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Three results in linear dynamics*

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ABSTRACT: In this article, first we show that the Fréchet space $H(\mathbb{D})$ cannot support strongly supercyclic weighted composition operators. Then we compute the constant ϵ for weighted backward shifts on ℓ^p $(1 \leq p < \infty)$ and c_0. This constant is used to find strongly hypercyclic scalar multiples of non-invertible strongly supercyclic Banach space operators. Finally, we give an affirmative answer to a recent open question concerning supercyclic vectors.

Key Words: Supercyclic, hypercyclic, strongly supercyclic, strongly hypercyclic.

Contents

1. Introduction

Suppose $\mathcal{X}$ is an infinite-dimensional second countable Baire topological vector space (over $\mathbb{C}$) and $\mathcal{S}$ is a multiplicative semigroup of maps on $\mathcal{X}$ (the binary operation is the composition of two maps). For a vector $x \in \mathcal{X}$ we put

$$\mathcal{S}x = \{f(x) : f \in \mathcal{S}\}.$$

A semigroup $\mathcal{S}$ is said to be *hypercyclic* if there is a nonzero vector $x \in \mathcal{X}$ such that $\overline{\mathcal{S}x} = X$. In this case, x is called a *hypercyclic vector* for $\mathcal{S}$. The set of all hypercyclic vectors for $\mathcal{S}$ is denoted by $HC(\mathcal{S})$. If $HC(\mathcal{S}) = \mathcal{X}\backslash\{0\}$, then $\mathcal{S}$ is called *hypertransitive*.

A semigroup $\mathcal{S}$ is called *supercyclic* if the semigroup

$$\mathbb{C}\mathcal{S} = \{bf : b \in \mathbb{C}, f \in \mathcal{S}\}$$

is hypercyclic. A vector $x \in \mathcal{X}$ is called a *supercyclic vector* for $\mathcal{S}$ whenever it is a hypercyclic vector for $\mathbb{C}\mathcal{S}$. The set of all supercyclic vectors for $\mathcal{S}$ is denoted by $SC(\mathcal{S})$.

We say that $\mathcal{S}$ is *topologically transitive* if, for any pair of nonempty open sets $U, V \subseteq \mathcal{X}$, there is some $f \in \mathcal{S}$ such that

$$f(U) \cap V \neq \emptyset.$$

Finally, $\mathcal{S}$ is said to be *strongly topologically transitive* if, for any nonempty open subset U of $\mathcal{X}$, we have that

$$\mathcal{X}\backslash\{0\} \subseteq \bigcup_{f\in\mathcal{S}} f(U).$$

It is clear that strong topological transitivity implies topological transitivity.

One can easily verify that a semigroup $\mathcal{S}$ is hypertransitive if and only if

$$\mathcal{X}\backslash\{0\} \subseteq \bigcup_{f\in\mathcal{S}} f^{-1}(U)$$

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for every nonempty open subset U of $\mathcal{X}$. Now assume that all members of $\mathcal{S}$ are bijections and put

$$S^{-1} = \{f^{-1} : f \in \mathcal{S}\}.$$

Then the following fact is obvious:

Fact 1.1 *Let $\mathcal{S}$ be a semigroup in which every map is a bijection. Then $\mathcal{S}$ is strongly topologically transitive if and only if $\mathcal{S}^{-1}$ is hypertransitive.*

Recall that by $L(\mathcal{X})$ we mean the set of all continuous linear operators on $\mathcal{X}$. From now on, by an operator, we mean a continuous linear operator. The most particular case among all multiplicative operator semigroups is the semigroup generated by a single operator. A continuous linear operator T on $\mathcal{X}$ is said to be hypercyclic (resp. supercyclic, topologically transitive) if the semigroup

