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A weak solution for a class of quasilinear elliptic Systems with nonstandard growth In musielak-Orlicz space

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ABSTRACT: Here we study existence of a weak solutions for some nonlinear elliptic systems like

$$\begin{cases} -\operatorname{div}\sigma(x,u,Du) = f(x,u,Du) \quad \text{in } \Omega \\ u(x) = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open domain. The Term f satisfy the growth and sign condition. We establish the existence solution by using the idea of Young measure.

Key Words: Musielak-Orlicz-Sobolev spaces, Elliptic problem, weak solutions, truncations.

Contents

1. Introduction, Essential Assumptions And Main Result

Let Ω be a bounded and smooth open subset of $\mathbb{R}^n$, and consider, as a model, the following nonlinear elliptic systems

$$\begin{cases} -\operatorname{div}\sigma(x,u,Du) = f(x,u,Du) \text{ in } \Omega \\ u(x) = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where $u : \Omega \to \mathbb{R}^m$ is a vector-valued function. We denote by $\mathbb{M}^{m\times n}$ the real vector space of $m \times n$ matrices equipped with the inner product $F : G = \sum_{i,j} F_{ij}G_{ij}$.

In some recent papers the problem (1.1) was studied when $u : \Omega \to \mathbb{R}$ by using different ideas. For instance in [6] El haji et al. have been showed the existence of entropy solution in weighted Orlicz spaces. In [18] the authors have been studied the existence result of sub-supsolution, nonlinear regularity theory and strong maximum principle. A regularity results for weak solutions have been proved by Pucci and Servadie (see [38])

The mathematical literature in the case $u : \Omega \to \mathbb{R}^m$ is massive; without the aim to be complete we refer the reader to [1,2,3,4,5,6,8,9,10,18,13,19,41,42] and references therein.

However, in this paper we are interested in existence results by using methods of Young Measure in order to identify weak limits. Furthermore, we get our solution under weak monotonicity conditions in

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the framework of Musielak-Orlicz space. the concept of Young measure is introduced by Young [39]. In this sense, many applications and developments were presented in nonlinear partial differential equations, optimal control theory and the calculus of variations.

Let P and φ two N-functions such that $P \ll \varphi$. Therefore, let us summarizing the main hypotheses :

$$\sigma : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n} \text{ and } f : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}^m \text{ are Carathéodory's functions (Continuity).} \tag{1.2}$$

There exist $0 < d_1(x) \in L_{\bar{\varphi}}(\Omega), d_2(x) \in L^1(\Omega), 0 < d_3(x) \in L_{\bar{\varphi}}(\Omega)$ and $\alpha, \beta, \gamma > 0$ such that

$$|\sigma(x,u,F)| \leq d_1(x) + \bar{\varphi}^{-1}P(x,\gamma|u|) + \bar{\varphi}^{-1}\varphi(x,\gamma|F|) \text{ (Growth)} \tag{1.3}$$

$$|f(x,u,F)| \leq d_3(x) + \bar{\varphi}^{-1}P(x,\gamma|u|) + \bar{\varphi}^{-1}\varphi(x,\gamma|F|) \text{ (Growth)} \tag{1.4}$$

$$\sigma(x,u,F) : F - f(x,u,F) \geq -d_2(x) + \alpha\varphi\left(x, \frac{|F|}{\beta}\right) \text{(coercivity).} \tag{1.5}$$

$$f(x,u,F)\cdot u \geq 0 \text{ (sign condition)} \tag{1.6}$$

σ satisfies one of the following conditions:

(a) For any $x \in \Omega$ and $u \in \mathbb{R}^m, F \mapsto p(x,u,F)$ is a C^1 and monotone, i.e.

$$(\sigma(x,u,F) - \sigma(x,u,G)) : (F-G) \geq 0 \text{ (Monotonicity)} \tag{1.7}$$

for all $x \in \Omega, u \in \mathbb{R}^m$ and $F, G \in \mathbb{M}^{m\times n}$.

(b) There exists a function

$$\begin{aligned} &W : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R} \text{ such that } \sigma(x,u,F) = (\partial W/\partial F)(x,u,F) \\ &\text{and } F \to W(x,u,F) \text{ is convex and } \mathcal{C}^1. \end{aligned} \tag{1.8}$$

(c) σ is strictly monotone, i.e. σ is monotone and

$$(\sigma(x,u,F) - \sigma(x,u,G)) : (F-G) = 0 \Rightarrow F = G. \tag{1.9}$$

(d) σ is strictly M-quasimonotone on $\mathbb{M}^{m\times n}$, i.e.

$$\int_{\mathbb{M}^{m\times n}} (\sigma(x,u,\lambda) - \sigma(x,u,\bar{\lambda})) : (\lambda - \bar{\lambda}) \mathrm{d}v(\lambda) > 0, \tag{1.10}$$

where $\bar{\lambda} = \langle v_x, id\rangle, v = \{v_x\}_{x\in\Omega}$ is any family of Young measures generated by a sequence in $L_\varphi(\Omega)$ and not a Dirac measure for a.e. $x \in \Omega$.

$$\text{For almost every } x \in \Omega \text{ and } u \in \mathbb{R}^m, \text{ the mapping } F \mapsto f(x,u,F) \text{ is linear.} \tag{1.11}$$

The aim of this paper is to demonstrate the existence of solutions for (1.1), without using the classical strict monotonicity. For example, the hypothesis (1.8) allows to take a potential $W(x,u,F)$, which is only convex but not strictly convex in $F \in \mathbb{M}^{m\times n}$, and to consider (1.1) with $\sigma(x,u,F) = (\partial W/\partial F)(x,u,F)$. Notice that if W is assumed to be strictly convex, then σ becomes strict monotone and the classical monotone method may apply.

