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On Characterization of Inclusion Properties for Grand Lorentz Spaces on a Finite Measure

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ABSTRACT: In this paper, we give some well-known results and investigate some inclusion theorems of the spaces $L^{p,q)}$ and $\Lambda_{p),\omega}$. Also, we consider equivalence assertions for these spaces. Finally, we present the approximation identities in $\Lambda_{p),\omega}$ by the boundedness of maximal operator.

Key Words: Grand Lorentz spaces, inclusion, approximate identity.

Contents

1. Introduction

Let (X, Σ, μ) and (X, Σ, ν) be two finite measure spaces. It is known that $l^p(X) \subseteq l^q(X)$ for $0 < p \leq q \leq \infty$. Subramanian [25] characterized all positive measures μ on (X, Σ) for which $L^p(\mu) \subseteq L^q(\mu)$ whenever $0 < p \leq q \leq \infty$. Also, Romero [23] investigated and developed several results of [25]. Moreover, Miamee [22] obtained the more general result as $L^p(\mu) \subseteq L^q(\nu)$ with respect to μ and ν. Aydin and Gurkanli [2] proved some inclusion results for which $L^{p(\cdot)}(\mu) \subseteq L^{q(\cdot)}(\nu)$. Moreover, these results was generalized by Gurkanli [14] and Kulak [19] to the classical and variable exponent Lorentz spaces. The classical Lorentz spaces were introduced by Lorentz [20] and [21]. The Lorentz spaces with power and logarithmic weights have had extensive use in many sectors of analysis, particularly in the theory of function spaces and interpolation. Nevertheless, the theory of classical and weak Lorentz spaces has so far not been finished even in the sense of characterization of all embeddings between them, see [6].

In 1992, Iwaniec and Sbordone [16] introduced grand Lebesgue spaces $L^{p)}(\Omega)$ $(1 < p < \infty)$ on bounded sets $\Omega \subset \mathbb{R}^d$. Later, Greco, Iwaniec and Sbordone in [13] obtained a generalized version $L^{p),\theta}(\Omega)$. Recently, these spaces have intensively studied for various applications including harmonic analysis, interpolation-extrapolation theory, analysis of PDE's etc. For more details, we can refer the reader to [3], [4], [10], [11], [12]. Gurkanli [15] studied the inclusion $L^{p),\theta}(\mu) \subseteq L^{q),\theta}(\nu)$ under some conditions for two different measures μ and ν on (X, Σ), and proved that $L^{p),\theta}(\mu)$ has no an approximate identities. Jain and Kumari [17] generalized the classical Lorentz spaces to grand ones. They gave some basic properties of the grand Lorentz spaces and investigated the boundedness of maximal operator for these spaces.

In this paper, we consider some properties of grand Lorentz spaces $L^{p,q)}(\Omega, \mu)$ and $\Lambda_{p),\omega}$. We also give the relationship between these spaces in the sense of [17]. Moreover, we investigate the inclusion theorems for $L^{p,q)}(\Omega, \mu)$ and $\Lambda_{p),\omega}$ and obtain more general results than [14]. Finally, we will discuss the approximate identities of $\Lambda_{p),\omega}$ regarding the boundedness of the maximal operator.

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2. Notations and Preliminaries

The space $L^1_{loc}\left(\mathbb{R}^d\right)$ is the space of all measurable functions f on K such that $f.\chi_K \in L^1\left(\mathbb{R}^d\right)$ for any compact subset $K \subset \mathbb{R}^d$ where χ_K is the characteristic function of K. A Banach function space (shortly BF-space) on $\mathbb{R}^d$ is a Banach space $(B, \|.\|_B)$ of measurable functions which is continuously embedded into $L^1_{loc}\left(\mathbb{R}^d\right)$, i.e. for any compact subset $K \subset \mathbb{R}^d$ there exists some constant $C_K > 0$ such that $\|f.\chi_K\|_{L^1} \leq C_K. \|f\|_B$ for all $f \in B$.

Definition 2.1 *(see [5]) Let (X, Σ, μ) be a finite measure space and let f be a measurable function on X. The distribution function of f is defined by*

$$\lambda_f(y) = \mu\{x \in X : |f(x)| > y\}$$

for all $y > 0$. Moreover, the rearrangement of f is defined by

$$f^*(t) = \inf\{y > 0 : \lambda_f(y) \leq t\} = \sup\{y > 0 : \lambda_f(y) > t\}, \qquad t > 0$$

where $\inf \emptyset = +\infty$. Also, the average function of f is defined by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds, \qquad t > 0.$$

It is clear that $\lambda_f(.)$, $f^*(.)$ and $f^{**}(.)$ are non-increasing and right continuous on $(0, \infty)$. Moreover, it is known that $f^* \leq f^{**}$.