$$\mathcal{S} = \{T^n : n \in \mathbb{N}_0\}$$

is hypercyclic (resp. supercyclic, topologically transitive). Here $\mathbb{N}_0$ is the set of all non-negative integers and $T^0 = I$, the identity operator on $\mathcal{X}$. We also say that x is a hypercyclic (resp. supercyclic) vector for T when it is a hypercyclic (resp. supercyclic) vector for $\mathcal{S}$, and we write $HC(T)$ (resp. $SC(T)$) for $HC(\mathcal{S})$ (resp. $SC(\mathcal{S})$). Notice that (since $\mathcal{X}$ has no isolated points) if x is a hypercyclic (resp. supercyclic) vector for T, then so is every $T^p x$ $(p \geq 1)$ (resp. $bT^p x$ $(0 \neq b \in \mathbb{C},\ p \geq 1)$), and this shows that $HC(T)$ (resp. $SC(T)$) is dense in $\mathcal{X}$.

Two well-known books which are always advised to study hypercyclic and supercyclic operators are [5,9].

We say that T is *strongly hypercyclic* (resp. *strongly supercyclic*) if the semigroup $\mathcal{S} = \{T^n : n \in \mathbb{N}_0\}$ (resp. $\mathcal{S} = \{bT^n : b \in \mathbb{C}, n \in \mathbb{N}_0\}$) is strongly topologically transitive. In other words, T is strongly hypercyclic (resp. strongly supercyclic) if

$$\mathcal{X}\backslash\{0\} \subseteq \bigcup_{n\in\mathbb{N}_0} T^n(U) \ \ \Big(\text{resp. } \mathcal{X}\backslash\{0\} \subseteq \bigcup_{b\in\mathbb{C}, n\in\mathbb{N}_0} bT^n(U)\Big)$$

for any nonempty open subset U of $\mathcal{X}$.

It is clear that strong hypercyclicity (resp. strong supercyclicity) is stronger than hypercyclicity (resp. supercyclicity). In fact, this claim is verified by taking a look at [5, Theorem 1.2] and its following remark (resp. [5, Theorem 1.12]). The reader is referred to [2,3] for more information on strong topological transitivity, strong hypercyclicity, and strong supercyclicity.

In Section 2, we show that there is no strongly supercyclic weighted composition operator on the function space $H(\mathbb{D})$. In Section 3, we compute the constant ϵ for weighted backward shift operators on ℓ^p $(1 \leq p < \infty)$ and c_0. In the last section, we give an affirmative answer to a recent open question about supercyclic vectors.

2. Weighted composition operators on $H(\mathbb{D})$

It is proved in [3] that no automorphism invariant weighted Hardy space $H^2(\beta)$ can support strongly supercyclic weighted composition operators. Some particular cases of theses spaces are the classical Hardy space $H^2(\mathbb{D})$ and the Bergman and Dirichlet spaces.

In the following theorem, we show that the same assertion is true for the Fréchet space $H(\mathbb{D})$. Recall that the space $H(\mathbb{D})$ is the set of all analytic functions on $\mathbb{D}$, equipped with the compact-open topology. It is worth mentioning that $H(\mathbb{D})$ supports hypercyclic (and hence supercyclic) weighted composition operators [11].

Assume that $w, \phi \in H(\mathbb{D})$ with $\phi(\mathbb{D}) \subseteq \mathbb{D}$. The weighted composition operator $C_{w,\phi}$ on $H(\mathbb{D})$ is defined by

$$C_{w,\phi}(f) = w(f \circ \phi), \text{ i.e., } C_{w,\phi}(f)(z) = w(z)f(\phi(z)).$$

Meanwhile, it is easy to see that the nth $(n \geq 2)$ iterate of $C_{w,\phi}$ is defined by

$$C^n_{w,\phi}(f) = w(w \circ \phi) \cdots (w \circ \phi_{n-1})(f \circ \phi_n)$$

for every $f \in H(\mathbb{D})$, where ϕ_k is the kth iterate of ϕ.