Our main existence result is the content of the following theorem whose proof will be given in Section 5.

Theorem 1.1 *Under the hypothesis (1.2)-(1.10), the Dirichlet problem (1.1) has a weak solution $u \in W_0^1 L_\varphi(\Omega; \mathbb{R}^m)$ satisfying*

$$\int_\Omega \sigma(x,u,Du) : Dw dx = \int_\Omega f(x,u,Du) \cdot w dx$$

for all $w \in W_0^1 L_\varphi(\Omega; \mathbb{R}^m)$.

This paper is organized as follows. In section 2, we mainly give the most important and necessary properties and notations of Musielak-Orlicz spaces. In section3, we give the definition of Young measures and some useful general properties Then the section 4 is devoted to the Galerkin approximating sequences, and to identify the limit of gradient sequences by means of Young measure. Finally in section 5 we manage to prove theorem 1.1.

2. Preliminaries

In this subsection, we shall recall only the most important and necessary properties and notations of Musielak-Orlicz spaces, and we refer the reader to (see [7]) for more details. Let Ω be an open subset of $\mathbb{R}^N$, a Musielak-Orlicz function φ is a real-valued function defined in $\Omega \times \mathbb{R}^+$ such that
a) $\varphi(x,.)$ is an N-function for all $x \in \Omega$ (i.e. convex, nondecreasing, continuous, $\varphi(x,0)=0$, $\varphi(x,t)>0$ for all $t>0$ and $\lim\limits_{t\to 0} \sup\limits_{x\in\Omega} \dfrac{\varphi(x,t)}{t} = 0$ and $\lim\limits_{t\to\infty} \inf\limits_{x\in\Omega} \dfrac{\varphi(x,t)}{t} = \infty$).
b) $\varphi(.,t)$ is a measurable function for all $t \geq 0$.
For a Musielak-Orlicz function φ, let $\varphi_x(t) = \varphi(x,t)$ and let φ_x^{-1} be the nonnegative reciprocal function with respect to t, i.e. the function that satisfies

$$\varphi_x^{-1}(\varphi(x,t)) = \varphi\left(x, \varphi_x^{-1}(t)\right) = t.$$

The Musielak-Orlicz function φ is said to satisfy the Δ_2 -condition if for some $k>0$, and a nonnegative function h, integrable in Ω, we have

$$\varphi(x,2t) \leq k\varphi(x,t) + h(x) \text{ for all } x \in \Omega \text{ and } t \geq 0. \tag{2.1}$$

When (2.1) holds only for $t \geq t_0 > 0$, then φ is said to satisfy the Δ_2-condition near infinity. Let φ and γ be two Musielak-Orlicz functions, we say that φ dominate γ and we write $\gamma \prec \varphi$, near infinity (resp. globally) if there exist two positive constants c and t_0 such that for a.e. $x \in \Omega$:

$$\gamma(x,t) \leq \varphi(x,ct) \text{ for all } t \geq t_0, \text{ (resp. for all } t \geq 0 \text{ i.e. } t_0 = 0).$$

We say that γ grows essentially less rapidly than φ at 0 (resp. near infinity) and we write $\gamma \prec\prec \varphi$ if for every positive constant c we have

$$\lim_{t\to 0}\left(\sup_{x\in\Omega} \frac{\gamma(x,ct)}{\varphi(x,t)}\right) = 0, \quad \text{(resp. } \lim_{t\to\infty}\left(\sup_{x\in\Omega} \frac{\gamma(x,ct)}{\varphi(x,t)}\right) = 0).$$

For a Musielak-Orlicz function φ and a measurable function $u : \Omega \longrightarrow \mathbb{R}$, we define the functional

$$\rho_{\varphi,\Omega}(u) = \int_\Omega \varphi(x, |u(x)|)\, dx.$$

The set $K_\varphi(\Omega) = \left\{u : \Omega \longrightarrow \mathbb{R} \text{ measurable}/ \ \rho_{\varphi,\Omega}(u) < \infty\right\}$ is called the Musielak-Orlicz class (or generalized Orlicz class). The Musielak-Orlicz space (the generalized Orlicz spaces) $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is, $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$. Equivalently

$$L_\varphi(\Omega) = \left\{u : \Omega \longrightarrow \mathbb{R} \text{ measurable}/ \ \rho_{\varphi,\Omega}\left(\frac{u}{\lambda}\right) < \infty, \text{ for some } \lambda > 0\right\}$$