Definition 2.2 *(see [17]) A measurable and locally integrable function $\omega : X \longrightarrow (0, \infty)$ is called a weight function. We say that $\omega_1 \prec \omega_2$ if only if there exists $c > 0$ such that $\omega_1(t) \leq c\omega_2(t)$ for all $t \in X$. Two weight functions are called equivalent and written $\omega_1 \sim \omega_2$ if $\omega_1 \prec \omega_2$ and $\omega_2 \prec \omega_1$. The classical Lorentz space $\Lambda_{p,\omega}$ consists of all measurable functions such that*

$$\|f\|_{\Lambda_{p,\omega}} = \left(\int_0^1 (f^*(t))^p\,\omega(t)dt\right)^{\frac{1}{p}} < \infty,$$

where $0 < p < \infty$ and ω is a weight function, see [17].

A special case of the space $\Lambda_{p,\omega}$, denoted by $L^{p,q}(X, \mu)$ (or shortly $L^{p,q}$), $0 < p < \infty$ and $0 < q \leq \infty$, consists of all measurable functions f on X such that

$$\|f\|_{p,q} = \begin{cases} \left(\frac{q}{p}\int_0^\infty t^{\frac{q}{p}-1}\left[f^*(t)\right]^q dt\right)^{\frac{1}{q}}, & 0 < q < \infty \\ \sup\limits_{0<t<\infty}\left[t^{\frac{1}{p}} f^*(t)\right], & q = \infty \end{cases}$$

is finite. The Lorentz space $L^{p,q}$ is a Banach space with respect to the norm $\|.\|_{p,q}$. It is known that $L^p(\mu) = L^{p,q}(X, \mu)$ whenever $p = q$ where $L^p(\mu)$ is the Lebesgue space. Moreover, the space $L^{p,q}$ is a subspace of $L^{p,s}$ for $q \leq s$. Let $0 < q \leq p \leq s \leq \infty$. Then, we have

$$L^{p,q} \subset L^p(\mu) \subset L^{p,s} \subset L^{p,\infty},$$

see [5].

Definition 2.3 *(see [16]) The construction of the Lorentz space $L^{p,q}$ seems to be inspired by the Lebesgue space L^p, where f is replaced by its non-increasing rearrangement. A generalization of Lebesgue space* *is the so-called grand Lebesgue space denoted by $L^{p)}$, which for $1 < p < \infty$ consists of all measurable functions f defined on $(0,1)$ such that*

$$\|f\|_{p)} = \sup_{0<\varepsilon<p-1} \left(\varepsilon \int_0^1 |f(x)|^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}} < \infty.$$

This norm makes the space $L^{p)}$ a BF-space. Moreover, we have $L^p \subset L^{p)} \subset L^{p-\varepsilon}$ for $\varepsilon \in (0, p-1)$, see [16].

Now, we are ready to give the definition of grand Lorentz space $L^{p,q)}$.

Definition 2.4 *(see [17]) Assume that $1 < p, q \leq \infty$ and $X = (0,1)$. The grand Lorentz space $L^{p,q)}(X,\mu)$ (shortly $L^{p,q)}$ or $L^{p,q)}(\mu)$) is the space of all measurable functions f on X such that*

$$\|f\|_{p,q),\mu} = \begin{cases} \sup\limits_{0<\varepsilon<q-1} \left(\frac{q}{p}\varepsilon \int_0^1 t^{\frac{q}{p}-1} \left[f^*(t)\right]^{q-\varepsilon} dt \right)^{\frac{1}{q-\varepsilon}}, & 1 < q < \infty \\ \sup\limits_{0<t<1} \left[t^{\frac{1}{p}} f^*(t) \right], & q = \infty \end{cases}$$

is finite. It is note that these spaces coincide with the grand Lebesgue spaces $L^{p)}$ whenever $q = p$. Also, the space $L^{p,q)}$ is a BF-space under the condition $1 < q < p < \infty$.

Throughout this paper, we will accept that (X, Σ) is a σ-finite measurable space where $X = (0,1)$. Moreover, we will assume that $1 < p, q, r, s \leq \infty$. Also, if two measures μ and ν are absolutely continuous with respect to each other (denoted by $\mu \ll \nu$ and $\nu \ll \mu$), then we will use the symbol $\mu \approx \nu$. We shall denote throughout the paper $W(t) = \int_0^t \omega(s)\, ds$ and $V(t) = \int_0^t \vartheta(s)\, ds$ for $t > 0$.

3. Inclusion Theorems of Grand Lorentz Spaces $L^{p,q)}$

In this section, we will discuss and investigate the existence of the inclusion $L^{p,q)}(X,\mu) \subseteq L^{r,s)}(X,\nu)$ where μ and ν are different measures.