Theorem 2.1 *The Fréchet space $H(\mathbb{D})$ cannot support strongly supercyclic weighted composition operators.*

Proof: To get a contradiction, suppose that $C_{w,\phi}$ is a strongly supercyclic operator on $H(\mathbb{D})$. Then we claim that $w(z) \neq 0$ for all $z \in \mathbb{D}$. To prove this claim, assume that g is the constant function $g(z) = 1$ ($z \in \mathbb{D}$). Then, by the definition of strong supercyclicity, we must have

$$g \in \bigcup_{b\in\mathbb{C}, n\in\mathbb{N}_0} bC^n_{w,\phi}(U) \tag{2.1}$$

for any nonempty open subset U of $H(\mathbb{D})$. Hence, there are some $f \in U$, $b \in \mathbb{C}$, and $n \in \mathbb{N}_0$ such that $g = bC^n_{w,\phi}f$. Without loss of generality, we can assume that $n \geq 1$. Indeed, if $n = 0$ then we can replace U by the open set $V = U\backslash\mathbb{C}g$ in (2.1). This shows that, for all $z \in \mathbb{D}$, we have that

$$1 = bw(z)w(\phi(z))\cdots w(\phi_{n-1}(z))f(\phi_n(z)),$$

which proves that $w(z) \neq 0$ for all $z \in \mathbb{D}$.

On the other hand, it is easy to see that ϕ cannot be a constant map. Indeed, if ϕ is constant, then we have

$$\{bC^n_{w,\phi}f : b \in \mathbb{C}, n \in \mathbb{N}_0\} \subseteq \mathbb{C}w$$

for any $f \in H(\mathbb{D})$. This contradicts our assumption that $C_{w,\phi}$ is strongly supercyclic.

Hence $C_{w,\phi}$ is injective and so it is a bijection by [3, Proposition 3.5]. Then, by [7, Theorem 2.2], ϕ is an automorphism of $\mathbb{D}$ (the proof of that theorem only uses the assumption that $C_{w,\phi}$ is a bijection). Now it is easily seen that $C_{w,\phi}C_{(1/w)\circ\phi^{-1},\phi^{-1}} = I$, and so we must have $C^{-1}_{w,\phi} = C_{(1/w)\circ\phi^{-1},\phi^{-1}}$. Note that $C^{-1}_{w,\phi}$ is a linear map on $H(\mathbb{D})$ which may not necessarily be continuous. Let us put $\psi = (1/w)\circ\phi^{-1}$ and $\rho = \phi^{-1}$ to simply write $C^{-1}_{w,\phi} = C_{\psi,\rho}$.

Now, in view of the definition of strong supercyclicity, the semigroup

$$\mathcal{S} = \{bC^n_{w,\phi} : b \in \mathbb{C}\backslash\{0\}, n \in \mathbb{N}_0\}$$

is strongly topologically transitive (if $\mathcal{S}$ is strongly topologically transitive, then so is $\mathcal{S}\backslash\{0\}$), and hence, the semigroup

$$\mathcal{S}^{-1} = \{bC^n_{\psi,\rho} : b \in \mathbb{C}\backslash\{0\}, n \in \mathbb{N}_0\}$$

is hypertransitive by Fact 1.1. So, every $f \in H(\mathbb{D})$ which is not identically zero is a hypercyclic vector for $\mathcal{S}^{-1}$.

We consider the two possible cases for the automorphism ρ:

(1) There is some $a \in \mathbb{D}$ such that $\rho(a) = a$. Then the function $f(z) = z - a$ ($z \in \mathbb{D}$) is a hypercyclic vector for $\mathcal{S}^{-1}$, and hence, there are sequences $(b_k)_k$ in $\mathbb{C}$ and $(n_k)_k$ in $\mathbb{N}_0$ such that $b_kC^{n_k}_{\psi,\rho}f \to g$ where $g(z) = 1$ ($z \in \mathbb{D}$). Thus, at $z = a$ we must have

$$0 = b_k\psi(a)\psi(\rho(a))\cdots\psi(\rho_{n_k-1}(a))f(\rho_{n_k}(a)) \to 1$$

which is not true.