For a Musielak-Orlicz function φ we put:

$$\overline{\varphi}(x,s)=\sup_{t>0}\{st-\varphi(x,t)\},$$

Note that $\overline{\varphi}$ is the Musielak-Orlicz function complementary to φ (or conjugate of φ) in the sense of Young with respect to the variable s. In the space $L_\varphi(\Omega)$ we define the following two norms:

$$\|u\|_{\varphi,\Omega}=\inf\left\{\lambda>0/\int_\Omega\varphi\left(x,\frac{|u(x)|}{\lambda}\right)dx\leq 1\right\}$$

which is called the Luxemburg norm and the so-called Orlicz norm by:

$$|||u|||_{\varphi,\Omega}=\sup_{\|v\|_{\overline{\varphi}}\leq 1}\int_\Omega|u(x)v(x)|\,dx$$

where $\overline{\varphi}$ is the Musielak-Orlicz function complementary to φ. These two norms are equivalent (see [7]). The closure in $L_\varphi(\Omega)$ of the bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_\varphi(\Omega)$, It is a separable space (see [7], Theorem 7.10).

We say that sequence of functions $u_n\in L_\varphi(\Omega)$ is modular convergent to $u\in L_\varphi(\Omega)$ if there exists a constant $\lambda>0$ such that

$$\lim_{n\to\infty}\rho_{\varphi,\Omega}\left(\frac{u_n-u}{\lambda}\right)=0.$$

The Musielak-Orlicz space $W^1L_\varphi(\Omega;\mathbb{R}^m)$ is the set of all $u\in L_\varphi(\Omega;\mathbb{R}^m)$ such that $Du\in L_\varphi(\Omega;\mathbb{M}^{m\times n})$, where Du is a matrix-valued function in which all components are distributional partial derivatives of u. It is a Banach space endowed with the norm

$$\|u\|_{W^1L_\varphi(\Omega;\mathbb{R}^m)}:=\|u\|_{1,\varphi}=\|u\|_{L_\varphi(\Omega;\mathbb{R}^m)}+\|Du\|_{L_\varphi(\Omega;\mathbb{M}^{m\times n})}.$$

If φ satisfies the Δ_2-condition, then there exists $\theta>0$ such that for all $u\in W_0^1L_\varphi(\Omega;\mathbb{R}^m)$,

$$\int_\Omega\varphi(x,|u|)\mathrm{d}x\leq\theta\int_\Omega\varphi(x,|Du|)\mathrm{d}x. \tag{2.2}$$

The symbol $\mathcal{C}_0^\infty(\Omega;\mathbb{R}^m)$ means the space of all $\mathcal{C}^\infty$-functions $u:\Omega\to\mathbb{R}^m$ with a compact support in Ω. Note that if $|\Omega|<\infty$ and φ satisfies the Δ_2-condition near infinity, then

$$W_0^1L_\varphi(\Omega;\mathbb{R}^m)=\overline{\mathcal{C}_0^\infty(\Omega;\mathbb{R}^m)}W^1L_\varphi(\Omega;\mathbb{R}^m)$$

and $W^{-1}L_{\bar\varphi}(\Omega;\mathbb{R}^m)=\left(W_0^1L_\varphi(\Omega;\mathbb{R}^m)\right)^*$. Moreover, if $\varphi,\bar\varphi\in\Delta_2$, then the spaces $W^1L_\varphi(\Omega;\mathbb{R}^m)$ and $W^{-1}L_{\bar\varphi}(\Omega;\mathbb{R}^m)$ are reflexive and separable. If we consider $\varphi(t)=|t|^p$ for $p\in(1,\infty)$, then $W_0^1L_\varphi(\Omega;\mathbb{R}^m)=W_0^{1,p}(\Omega;\mathbb{R}^m)$. We say that u_k converges to u for the modular convergence in $W^1L_\varphi(\Omega;\mathbb{R}^m)$ if for some $\beta>0$,

$$\int_\Omega\varphi\left(x,\frac{D^\alpha u_k-D^\alpha u}{\beta}\right)\mathrm{d}x\to 0\text{ as }k\to\infty,\forall|\alpha|\leq 1.$$

Furthermore, if φ satisfies the Δ_2-condition (near infinity only when Ω has finite measure), then modular convergence coïncides with the norm convergence. For two complementary Musielak-Orlicz functions φ and $\overline{\varphi}$, let $u\in L_\varphi(\Omega)$ and $v\in L_{\overline{\varphi}}(\Omega)$, then we have the Hölder inequality (see [7]):

$$\left|\int_\Omega u(x)v(x)\,dx\right|\leq\|u\|_{\varphi,\Omega}|||v|||_{\overline{\varphi},\Omega}. \tag{2.3}$$

Lemma 2.1
Let $u_k:\Omega\to\mathbb{R}^m$ be a measurable sequence. Then u_k converge in modular to u in $L_\varphi(\Omega;\mathbb{R}^m)$ if and only if $u_k\to u$ in measure and there exists some $\gamma>0$ such that $\{\varphi(x,\gamma u_k)\}_k$ is uniformly integrable, i.e.

$$\lim_{L\to\infty}\sup_k\int_{\{x\in\Omega:|\varphi(x,\gamma u_k)|\geq L\}}\varphi(x,\gamma u_k)\,\mathrm{d}x=0.$$

3. A Review on Young Measures

In the following, $\mathcal{C}_0(\mathbb{R}^m)$ denotes the closure of the space of continuous functions on $\mathbb{R}^m$ with compact support with respect to the $\|\cdot\|_\infty$-norm. Its dual space can be identified with $\mathcal{M}(\mathbb{R}^m)$, the space of signed Radon measures with finite mass. The related duality pairing is given by

$$\langle v, f\rangle = \int_{\mathbb{R}^m} f(\lambda) dv(\lambda).$$

We mean by supp v, the support of the Young measure v. Note that if $f \equiv id$ then $\langle v, id\rangle = \displaystyle\int_{\mathbb{R}^m} \lambda dv(\lambda)$. For more details on this notion, the readers are requested to see [15,22,35,43] and other references therein.