Theorem 3.1 *Let $1 < q < p$. If $(f_n)_{n\in\mathbb{N}}$ converges to f in $L^{p,q)}(X,\mu)$ then $(f_n)_{n\in\mathbb{N}}$ converges to f in measure.*

Proof: Let $(f_n)_{n\in\mathbb{N}}$ convergences to f in $L^{p,q)}(X,\mu)$. Then, for $\varepsilon \in (0, q-1)$ we have

$$\left(\frac{q}{p}\varepsilon \int_0^1 t^{\frac{q}{p}-1} \left((f_n - f)^*(t)\right)^{q-\varepsilon} d\mu(t) \right)^{\frac{1}{q-\varepsilon}} \leq \|f_n - f\|_{p,q),\mu} \longrightarrow 0 \tag{3.1}$$

as $n \longrightarrow \infty$. Since the space $L^{p,q)}(X,\mu)$ is a BF-space, we have

$$\int_X |f_n(x) - f(x)|\, d\mu(x) \leq C_X \|f_n - f\|_{p,q),\mu}. \tag{3.2}$$

By (3.1) and (3.2), we get that $(f_n)_{n\in\mathbb{N}}$ converges to f in $L^1(X)$. This completes the proof. □

Theorem 3.2 *Assume that $1 < p, r < \infty$ and $1 < q, s \leq \infty$. Then the space $L^{p,q)}(X,\mu)$ is a subspace of $L^{r,s)}(X,\nu)$ in sense of equivalence classes if and only if $\mu \approx \nu$ and the space $L^{p,q)}(X,\mu)$ is a subspace of $L^{r,s)}(X,\nu)$ in sense of individual functions.*

Proof: The sufficient condition of the theorem is clear. Now, we assume that $L^{p,q)}(X,\mu) \subseteq L^{r,s)}(X,\nu)$ holds in the sense of equivalence classes. Let $f \in L^{p,q)}(X,\mu)$ be any individual function. This implies $f \in L^{p,q)}(X,\mu)$ in the sense of equivalent classes thus $f \in L^{r,s)}(X,\nu)$ in the sense of equivalent classes by the assumption. Hence we have $f \in L^{r,s)}(X,\nu)$ in the sense of individual functions. This shows that

$$L^{p,q)}(X,\mu) \subseteq L^{r,s)}(X,\nu)$$

in sense of individual functions. Now, let us take any $A \in \Sigma$ with $\mu(A) = 0$. Then $\chi_A = 0$ μ-almost everywhere. By $L^{p,q)}(X,\mu) \subseteq L^{r,s)}(X,\nu)$, it is easy to see that $\chi_A = 0$ ν-almost everywhere. This implies that $\nu(A) = 0$. Hence, we obtain that $\nu \ll \mu$. One can prove that $\mu \ll \nu$ with the similar method. This completes the proof. □

Theorem 3.3 *The space $L^{p,q)}(X,\mu)$ is a subspace of $L^{r,s)}(X,\nu)$ in sense of equivalence classes if and only if $\mu \approx \nu$ and there is a $C > 0$ such that*

$$\|f\|_{r,s),\nu} \leq C \|f\|_{p,q),\mu} \tag{3.3}$$

for every $f \in L^{p,q)}(X,\mu)$.

Proof: Let $1 < q, s < \infty$. Assume that $L^{p,q)}(X,\mu) \subseteq L^{r,s)}(X,\nu)$ in sense of equivalence classes. Moreover, we define the sum norm on $L^{p,q)}(X,\mu)$

$$|||.||| = \|.\|_{p,q),\mu} + \|.\|_{r,s),\nu}\,.$$

The space $L^{p,q)}(X,\mu)$ is a Banach space with respect to $|||.|||$. To prove this, we assume that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^{p,q)}(X,\mu)$. Then for all $\beta > 0$ there exists $N(\beta) > 0$ whenever $n, m > N(\beta)$ such that

$$\|f_n - f_m\|_{p,q),\mu} = \sup_{0<\varepsilon<q-1} \left(\frac{q}{p}\varepsilon \int_0^1 t^{\frac{q}{p}-1} \left[(f_n - f_m)^*(t) \right]^{q-\varepsilon} dt \right)^{\frac{1}{q-\varepsilon}} < \beta$$

and

$$\|f_n - f_m\|_{r,s),\nu} = \sup_{0<\varepsilon<s-1} \left(\frac{s}{r}\varepsilon \int_0^1 t^{\frac{s}{r}-1} \left[(f_n - f_m)^*(t) \right]^{s-\varepsilon} dt \right)^{\frac{1}{s-\varepsilon}} < \beta.$$