(2) The map ρ has no fixed point in $\mathbb{D}$. Then, by the well-known Denjoy-Wolff theorem, there is some $a \in \mathbb{T}$ (the unit circle $\partial\mathbb{D}$) such that $\rho_n \to a$ uniformly on compact subsets of $\mathbb{D}$. If we put $A = \{\rho_n(0) : n \in \mathbb{N}\}$, then there is some $f \in H(\mathbb{D})$ such that $Z(f) = A$ [10, Theorem 15.11], where $Z(f)$ is the set of all zeros of f. Now f is a hypercyclic vector for $\mathcal{S}^{-1}$ and so, for the constant function $g(z) = 1$ ($z \in \mathbb{D}$), there exist sequences $(b_k)_k$ in $\mathbb{C}$ and $(n_k)_k$ in $\mathbb{N}_0$ such that $b_kC^{n_k}_{\psi,\rho}f \to g$. Then we have that

$$0 = b_k\psi(0)\psi(\rho(0))\cdots\psi(\rho_{n_k-1}(0))f(\rho_{n_k}(0)) \to 1,$$

which is impossible.

Therefore, the assumption of strong supercyclicity of $C_{w,\phi}$ cannot be true and the proof is complete. □

3. The constant ϵ for weighted backward shifts

Let $\mathcal{X}$ be a Banach space and T be a bounded linear operator on $\mathcal{X}$. In [3], the constant $\epsilon(T)$ is defined by

$$\epsilon(T) = \inf\{\|y\| : y \in \mathcal{X}\backslash T(B)\},$$

where B is the open unit ball of $\mathcal{X}$. Then it is proved that:

Theorem 3.1 *[3, Theorem 4.2] Assume that $T \in B(\mathcal{X})$ is not invertible. Then T is surjective with dense generalized kernel if and only if cT is strongly hypercyclic for all $c \in \mathbb{C}$ with $|c| > 1/\epsilon(T)$.*

Corollary 3.1 *[3, Corollary 4.3] If T is not invertible then T is strongly supercyclic if and only if it is surjective and has dense generalized kernel.*

Thus, the evaluation of $\epsilon(T)$ is important to find strongly hypercyclic scalar multiples of non-invertible strongly supercyclic Banach space operators.

Remark 3.1 It is worth mentioning that, while in view of Theorem 3.1 and Corollary 3.1, non-invertible strongly supercyclic and strongly hypercyclic Banach space operators are scalar multiples of one another, there are (non-invertible) supercyclic Banach space operators whose scalar multiples are never hypercyclic [5, Example 1.15].

In the next result, we compute the constant ϵ for weighted backward shifts on ℓ^p $(1 \leq p < \infty)$ and c_0. Recall that ℓ^p $(1 \leq p < \infty)$ is the space of all complex sequences $(a_n)_{n\geq 0}$ satisfying $\sum_{n=0}^{\infty} |a_n|^p < \infty$. The norm of $x = (a_n)_n$ in ℓ^p is defined by $\|x\| = (\sum_{n=0}^{\infty} |a_n|^p)^{1/p}$. The space c_0 is comprised of all sequences $(a_n)_{n\geq 0}$ in $\mathbb{C}$ such that $a_n \to 0$, and the norm of $x = (a_n)_n$ in c_0 is defined by $\|x\| = \sup_{n\geq 0} |a_n|$.

The weighted backward shift B_W on $\mathcal{X} = \ell^p$ $(1 \leq p < \infty)$ or c_0 is defined by $B_W(e_0) = 0$, and $B_W(e_n) = w_n e_{n-1}$ $(n = 1, 2, 3, \cdots)$, where $W = (w_n)_{n\geq 1}$ is a bounded sequence of positive numbers and $(e_n)_{n\geq 0}$ is the canonical basis of $\mathcal{X}$. One can readily verify that B_W is surjective if and only if $(w_n)_n$ is bounded-away from zero, i.e., $\inf_{n\geq 1} w_n > 0$.