Definition 3.1
Assume that the sequence $\{f_j\}_{j\geq 1}$ is bounded in $L^\infty(\Omega; \mathbb{R}^m)$. Then, there exists a subsequence $\{f_k\}_k$ and a Borel probability measure v_x on $\mathbb{R}^m$ for a.e. $x \in \Omega$, such that for almost each $g \in C(\mathbb{R}^m)$, we have

$$g(f_k) \rightarrow^* \bar{g} \text{ weakly in } L^\infty(\Omega),$$

where

$$\bar{g}(x) = \int_{\mathbb{R}^m} g(\lambda) dv_x(\lambda).$$

We call $\{v_x\}_{x\in\Omega}$ the family of Young measure associated with the subsequence $\{f_k\}_k$.

Remark 3.1
The notion of gradient Young measures arises when we replace f_k in the above definition by ∇u_k where $u_k : \Omega \to \mathbb{R}^m$. In this case, ∇u_k take their values in $\mathbb{M}^{m\times n}$.

We are now ready to begin state the fundamental lemma on Young measures as following:

Lemma 3.1 *[16]*
Let $\Omega \subset \mathbb{R}^n$ be Lebesgue measurable (not necessarily bounded) and $w_j : \Omega \to \mathbb{R}^m, j = 1, ..$ be a sequence of Lebesgue measurable functions. Then, there exists a subsequence w_k and a family $\{v_x\}_{x\in\Omega}$ of nonnegative Radon measures on $\mathbb{R}^m$, such that

(i) $\|v_x\|_{\mathcal{M}(\mathbb{R}^m)} := \displaystyle\int_{\mathbb{R}^m} dv_x \leq 1$ for almost $x \in \Omega$.

(ii) $\varphi(w_k) \rightarrow^* \bar{\varphi}$ weakly in $L^\infty(\Omega)$ for all $\varphi \in \mathcal{C}_0(\mathbb{R}^m)$, where $\bar{\varphi}(x) = \langle v_x, \varphi\rangle$.

(iii) If for all $R > 0$

$$\lim_{L\to\infty} \sup_k |\{x \in \Omega \cap B_R(0) : |w_k(x)| \geq L\}| = 0, \tag{3.1}$$

then $\|v_x\| = 1$ for a.e. $x \in \Omega$, and for all measurable $\Omega' \subset \Omega$ there holds $\varphi(w_k) \to \bar{\varphi} = \langle v_x, \varphi\rangle$ weakly in $L^1(\Omega')$ for a continuous function φ provided the sequence $\varphi(w_k)$ is weakly precompact in $L^1(\Omega')$.

Remark 3.2
(a) In [16], it is shown that under hypothesis (3.1) for any measurable $A \subset \Omega$,

$$g(., u_k) \to \langle v_x, g(x, .)\rangle \quad \text{in } L^1(A),$$

for every Carathéodory function $g : A \times \mathbb{R}^m \to \mathbb{R}$ such that $\{g(., u_k)\}$ is sequentially weakly relative compact in $L^1(A)$.

(b) A family $\{v_x\}$ satisfying (i)-(iii) always exists and v_x is a probability measure if Eq. (3.1) holds.

The following lemmas are useful for us and can be considered as the applications of Lemma 3.2

Lemma 3.2 *[26]*
(i) If $|\Omega|<\infty$ and v_x is the Young measure generated by the (whole) sequence u_k, then there holds

$$u_k \to u \text{ in measure } \Leftrightarrow v_x=\delta_{u(x)} \text{ for a.e. } x\in\Omega. \tag{3.2}$$

(ii) If the sequences $u_k:\Omega\to\mathbb{R}^m$ and $v_k:\Omega\to\mathbb{R}^d$ generate the Young measures $\delta_{u(x)}$ and v_x, respectively, then (u_k,v_k) generates the Young measure $\delta_{u(x)}\otimes v_x$.

Lemma 3.3 *([28] Generalized Fatou's Lemma)*
Let $F:\Omega\times\mathbb{R}^m\times\mathbb{M}^{m\times n}\to\mathbb{R}$ be a Carathéodory function and $\{u_k\}$ be a sequence of measurable functions, where $u_k:\Omega\to\mathbb{R}^m$, such that $u_k\to u$ in measure and such that Du_k generates the Young measure v_x. Then,

$$\liminf_{k\to\infty}\int_\Omega F\left(x,u_k,Du_k\right)dx\geq\int_\Omega\int_{\mathbb{M}^{m\times n}}F(x,u,\lambda)dv_x(\lambda)dx$$

provided that the negative part $F^-\left(x,u_k,Du_k\right)$ is equiintegrable.