This yields that $(f_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence in $L^{p,q)}(X,\mu)$ and $L^{r,s)}(X,\nu)$, and $(f_n)_{n\in\mathbb{N}}$ converges to functions $f \in L^{p,q)}(X,\mu)$ and $g \in L^{r,s)}(X,\nu)$, respectively. By the definition of the norm of Lorentz space, we have

$$L^{p,q)}(X,\mu) \hookrightarrow L^{p,q-\varepsilon}(X,\mu) \tag{3.4}$$

for $\varepsilon \in (0, q-1)$ and

$$L^{r,s)}(X,\nu) \hookrightarrow L^{r,s-\varepsilon}(X,\nu) \tag{3.5}$$

for $\varepsilon \in (0, s-1)$. If we use (3.4), then we obtain that there is a subsequence $(f_{n_i})_{i\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ such that $f_{n_i} \longrightarrow f$ (μ-almost everywhere). Moreover, it is easy to prove that $(f_{n_i})_{n\in\mathbb{N}}$ converges to g in $L^{r,s)}(X,\nu)$ and $f_{n_i} \longrightarrow g$ (ν-almost everywhere) due to (3.5). Hence, one can find a subsequence $\left(f_{n_{i_k}}\right)$ of (f_{n_i}) such that $f_{n_{i_k}} \longrightarrow g$ (ν-almost everywhere). If we consider the space $L^{p,q)}(X,\mu)$ as a subspace of $L^{r,s)}(X,\nu)$ in the sense of equivalence classes, then we have $\mu \approx \nu$ by Theorem 3.2. This implies that

$$|f(x) - g(x)| \leq \left| f(x) - f_{n_{i_k}}(x) \right| + \left| f_{n_{i_k}}(x) - g(x) \right|,$$

so that we have $f = g$ (μ-almost everywhere). Since $\mu \approx \nu$, we obtain $f = g$ (ν-almost everywhere), and $f_n \longrightarrow f$ in $L^{p,q)}(X,\mu)$ with respect to the norm $|||.|||$. Now, we define the unit function I from $\left(L^{p,q)}(X,\mu), |||.|||\right)$ into $\left(L^{p,q)}(X,\mu), \|.\|_{p,q),\mu}\right)$. Since

$$\|I(f)\|_{p,q),\mu} = \|f\|_{p,q),\mu} \leq |||f|||,$$

then I is continuous. If we consider the Banach's theorem, then I is a homeomorphism, see [7]. This yields that the norms $|||.|||$ and $\|.\|_{p,q),\mu}$ are equivalent. Thus there exists a constant $C > 0$ such that

$$|||f||| \leq C \|f\|_{p,q),\mu}$$

for all $f \in L^{p,q)}(X,\mu)$. Finally, we have

$$\|f\|_{r,s),\nu} \leq |||f||| \leq C \|f\|_{p,q),\mu}.$$

This completes the necessity part of the proof. Now, we suppose that $\mu \approx \nu$ and the inequality (3.3) holds for $f \in L^{p,q)}(X,\mu)$. Then, we have $L^{p,q)}(X,\mu) \subseteq L^{r,s)}(X,\nu)$ in the sense of individual functions. By Theorem 3.2, the space $L^{p,q)}(X,\mu)$ is a subspace of $L^{r,s)}(X,\nu)$ in the sense of equivalence classes. □

Theorem 3.4 *The following statements are equivalent.*

(i) The inclusion $L^{p,q)}(X,\mu) \subseteq L^{p,q)}(X,\nu)$ holds.

(ii) $\mu \approx \nu$ and there is a constant $C > 0$ such that

$$\nu(A) \leq C\mu(A)$$

for all $A \in \Sigma$.

(iii) $L^1(\mu) \subseteq L^1(\nu)$.

Proof: *(i) $\Longrightarrow$ (ii):* By Theorem 3.3, we have $\mu \approx \nu$ and there is a $C > 0$ such that

$$\|f\|_{p,q),\nu} \leq C_1 \|f\|_{p,q),\mu} \tag{3.6}$$

for all $f \in L^{p,q)}(X,\mu)$. Let $A \in \Sigma$. In particular, if we consider in (3.6) the function $f = \chi_A$, we have

$$(q-1)(\nu(A))^{\frac{1}{p}} \leq C_1 (q-1)(\mu(A))^{\frac{1}{p}}$$

This implies

$$\nu(A) \leq C\mu(A)$$

where $C = C_1^p > 0$.

(ii) $\Longrightarrow$ (iii): This proof is an immediate result of [22, Proposition 1].