Proposition 3.1 *Suppose that B_W is a weighted backward shift on $\mathcal{X} = \ell^p$ $(1 \leq p < \infty)$ or c_0 with the weight sequence $(w_n)_n$. Then we have that $\epsilon(B_W) = \inf_{n\geq 1} w_n$.*

Proof: Put $\inf_{n\geq 1} w_n = r$. If $r = 0$ then there exists a strictly increasing sequence $(n_k)_k$ of positive integers such that $w_{n_k} \to 0$ as $k \to \infty$. For each $k \geq 1$, put $x_k = (0, 0, \cdots, 1, 0, 0, \cdots)$, where the number 1 has been set at the n_k-th position (remember that the position numbering starts with zero). Now, if we put $y_k = B_W x_k$ $(k \geq 1)$, then it is easy to see that $y_k \in \mathcal{X}\backslash B_W(B)$. In fact, if $z = (b_i)_{i\geq 0}$ and $y_k = B_W z$ for a fixed $k \geq 1$, then the equality $y_k = B_W x_k$ shows that $b_{n_k} = 1$, $b_i = 0$ for all $0 < i \neq n_k$, and b_0 could be any complex number, and hence, $\|z\| \geq 1$. Now, since $\|y_k\| = w_{n_k} \to 0$, we conclude that $\epsilon(B_W) = 0$.

Now assume that $r > 0$. Then B_W is surjective. Let $y \in \mathcal{X}\backslash B_W(B)$ be an arbitrary vector. Then there exists a vector $x = (a_0, a_1, \cdots) \notin B$ such that $y = B_W x = (w_1 a_1, w_2 a_2, \cdots)$. If we set $\hat{x} = (0, a_1, a_2, \cdots)$ then we must have $\|\hat{x}\| \geq 1$, because otherwise, $y = B_W x = B_W \hat{x} \in B_W(B)$ which contradicts our assumption. Then $\|y\| \geq r\|\hat{x}\| \geq r$, and hence, $\epsilon(B_W) \geq r$.

To complete our proof, we show that $\epsilon(B_W) \leq r$. Let $\delta > 0$ be an arbitrary number. Then there is some $k \geq 1$ such that $w_k < r + \delta$. Choose the vector $x = (a_i)_{i\geq 0}$ for which $a_k = (r + \delta)w_k^{-1}$ and $a_i = 0$ for all $0 \leq i \neq k$, and let $y = B_W x$. Then $\|x\| = a_k > 1$ and meanwhile, we claim that $y \notin B_W(B)$. Indeed, if $z = (b_i)_{i\geq 0}$ and $y = B_W z$ then we have that $b_k = a_k$, $b_i = 0$ for all $0 < i \neq k$, and b_0 could be any complex number. Thus, $\|z\| > 1$. Now, since $y \notin B_W(B)$ and $\|y\| = r + \delta$, we have $\epsilon(B_W) \leq r + \delta$. Finally, since $\delta > 0$ was arbitrary, we conclude that $\epsilon(B_W) \leq r$. □

4. A positive answer to an open question

We finish this paper by giving an affirmative answer to a recently asked open question concerning supercyclic vectors. It should be mentioned that this section has appeared as an arXiv preprint [1].

For a subset M of $\mathcal{X}$, by $\bigvee M$ we mean the closed linear span of M, i.e., $\bigvee M = \overline{\text{span} M}$. It is clear that if x is a supercyclic vector for an operator T on $\mathcal{X}$, then

$$\bigvee\{T^n x : n \in \mathbb{N}_0\} = \mathcal{X}.$$

A natural question which may be asked here is that whether there is a strictly increasing sequence $(n_k)_k$ of positive integers such that

$$\bigvee\{T^{n_k} x : k \geq 1\} \neq \mathcal{X}.$$

In their recently published paper, Faghih-Ahmadi and Hedayatian [8] have proved the following interesting result. Recall that for a normed linear space $\mathcal{X}$, we often write $B(\mathcal{X})$ instead of $L(\mathcal{X})$.