4. Existence of weak solutions

4.1. Galerkin approximation

Let $V_1\subset V_2\subset\cdots\subset W_0^1L_\varphi\left(\Omega;\mathbb{R}^m\right)$ be a sequence of finite dimensional subspaces with the property that $\cup_{i\in\mathbb{N}}V_i$ is dense in $W_0^1L_\varphi\left(\Omega;\mathbb{R}^m\right)$. Note that (V_i) exist since $W_0^1L_\varphi\left(\Omega;\mathbb{R}^m\right)$ is separable. We define the operator

$$\begin{aligned}&T:W_0^1L_\varphi\left(\Omega;\mathbb{R}^m\right)\to W^{-1}L_{\bar\varphi}\left(\Omega;\mathbb{R}^m\right)\\&\qquad u\mapsto\left(w\mapsto\int_\Omega\sigma(x,u(x),Du(x)):Dw\ \mathrm{d}x-\int_\Omega f(x,u(x),Du(x)).w\ \mathrm{d}x\right),\end{aligned}$$

where $\langle .,\ .\ \rangle$ denotes the dual pairing of $W^{-1}L_{\bar\varphi}\left(\Omega;\mathbb{R}^m\right)$ and $W_0^1L_\varphi\left(\Omega;\mathbb{R}^m\right)$. The following assertions allow the construction of the approximating solutions.

4.2. Step I

: The operator $T(u)$ is linear, well defined and bounded for arbitrary $u\in W_0^1L_\varphi\left(\Omega;\mathbb{R}^m\right)$

Trivially $T(u)$ is linear. By the growth condition in (1.3), together with Holder inequality, we have we get

$$\begin{aligned}&\int_\Omega\sigma(x,u,Du):Dwdx\\&\quad\leq2\left(\int_\Omega\bar\varphi(x,|\sigma(x,u,Du)|)dx\right)\left(\int_\Omega\varphi(x,|Dw|)dx\right)\\&\quad\leq c\left(\int_\Omega\bar\varphi\left(x,d_1(x)\right)+P(x,\gamma|u|)+\varphi(x,\gamma|Du|)dx\right)\left(\int_\Omega\varphi(x,|Dw|)dx\right)\end{aligned}$$

and similarly

$$\begin{aligned}&\int_\Omega f(x,u,Du)\cdot wdx\\&\quad\leq c\left(\int_\Omega\bar\varphi\left(x,d_3(x)\right)+P(x,\gamma|u|)+\varphi(x,\gamma|Du|)dx\right)\left(\int_\Omega\varphi(x,|w|)dx\right)\end{aligned}$$

where c is a positive constant. Since $u,w\in W_0^1L_\varphi\left(\Omega;\mathbb{R}^m\right),\varphi\in\Delta_2,P\ll\varphi$ and (2.2), we can infer from the above inequalities that $T(u)$ is bounded.

4.3. Step II

The restriction of T to a finite subspace V of $W_0^1 L_\varphi(\Omega;\mathbb{R}^m)$ is continuous.

Proof Let V be a subspace of $W_0^1 L_\varphi(\Omega;\mathbb{R}^m)$ with $\dim V = r$ and $(w_i)_{i=1}^r$ a basis of V. Let $\left(u_k = a_k^i w_i\right)$ be a sequence in V which converges to $u = a^i w_i$ in V (with conventional summation). Then on the one hand the sequence (a_k) converges to a in $\mathbb{R}^r$ and so $u_k \to u$ and $Du_k \to Du$ almost everywhere. On the other hand $\|u_k\|_{\varphi,\Omega}$ and $\|Du_k\|_{\varphi,\mathbb{M}^{m\times n}}$ are bounded by a constant C. By the continuity condition (1.2), it follows that $\sigma(x,u_k,Du_k) : Dw \to \sigma(x,u,Du) : Dw$ and $f(x,u_k,Du_k)$. $w \to f(x,u,Du)\cdot w$ almost everywhere. Let $\Omega' \subset \Omega$ be a measurable subset and $w \in W_0^1 L_\varphi(\Omega;\mathbb{R}^m)$. From the inequalities in the proof of Step 4.2 together with the Eq. (2.2), we obtain

$$\begin{aligned}&\int_{\Omega'} |\sigma(x,u_k,Du_k) : Dw|\, dx \\ &\quad \le c(\|d_1\|_\varphi + \theta \underbrace{\|Du_k\|_{\varphi,\mathbb{M}^{m\times n}}}_{\le C} + \underbrace{\|Du_k\|_{\varphi,\mathbb{M}^{m\times n}}}_{\le C}) \left(\int_{\Omega'} \varphi(x,|Dw|)dx\right)\end{aligned}$$

and

$$\begin{aligned}&\int_{\Omega'} |f(x,u_k,Du_k)\cdot w|\, dx \\ &\quad \le c\theta(\|d_3\|_\varphi + \theta \underbrace{\|Du_k\|_{\varphi,\mathbb{M}^{m\times n}}}_{\le C} + \underbrace{\|Du_k\|_{\varphi,\mathbb{M}^{m\times n}}}_{\le C}) \left(\int_{\Omega'} \varphi(x,|Dw|)dx\right)\end{aligned}$$

by the Holder inequality. Note that $\left(\int_{\Omega'} \varphi(x,|Dw|)dx\right)$ is arbitrary small if the measure of Ω' is chosen small enough. As a consequence, the sequences $(\sigma(x,u_k,Du_k) : Dw)$

and $(f(x,u_k,Du_k)\cdot w)$ are equiintegrable. Applying the Vitali Theorem, it follows that for all $w \in W_0^1 L_\varphi(\Omega;\mathbb{R}^m)$ we have $\lim_{k\to\infty} \langle T(u_k), w\rangle = \langle T(u), w\rangle$.