(iii) $\Longrightarrow$ (i): By the assumption, there exists a $C_1 > 0$ such that

$$\|h\|_{1,\nu} \leq C_1 \|h\|_{1,\mu} \tag{3.7}$$

for all $h \in L^1(\mu)$. Let $f \in L^{p,q)}(X,\mu)$ be given. Therefore, we obtain

$$\left(\frac{q}{p}\varepsilon \int_0^1 t^{\frac{q}{p}-1} (f^*(t))^{q-\varepsilon} d\mu(t)\right)^{\frac{1}{q-\varepsilon}} \leq \|f\|_{p,q),\mu} < \infty$$

for all $0 < \varepsilon < q-1$. This yields $\chi_{(0,1)}(t)^{\frac{q}{p}-1}(f^*(t))^{q-\varepsilon} \in L^1(\mu)$ for all $0 < \varepsilon < q-1$. By (3.7), we get $\chi_{(0,1)}(t)^{\frac{q}{p}-1}(f^*(t))^{q-\varepsilon} \in L^1(\nu)$ and

$$\int_0^1 t^{\frac{q}{p}-1} (f^*(t))^{q-\varepsilon} d\nu(t) \leq C_1 \int_0^1 t^{\frac{q}{p}-1} [f^*(t)]^{q-\varepsilon} d\mu(t).$$

This implies

$$\left(\frac{q}{p}\varepsilon\int_0^1 t^{\frac{q}{p}-1}\left(f^*(t)\right)^{q-\varepsilon}d\nu(t)\right)^{\frac{1}{q-\varepsilon}} \leq C\left(\frac{q}{p}\varepsilon\int_0^1 t^{\frac{q}{p}-1}\left(f^*(t)\right)^{q-\varepsilon}d\mu(t)\right)^{\frac{1}{q-\varepsilon}} \tag{3.8}$$

where $C = C_1^{\frac{1}{q-\varepsilon}}$. If we take the supremum over $0 < \varepsilon < q-1$ in (3.8), then we have

$$\|f\|_{p,q),\nu} \leq C\,\|f\|_{p,q),\mu} < \infty$$

for every $f \in L^{p,q)}(X,\mu)$. □

Theorem 3.5 *Let $1 < q \leq p < r \leq s < \infty$. If the inclusion $L^{p,q)}(X,\mu) \subseteq L^{r,s)}(X,\mu)$ holds, then there is a constant $M > 0$ such that $\mu(A) \geq M$ for every μ-non-null set $A \in \Sigma$.*

Proof: Assume that $L^{p,q)}(X,\mu) \subseteq L^{r,s)}(X,\mu)$. By Theorem 3.3, there is $C > 0$ such that

$$\|f\|_{r,s),\mu} \leq C\,\|f\|_{p,q),\mu} \tag{3.9}$$

for all $f \in L^{p,q)}(X,\mu)$. Now, let $A \in \Sigma$ be a μ-non-null set. Since $p < r$, we have $\frac{1}{p} - \frac{1}{r} > 0$. If we consider in (3.9) the function $f = \chi_A$, we obtain

$$(s-1)\,\mu(A)^{\frac{1}{r}} \leq C\,(q-1)\,\mu(A)^{\frac{1}{p}}$$

or equivalently

$$\frac{s-1}{C(q-1)} \leq \mu(A)^{\frac{1}{p}-\frac{1}{r}}.$$

If we set $M = \left(\frac{s-1}{C(q-1)}\right)^{\frac{rp}{r-p}}$, we get $\mu(A) \geq M$. □

4. Inclusion Theorems of $\Lambda_{p),\omega}$

In [17], the authors unified the concepts of the classical Lorentz space and the grand Lebesgue space that would result in a new space $\Lambda_{p),\omega}$ denoted as follows.

Definition 4.1 *Assume that $1 < p < \infty$ and ω is a weight function. The grand Lorentz space $\Lambda_{p),\omega}$ is to be space of all measurable functions f defined on $(0,1)$ such that*

$$\|f\|_{\Lambda_{p),\omega}} = \sup_{0<\varepsilon<p-1}\left(\varepsilon\int_0^1\left(f^*(t)\right)^{p-\varepsilon}\omega(t)\,dt\right)^{\frac{1}{p-\varepsilon}} < \infty.$$

The space $\Lambda_{p),\omega}$ is a BF-space when ω is decreasing integrable weight. Moreover, the following inclusion holds

$$\Lambda_{p,\omega} \hookrightarrow \Lambda_{p),\omega} \hookrightarrow \Lambda_{p-\varepsilon,\omega} \tag{4.1}$$

for all $0 < \varepsilon < p-1$. Also, if we consider $\omega(t) = \frac{q}{p}t^{\frac{q}{p}-1}$, then we can say that the space $L^{p,q)}(X,\mu)$ is a special case of $\Lambda_{p),\omega}$, see [17].