Theorem 4.1 (Theorem 1 of [8]) *Let $\mathcal{H}$ be an infinite-dimensional Hilbert space. If x is a supercyclic vector for $T \in B(\mathcal{H})$, then there is a (strictly increasing) sequence $(n_k)_k$ of positive integers such that $\bigvee\{T^{n_k} x : k \geq 1\} \neq \mathcal{H}$.*

Then they have asked whether the assertion is true for locally convex spaces or at least for Banach spaces [8, Question 1].

We show that the assertion is true for normed linear spaces. To prove our result, we use Lemma 4.2 which is an analogue of the following lemma.

Lemma 4.1 (Lemma 2.3 of [4]) *Let $\mathcal{A}$ be a dense subset of an infinite-dimensional Banach space $\mathcal{X}$ and e be a fixed element with $\|e\| > 1$. Then, for every finite-dimensional subspace $Y \subset \mathcal{X}$ with $dist(e, Y) > 1$, for every $\epsilon > 0$ and $y \in Y$, there is an $a \in \mathcal{A}$ such that $\|y - a\| < \epsilon$ and $dist(e, span\{Y, a\}) > 1$.*

The proof of the above lemma shows that it can also be stated for infinite-dimensional normed linear spaces. In fact, the lemma is proved by using Lemma 2.2 of [4] whose proof uses the fact that the weak closure of the unit sphere is the unit ball. Thus, we can give the following modified version.

Lemma 4.2 *Let $\mathcal{A}$ be a dense subset of an infinite-dimensional normed linear space $\mathcal{X}$ and $e \in \mathcal{X}$ be a fixed element with $\|e\| > 1$. Then, for every finite-dimensional subspace Y of $\mathcal{X}$ with $dist(e, Y) > 1$, there is an $a \in \mathcal{A}$ such that $dist(e, span\{Y, a\}) > 1$.*

Now we are ready to answer Question 1 of [8] for normed linear spaces.

Theorem 4.2 *Let $\mathcal{X}$ be an infinite-dimensional normed linear space. If x is a supercyclic vector for $T \in B(\mathcal{X})$ then there is a strictly increasing sequence $(n_k)_k$ of positive integers such that $\bigvee\{T^{n_k} x : k \geq 1\} \neq \mathcal{X}$.*

Proof: It is clear that every nonzero scalar multiple of x is also a supercyclic vector for T. On the other hand, since x and Tx are linearly independent vectors, we have that $\text{dist}(x, \text{span}\{Tx\}) > 0$. Thus, without loss of generality, we can assume that $\|x\| > 1$ and $\text{dist}(x, \text{span}\{Tx\}) > 1$. Now, if we put

$$\mathcal{A} = \{cT^n x : c \in \mathbb{C}, n > 1\}, Y = \text{span}\{Tx\},$$

and $e = x$, then $\mathcal{A}$ is dense in $\mathcal{X}$ because

$$\mathcal{A} = \{cT^n x : c \in \mathbb{C}, n \geq 0\} \backslash \{cT^n x : c \in \mathbb{C}, n = 0, 1\}$$

and $\{cT^n x : c \in \mathbb{C}, n = 0, 1\}$ is nowhere dense in $\mathcal{X}$, and hence, in view of Lemma 4.2, there is some $a = cT^{n_2} x \in \mathcal{A}$ such that

$$\text{dist}(x, \text{span}\{Tx, T^{n_2} x\}) = \text{dist}(x, \text{span}\{Y, a\}) > 1.$$

Now let

$$\mathcal{A}_2 = \{cT^n x : c \in \mathbb{C}, n > n_2\}, Y_2 = \text{span}\{Tx, T^{n_2} x\}.$$

Then, by Lemma 4.2, there is some $a_2 = c_2 T^{n_3} x \in \mathcal{A}_2$ such that

$$\operatorname{dist}(x, \operatorname{span}\{Tx, T^{n_2}x, T^{n_3}x\}) = \operatorname{dist}(x, \operatorname{span}\{Y_2, a_2\}) > 1.$$