Now, let fix some k and assume that $w_1,\ldots,w_r$ is a basis of V_k with $\dim V_k = r$. In the following lemma, we write for simplicity $\sum_{1\le i\le r} a^i w_i = a^i w_i$ and we define the map

$$\begin{aligned}G &: \mathbb{R}^r \to \mathbb{R}^r \\ (a^1,\ldots,a^r) &\mapsto \left(\langle T(a^i w_i), w_j\rangle\right)_{j=1}^r\end{aligned}$$

4.4. Step III

G is continuous and $G(a)\cdot a \to \infty$ as $\|a\|_{\mathbb{R}^r} \to \infty$ (the dot $\cdot$ is the inner product in $\mathbb{R}^r$).

Proof Let $u_k = a_k^i w_i, u_0 = a_0^i w_i \in V_k$. Then $\|a_k\|_{\mathbb{R}^r}$ (resp. $\|a_0\|_{\mathbb{R}^r}$) is equivalent to $\|u_k\|_{1,\varphi}$ (resp. $\|u_0\|_{1,\varphi}$). On the one hand, by Holder inequality, we have

$$\begin{aligned}\left|(G(a_k) - G(a))_j\right| &= \left|\langle T(a_k^i w_i) - T(a_0^i w_i), w_j\rangle\right| \\ &\le 2\|T(u_k) - T(u_0)\|_{-1,\bar\varphi} \|w_j\|_{1,\varphi}\end{aligned}$$

By vertue of Step 4.3, the continuity of G follows. On the other hand, consider

$u = a^i w_i \in V_k$, then $\|a\|_{\mathbb{R}^r} \to \infty$ is equivalent to $\|u\|_{1,\varphi} \to \infty$ and

$$G(a)\cdot a = \langle T(a^i w_i), a^i w_i\rangle = \langle T(u), u\rangle.$$

From the coercivity condition in (1.5), we obtain

$$\begin{aligned}G(a)\cdot a = \langle T(u), u\rangle &= \int_\Omega (\sigma(x,u,Du) : Du - f(x,u,Du)\cdot u)dx \\ &\ge \alpha \int_\Omega \varphi\left(x, \frac{|Du|}{\beta}\right) dx - \int_\Omega d_2(x)dx \\ &\ge c\|Du\|_{\varphi,\mathbb{M}^{m\times n}} + c.\end{aligned}$$

Therefore $\langle T(u), u\rangle \rightarrow \infty$ as $\|u\|_{1,\varphi} \rightarrow \infty$ Now, we can construct the sequence of approximating solutions in the following way: According to Lemma (4.2), there exists $R>0$ such that for all $a \in \partial B_R(0) \subset \mathbb{R}^r$ we have $G(a).a>0$. Thanks to the standard method used in [36], we obtain the existence of $x \in B_R(0)$ such that $G(x)=0$. Consequently, for all k there exists $u_k \in V_k$ such that

$$\langle T\left(u_k\right), w\rangle=0 \text{ for all } w \in V_k. \tag{4.1}$$

4.5. Weak limits by gradient Young measures

The following lemma is concerned with the weak limit points of gradient sequences by means of the Young measure in Musielak- orlicz spaces.

Lemma 4.1
(i) If the sequence $\left\{Du_k\right\}_k$ is bounded in $L_{\varphi}(\Omega;\mathbb{M}^{m\times n})$, then there is a Young measure v_x generated by $\{Du_k\}$ satisfying $\|v_x\|_{\mathcal{M}(\mathbb{M}^{m\times n})}=1$ and the weak L^1-limit of Du_k is $\displaystyle\int_{M^{m\times n}} \lambda \mathrm{d}v_x(\lambda)$.
(ii) For almost every $x\in\Omega, v_x$ satisfies

$$\langle v_x, id\rangle = Du(x) \quad \textit{for a.e. } x\in\Omega.$$

Proof:
(i) It is sufficient to show that $\{Du_k\}$ satisfies Equation (3.1) in Lemma 3.1 Putting

$$\delta(L)=\min_{|\xi|=L}(\varphi(x,\xi)/|\xi|).$$

We have $\langle T(u), u\rangle \rightarrow \infty$ as $\|u\|_{1,\varphi} \rightarrow \infty$, then there exists $R>0$ with the property, that $\langle T(u), u\rangle>1$ whenever $\|u\|_{1,\varphi}>R$. Hence, for the sequence of the Galerkin approximations $u_k \in V_k$ constructed in Section 3, which satisfy $\langle T\left(u_k\right), u_k\rangle=0$, we obtain the uniform bound