It is easy to see that $\Lambda_{p),\vartheta} \hookrightarrow \Lambda_{p),\omega}$ if $\omega \prec \vartheta$. Moreover, as a direct result of the definition of $\omega \sim \vartheta$, we obtain $\Lambda_{p),\vartheta} = \Lambda_{p),\omega}$. Now, we will present several advanced inclusion theorems for grand Lorentz spaces.

Theorem 4.1 *Let $1 < p \leq q < \infty$. Then we have $\Lambda_{p),\omega} \hookrightarrow \Lambda_{q),\omega}$ if and only if*

$$\sup_{x>0} W(x)^{\frac{1}{q-\varepsilon}-\frac{1}{p-\varepsilon}} < \infty \tag{4.2}$$

holds for $0 < \varepsilon < p-1$, where $W(x) = \int_0^x \omega(t)\, dt$.

Proof: The necessary part is trivial by substituting $f = \chi_{[0,x]}$ in the inequality

$$\|f\|_{\Lambda_{q),\omega}} \leq C \|f\|_{\Lambda_{p),\omega}}$$

where $\chi_A(y) = 1$ for $y \in A$ and $\chi_A(y) = 0$ for $y \notin A$. Indeed, since the functions $\varepsilon^{\frac{1}{p-\varepsilon}}$ and $\varepsilon^{\frac{1}{q-\varepsilon}}$ are increasing for $0 < \varepsilon < p-1 < q-1$, we obtain

$$\begin{aligned}(q-1) \sup_{0<\varepsilon<q-1} \left(\int_0^x \omega(t)\, dt \right)^{\frac{1}{q-\varepsilon}} &\leq C(p-1) \sup_{0<\varepsilon<p-1} \left(\int_0^x \omega(t)\, dt \right)^{\frac{1}{p-\varepsilon}} \\ &\leq C(p-1) \sup_{0<\varepsilon<q-1} \left(\int_0^x \omega(t)\, dt \right)^{\frac{1}{p-\varepsilon}}.\end{aligned}$$

This yields that

$$\sup_{0<\varepsilon<q-1} \left(\int_0^x \omega(t)\, dt \right)^{\frac{1}{q-\varepsilon}-\frac{1}{p-\varepsilon}} \leq C < \infty.$$

If we take the supremum over $x > 0$ in both sides, then the necessary part of the proof holds. Now, let $f \in \Lambda_{p),\omega}$ be given. By (4.1), we get $f \in \Lambda_{p-\varepsilon,\omega}$. Since (4.2) holds, we have $\Lambda_{p-\varepsilon,\omega} \hookrightarrow \Lambda_{q-\varepsilon,\omega}$ for $0 < \varepsilon < p-1$, see [6, Theorem 3.1]. This yields that there exists $C(\varepsilon) > 0$ such that

$$\|f\|_{\Lambda_{q-\varepsilon,\omega}} \leq C(\varepsilon) \|f\|_{\Lambda_{p-\varepsilon,\omega}} \tag{4.3}$$

for $f \in \Lambda_{p-\varepsilon,\omega}$ and $\varepsilon \in (0, p-1)$. It is note that identity operator does not exceed $\mu(X) + 1$, see [18]. By (4.3), we have

$$\left(\varepsilon \int_0^1 (f^*(t))^{q-\varepsilon} \omega(t) dt \right)^{\frac{p-\varepsilon}{q-\varepsilon}} \leq 2^{p-\varepsilon} \varepsilon^{\frac{p-\varepsilon}{q-\varepsilon}} \int_0^1 (f^*(t))^{p-\varepsilon} \omega(t) dt.$$

This obtain

$$\left(\varepsilon \int_0^1 (f^*(t))^{q-\varepsilon} \omega(t) dt \right)^{\frac{1}{q-\varepsilon}} \leq 2\varepsilon^{\frac{1}{q-\varepsilon}} \varepsilon^{-\frac{1}{p-\varepsilon}} \left(\varepsilon \int_0^1 (f^*(t))^{p-\varepsilon} \omega(t) dt \right)^{\frac{1}{p-\varepsilon}}$$

or equivalently

$$\varepsilon^{\frac{q-p}{(p-\varepsilon)(q-\varepsilon)}} \left(\varepsilon \int_0^1 (f^*(t))^{q-\varepsilon} \omega(t) dt \right)^{\frac{1}{q-\varepsilon}} \leq 2 \left(\varepsilon \int_0^1 (f^*(t))^{p-\varepsilon} \omega(t) dt \right)^{\frac{1}{p-\varepsilon}}. \tag{4.4}$$

If we take the supremum over $0 < \varepsilon < p-1 < q-1$ in both sides of (4.4), then we get

$$\|f\|_{\Lambda_{q),\omega}} \leq C \|f\|_{\Lambda_{p),\omega}}$$

where $\sup\limits_{0<\varepsilon<q-1} \varepsilon^{\frac{q-p}{(p-\varepsilon)(q-\varepsilon)}} < \infty$. This completes the proof. □

By the similar method in Theorem 4.1, we have

Theorem 4.2 *Let $1 < p \leq q < \infty$. Then we have $\Lambda_{p),\vartheta} \hookrightarrow \Lambda_{q),\omega}$ if and only if*

$$\sup_{x>0} W(x)^{\frac{1}{q-\varepsilon}} V(x)^{-\frac{1}{p-\varepsilon}} < \infty$$

holds for $0 < \varepsilon < p-1$.

