}}{\Gamma(q+r+1)}\right)\right\} \\ &\simeq 0.941679 < 1. \end{aligned}$$

Obviously all the conditions of Theorem 3.7 are satisfied with $\big(L_1+\overline{\psi}L_2T\big)\left(\Omega - \frac{T^{q+r}}{\Gamma(q+r+1)}\right) \simeq 0.941679 < 1$. *Hence, by Theorem 3.7, the fractional order boundary value problem* (4.3) *has at least one solution on* $[0,1]$.

5. Conclusion

In this paper, we have presented some existence results for a class of fractional integro-differential boundary value problems involving both Riemann–Liouville and Caputo fractional derivatives with multi-point and nonlocal Riemann-Liouville fractional integral and Caputo fractional derivative boundary conditions. We have established the existence of a unique solution by using the Banach contraction mapping principle, while Schaefer's fixed point theorem and Krasnoselskii's fixed point theorem are used to obtain the existence result for the proposed problem. Also a special case is produced by fixing $q=r=\nu=p=1$, and $\lambda = 0$. Furthermore, we have also discussed the Ulam–Hyers stability for the problem at hand. At the end of this paper, some examples are presented to show the applicability of the obtained results. We emphasize that the novelty of our work will contribute significantly to the existing literature on the topic of research.

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