$$\left\|u_k\right\|_{1,\varphi} \leq R \quad \text{ for all } k \tag{4.2}$$

Then there is $c\geq 0$ such that for any $r>0$,

$$\begin{aligned}
c \geq \int_{\Omega} \varphi\left(x,\left|Du_k\right|\right) \mathrm{d}x & \geq \int_{\left[x\in\Omega\cap B_r(0):\left|Du_k(x)\right|\geq L\right\}} \varphi\left(x,\left|Du_k\right|\right) \mathrm{d}x \\
& \geq \delta(L) \int_{\left\{x\in\Omega\cap B_r(0):\left|Du_k(x)\right|\geq L\right\}}\left|Du_k\right| \mathrm{d}x \\
& \geq L\delta(L)\left|\left\{x\in\Omega\cap B_r(0):\left|Du_k(x)\right|\geq L\right\}\right|
\end{aligned}$$

Thus

$$\sup_k\left|\left\{x\in\Omega\cap B_r(0):\left|Du_k(x)\right|\geq L\right\}\right| \leq \frac{c}{L\delta(L)} \longrightarrow 0 \quad \text{ as } L\rightarrow\infty.$$

By Lemma 3.1 (iii), it follows that $\|v_x\|_{\mathcal{M}}=1$ (i.e. v_x is a probability measure). As $L_{\varphi}\left(\Omega;\mathbb{M}^{m\times n}\right)$ is reflexive $\left(\mathbb{M}^{m\times n}\cong\mathbb{R}^{mn}\right)$, then there is a subsequence (still denoted by $\{Du_k\}$) weakly convergent in $L_{\varphi}\left(\Omega;\mathbb{M}^{m\times n}\right)\subset L^1\left(\Omega;\mathbb{M}^{m\times n}\right)$, thus weakly convergent in $L^1\left(\Omega;\mathbb{M}^{m\times n}\right)$. Return to Lemma 3.1 and take φ as the identity mapping, we obtain then

$$Du_k \rightarrow \langle v_x, id\rangle=\int_{\mathbb{M}^{m\times n}} \lambda dv_x(\lambda) \quad \text{ weakly in } L^1\left(\Omega;\mathbb{M}^{m\times n}\right)$$

(ii) By (4.2), we have $u_k\rightarrow u$ in $W_0^1L_{\varphi}\left(\Omega;\mathbb{R}^m\right)$ and $u_k\rightarrow u$ in $L_{\varphi}\left(\Omega;\mathbb{R}^m\right)$ (for a subsequence), thus

$$Du_k \rightarrow Du \quad \text{ in } L_{\varphi}\left(\Omega;\mathbb{M}^{m\times n}\right)$$

Or $L_{\varphi}\left(\Omega;\mathbb{M}^{m\times n}\right)\subset L^1\left(\Omega;\mathbb{M}^{m\times n}\right)$, then $Du_k\rightarrow Du$ in $L^1\left(\Omega;\mathbb{M}^{m\times n}\right)$. Owing to (i), we conclude by the uniqueness of limit that

$$Du(x)=\langle v_x, id\rangle \quad \text{ for a.e. } x\in\Omega$$

The following lemma is the main technical to pass to the limit in the approximating equations and to prove that the weak limit u of the Galerkin approximations u_k is indeed a solution of (1.1).

Lemma 4.2 *(div-curl inequality):*
The Young measure v_x generated by the gradient Du u_k is satisfying the following inequality:

$$\int_{\Omega}\int_{M^{m\times n}}(\sigma(x,u,\lambda)-\sigma(x,u,Du)):(\lambda-Du)\mathrm{d}v_x(\lambda)\mathrm{d}x\leq 0. \tag{4.3}$$

Proof: Let us consider the sequence

$$\begin{aligned} I_k &:= (\sigma(x,u_k,Du_k)-\sigma(x,u,Du)):(Du_k-Du)\\ &= \sigma(x,u_k,Du_k):(Du_k-Du)-\sigma(x,u,Du):(Du_k-Du)\\ &=: I_{k,1}+I_{k,2}. \end{aligned}$$

By the growth condition in (1.3)

$$\int_{\Omega}\bar{\varphi}(x,|\sigma(x,u,Du)|)dx\leq c\int_{\Omega}(\bar{\varphi}(x,d_1(x))+P(x,\gamma|u|)+\varphi(x,\gamma|Du|))\,dx<\infty$$

Since $u\in W_0^1L_{\varphi}(\Omega;\mathbb{R}^m)$, $P\ll\varphi$ and by the equation (2.2), it follows that $\sigma\in L_{\bar{\varphi}}(\Omega;\mathbb{M}^{m\times n})$. According to a weak convergence defined in Lemma 4.1 , we obtain that

$$\liminf_{k\to\infty}\int_{\Omega}I_{k,2}dx=\int_{\Omega}\sigma(x,u,Du):\left(\int_{\mathbb{M}^{m\times n}}\lambda dv_x(\lambda)-Du\right)dx=0$$