Theorem 4.3 *Let $1 < q < p < \infty$ and let r be given by $\frac{1}{r-\varepsilon} = \frac{1}{q-\varepsilon} - \frac{1}{p-\varepsilon}$. Then we have $\Lambda_{p),\vartheta} \hookrightarrow \Lambda_{q),\omega}$ if*

$$\left(\int_0^1 \left(\frac{W(t)}{V(t)} \right)^{\frac{r-\varepsilon}{p-\varepsilon}} \omega(t)\, dt \right)^{\frac{1}{r-\varepsilon}} < \infty \tag{4.5}$$

holds for $0 < \varepsilon < q-1$.

Proof: Let $f \in \Lambda_{p),\vartheta}$ be given. By (4.1), we get $f \in \Lambda_{p-\varepsilon,\vartheta}$. Moreover, we assume that (4.5) holds. Thus, we have $\Lambda_{p-\varepsilon,\vartheta} \hookrightarrow \Lambda_{q-\varepsilon,\omega}$ for $0 < \varepsilon < q-1$, see [6, Theorem 3.1]. This yields that there exists $C(\varepsilon) > 0$ such that

$$\|f\|_{\Lambda_{q-\varepsilon,\omega}} \leq C(\varepsilon) \|f\|_{\Lambda_{p-\varepsilon,\vartheta}} \tag{4.6}$$

for $f \in \Lambda_{p-\varepsilon,\vartheta}$ and $\varepsilon \in (0, p-1)$. It is note that identity operator does not exceed $\mu(X) + 1$, see [18]. By (4.6), we have

$$\left(\varepsilon \int_0^1 (f^*(t))^{q-\varepsilon} \omega(t) dt \right)^{\frac{p-\varepsilon}{q-\varepsilon}} \leq 2^{p-\varepsilon} \varepsilon^{\frac{p-\varepsilon}{q-\varepsilon}} \int_0^1 (f^*(t))^{p-\varepsilon} \vartheta(t) dt.$$

Thus we obtain

$$\left(\varepsilon \int_0^1 (f^*(t))^{q-\varepsilon} \omega(t) dt \right)^{\frac{1}{q-\varepsilon}} \leq 2\varepsilon^{\frac{p-q}{(p-\varepsilon)(q-\varepsilon)}} \left(\varepsilon \int_0^1 (f^*(t))^{p-\varepsilon} \vartheta(t) dt \right)^{\frac{1}{p-\varepsilon}}. \tag{4.7}$$

If we take the supremum over $0 < \varepsilon < q-1 < p-1$ in both sides of (4.7), then we get

$$\|f\|_{\Lambda_{q),\omega}} \leq C \|f\|_{\Lambda_{p),\vartheta}}$$

where $\sup\limits_{0<\varepsilon<q-1} \varepsilon^{\frac{p-q}{(p-\varepsilon)(q-\varepsilon)}} < \infty$. This completes the proof. □

Since the space $L^{p,q)}(X,\mu)$ is a special case of $\Lambda_{p),\omega}$, we have the following result.

Corollary 4.1 *The following inclusions hold.*

For $q_1 \leq p \leq s_1$, under the condition (4.2), we have

$$L^{p,q_1)} \subset L^{p)}(\mu) \subset L^{p,s_1)}.$$

For $s_2 \leq p \leq q_2$, under the condition (4.5), we have

$$L^{p,q_2)} \subset L^{p)}(\mu) \subset L^{p,s_2)}.$$

5. Approximate Identities in $\Lambda_{p),\omega}$

Let $\Omega \subset \mathbb{R}^d$ be a bounded and open set. It is well known that the classical Lebesgue space $L^p(\Omega)$ has a bounded approximate identity in $L^1(\Omega)$. Gurkanli proved that $L^{p),\theta}(\Omega)$ does not admit a bounded approximate identity in $L^1(\Omega)$ in [15, Theorem 4], and also that $[L^p(\Omega)]_{p),\theta}$, the closure of $C_0^\infty(\Omega)$ in $L^{p),\theta}(\Omega)$, admits a bounded approximate identity in $L^1(\Omega)$ in [15, Theorem 6]. It is known that when $\theta = 1$ the space $L^{p),1}(\Omega)$ reduces to the grand Lebesgue space $L^{p)}(\Omega)$.