By continuing this construction, assume that for some $k \geq 2$, the dense set $\mathcal{A}_k$ and the finite-dimensional subspace Y_k have been presented and (by using Lemma 4.2) we have found an element $a_k = c_k T^{n_{k+1}} x \in \mathcal{A}_k$ such that

$$\operatorname{dist}(x, \operatorname{span}\{Tx, T^{n_2}x, \cdots, T^{n_{k+1}}x\}) = \operatorname{dist}(x, \operatorname{span}\{Y_k, a_k\}) > 1.$$

Then we put

$$\mathcal{A}_{k+1} = \{cT^n x : c \in \mathbb{C}, n > n_{k+1}\}, Y_{k+1} = \operatorname{span}\{Tx, T^{n_2}x, \cdots, T^{n_{k+1}}x\}.$$

Again, by Lemma 4.2, there is some $a_{k+1} = c_{k+1} T^{n_{k+2}} x \in \mathcal{A}_{k+1}$ such that

$$\operatorname{dist}(x, \operatorname{span}\{Tx, T^{n_2}x, \cdots, T^{n_{k+2}}x\}) = \operatorname{dist}(x, \operatorname{span}\{Y_{k+1}, a_{k+1}\}) > 1.$$

This inductive procedure gives a strictly increasing sequence $(n_k)_k$ (with $n_1 = 1$) and it is easily seen that

$$\operatorname{dist}(x, \bigvee\{T^{n_k}x : k \geq 1\}) \geq 1.$$

Indeed, let $M = \operatorname{span}\{T^{n_k}x : k \geq 1\}$ and suppose, to get a contradiction, that $\operatorname{dist}(x, \overline{M}) < 1$. Then there is some $y \in M$ such that $\operatorname{dist}(x, y) < 1$. But it is clear that $y \in \operatorname{span}\{Y_k, a_k\}$ for some $k \geq 1$ and we have already seen that $\operatorname{dist}(x, \operatorname{span}\{Y_k, a_k\}) > 1$. Therefore, the assumption $\operatorname{dist}(x, \overline{M}) < 1$ cannot be true and this shows that $x \notin \bigvee\{T^{n_k}x : k \geq 1\}$. □

We need to mention that the authors in [6] have also answered the above-mentioned open question independently. The interested readers are invited to investigate Question 1 of [8] for operators on locally convex spaces.

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References

1. Ansari, M., *Supercyclic vectors of operators on normed linear spaces*, preprint, https://arxiv.org/abs/2206.01508.
2. Ansari, M., *Strong topological transitivity of some classes of operators*, Bull. Belg. Math. Soc. Simon Stevin, **25**, 677-685, (2018).
3. Ansari, M., Hedayatian, K., Khani-Robati, B., *Strong hypercyclicity of Banach space operators*, J. Korean Math. Soc. **58** (1), 91-107, (2021).
4. Bamerni, N., Kadets, V., Kilicman, A., *Hypercyclic operators are subspace hypercyclic*, J. Math. Anal. Appl. **435**, 1812-1815, (2016).
5. Bayart, F., Matheron, E., *Dynamics of linear operators*, Cambridge University Press, **179**, (2009).
6. Bernal-González, L., Bonilla, A., *Total and non-total suborbits for hypercyclic operators*, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. **117**, 17, (2023).
7. Bourdon, P.S., *Invertible weighted composition operators*, Proc. Amer. Math. Soc. **142** ,289-299, (2014).
8. Faghih-Ahmadi, M., Hedayatian, K., *A note on supercyclic vectors of Hilbert space operators*, J. Math. Anal. Appl. **505** (2022).
9. Grosse-Erdmann, K.-G., Peris Manguillot, A., *Linear chaos*, Universitext, Springer-Verlag London Limited, (2011).
10. Rudin, W., *Real and complex analysis*, McGraw-Hill International Editions, (1987).
11. Yousefi, B., Rezaei, H., *Hypercyclic property of weighted composition operators*, Proc. Amer. Math. Soc. **135**(10), 3263–3271, (2007).

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