For $x \in X$ and $r > 0$, we denote an open ball with center x and radius r by $B(x,r)$. For $f \in L^1_{loc}(\Omega)$, we denote the (centered) Hardy-Littlewood maximal operator Mf of f by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy,$$

where the supremum is taken over all balls $B(x,r)$. In [17], the authors proved that the Hardy-Littlewood maximal operator is bounded in $\Lambda_{p),\vartheta}$.

Definition 5.1 *Assume that φ is an integrable function defined on $\mathbb{R}^d$ such that $\int_{\mathbb{R}^d} \varphi(x)dx = 1$. For each $t > 0$, define the function $\varphi_t(x) = t^{-d}\varphi\left(\frac{x}{t}\right)$. The sequence $\{\varphi_t\}$ is referred to as an approximate identity. It is known that for $1 < p < \infty$, the sequence $\{\varphi_t * f\}$ converges to f in $L^p(\Omega)$, i.e.*

$$\lim_{t \longrightarrow \infty} \|\varphi_t * f - f\|_{p,\Omega} = 0,$$

see [24, p. 63]. If we impose additional conditions on φ, then the entire sequence converges almost everywhere to f. Define the radial majorant of φ as the function

$$\widetilde{\varphi}(x) = \sup_{|y| \geq |x|} |\varphi(y)|.$$

If the function $\widetilde{\varphi}$ is integrable, then $\{\varphi_t\}$ is called a potential-type approximate identity, see [8].

Theorem 5.1 *(see [9, p. 31]) Let $f \in L^1_{loc}(X)$ be given. Then we have*

$$\sup_{t>0} |\varphi_t * f(x)| \leq Mf(x).$$

Theorem 5.2 *If $f \in \Lambda_{p,\omega}$, then we have $\varphi_t * f \longrightarrow f$ in $\Lambda_{p,\omega}$ as $t > 0$.*

Proof: Let $f \in \Lambda_{p,\omega}$ and $\varepsilon > 0$ be given. By Theorem 5.1 and the boundedness of maximal function in $\Lambda_{p,\omega}$, we have

$$\|\varphi_t * f\|_{\Lambda_{p,\omega}} \leq C^* \|Mf\|_{\Lambda_{p,\omega}} \leq C \|f\|_{\Lambda_{p,\omega}} < \infty$$

and then $\varphi_t * f \in \Lambda_{p,\omega}$ for all $t > 0$. If we use the similar method in [26], then it is easy to see that the space $C_c(X)$ is dense in $\Lambda_{p,\omega}$. Therefore, there exists a function $g \in C_c(X)$ such that

$$\|f - g\|_{\Lambda_{p,\omega}} < \varepsilon \tag{5.1}$$

as $t > 0$. Moreover, for all $t > 0$, we have $\varphi_t * g \in C_0^\infty(X)$, see [1, Theorem 2.29]. It is easily seen that $\varphi_t * g \longrightarrow g$ uniformly on compact sets as $t \longrightarrow 0^+$. Hence we have

$$\|\varphi_t * g - g\|_{\Lambda_{p,\omega}} < \varepsilon. \tag{5.2}$$

Finally by using (5.1) and (5.2),

$$\begin{aligned} \|f - \varphi_t * f\|_{\Lambda_{p,\omega}} &\leq \|f - g\|_{\Lambda_{p,\omega}} + \|g - \varphi_t * g\|_{\Lambda_{p,\omega}} + \|\varphi_t * g - \varphi_t * f\|_{\Lambda_{p,\omega}} \\ &< \varepsilon. \end{aligned}$$

This completes the proof. □

Now, we are ready to present the main theorem of this section for the space $\Lambda_{p),\omega}$.

Theorem 5.3 *Let $\{\varphi_t\}$ be a potential-type approximate identity. If p is finite, then we have*

$$\|\varphi_t * f - f\|_{\Lambda_{p),\omega}} \longrightarrow 0$$

as $t \longrightarrow 0^+$ for $f \in \Lambda_{p),\omega}$.

Proof: Let $f \in \Lambda_{p),\omega}$ be given. By the boundedness of maximal operator in $\Lambda_{p),\omega}$, we have $\varphi_t * f \in \Lambda_{p),\omega}$ for all $t > 0$. By Theorem 5.2, for every $\eta > 0$ there exists an $h > 0$ such that

$$\|\varphi_t * f - f\|_{\Lambda_{p-\varepsilon,\omega}} < \eta$$

for all t satisfying $t < h$ and $0 < \varepsilon < p - 1$. This follows that

$$\|\varphi_t * f - f\|_{\Lambda_{p),\omega}} < \eta \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} = (p-1)\,\eta.$$

That is the desired result. □

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