We have $(\sigma(x,u_k,Du_k):Du)^-$ is equiintegrable (see the proof of Lemma Step 4.3 if necessary). The sequence $(\sigma(x,u_k,Du_k):Du_k)^-$is easily seen to be equiintegrable. Indeed, by the coercivity condition in 1.5, we have

$$\begin{aligned} \sigma(x,u_k,Du_k):Du_k &\geq f(x,u_k,Du_k)\cdot u_k+\alpha\varphi\left(x,\frac{|Du_k|}{\beta}\right)-d_2(x)\\ &\geq \alpha\varphi\left(x,\frac{|Du_k|}{\beta}\right)-d_2(x) \end{aligned}$$

where we have used the sign condition $f(x,u_k,Du_k)\cdot u_k\geq 0$. Therefore

$$\begin{aligned} &\int_{\Omega^l}|\min(\sigma(x,u_k,Du_k):Du_k,0)|\,dx\\ &\quad\leq\int_{\Omega'}|d_2(x)|\,dx+\alpha\int_{\Omega'}\varphi\left(\frac{x,|Du_k|}{\beta}\right)dx<\infty \end{aligned}$$

We have by (4.2), $u_k\to u$ in $L_{\varphi}(\Omega;\mathbb{R}^m)$, and by vertue of Lemma 2.1 , we get $u_k\to u$ in measure. Hence, we may use lemma 3.3 which gives

$$I:=\liminf_{k\to\infty}\int_{\Omega}I_{k,1}dx\geq\int_{\Omega}\int_{\mathbb{M}^{m\times n}}\sigma(x,u,\lambda):(\lambda-Du)dv_x(\lambda)dx.$$

Now, we prove that $I\leq 0$. According to Mazur's theorem (see, e.g., [[40], Theorem 2 , page 120]) there exists a sequence v_k in $W_0^1L_{\varphi}(\Omega;\mathbb{R}^m)$ where each v_k is a convex linear combination of $\{u_1,\ldots,u_k\}$ such that $v_k\to u$ in $W_0^1L_{\varphi}(\Omega;\mathbb{R}^m)$. This significant that v_k belongs to the same space V_k as u_k. By taking u_k-v_k as a test function in (4.1), we obtain

$$\int_{\Omega}\sigma(x,u_k,Du_k):(Du_k-Dv_k)\,dx=\int_{\Omega}f(x,u_k,Du_k)\cdot(u_k-v_k)\,dx \tag{4.4}$$

From the growth condition in (1.4) and the Holder inequality, it follows that

$$\begin{aligned} &\left|\int_{\Omega}f(x,u_k,Du_k)\cdot(u_k-v_k)\,dx\right|\\ &\quad\leq c\left(\int_{\Omega}\bar{\varphi}(x,d_3(x))+P(x,\gamma|u_k|)+\varphi(x,\gamma|Du_k|)\,dx\right)\left(\int_{\Omega}\varphi(x,|u_k-v_k|)\,dx\right). \end{aligned}$$

The right hand side of this inequality vanishes as $k \to \infty$, since by the construction of v_k, we have

$$\|u_k - v_k\|_{\varphi,\Omega} \le \|u_k - u\|_{\varphi,\Omega} + \|v_k - u\|_{\varphi,\Omega} \to 0 \quad \text{as} \quad k \to \infty$$

Hence, the left hand side in (4.4) tends to zero as $k \to \infty$. Using this result and the fact that $v_k \to u$ in $W_0^1 L_\varphi(\Omega;\mathbb{R}^m)$ to deduce the following

$$\begin{aligned}
I &= \liminf_{k\to\infty} \int_\Omega \sigma(x,u_k,Du_k) : (Du_k - Du)\,dx \\
&= \liminf_{k\to\infty} \left(\int_\Omega \sigma(x,u_k,Du_k) : (Du_k - Dv_k)\,dx + \int_\Omega \sigma(x,u_k,Du_k) : (Dv_k - Du)\,dx \right) \\
&= \liminf_{k\to\infty} \int_\Omega \sigma(x,u_k,Du_k) : (Dv_k - Du)\,dx \\
&\le \liminf_{k\to\infty} c\,\||\sigma(x,u_k,Du_k)|\|_{\bar{\varphi},\mathbb{M}^{m\times n}} \|v_k - u\|_{1,\varphi} = 0.
\end{aligned}$$

In view of Lemma 4.1, we have

$$\begin{aligned}
&\int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,Du) : (\lambda - Du) dv_x(\lambda) dx \\
&\quad = \int_\Omega \sigma(x,u,Du) : \left(\int_{\mathbb{M}^{m\times n}} \lambda dv_x(\lambda) - Du \right) dx = 0
\end{aligned}$$

and together with $I \le 0$ we finish the proof of Lemma 4.2.

Remark 4.1 For almost every $x \in \Omega$, we have

$$(\sigma(x,u,\lambda) - \sigma(x,u,Du)) : (\lambda - Du) = 0 \text{ on } \operatorname{supp} v_x.$$

Proof
We have by Lemma 4.2

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} (\sigma(x,u,\lambda) - \sigma(x,u,Du)) : (\lambda - Du) dv_x(\lambda) dx \le 0$$

We infer from the monotonicity of σ that the integrand in the above inequality is nonnegative. Thus, must vanish with respect to the product measure $dv_x(\lambda) \otimes dx$. It follows that for almost every $x \in \Omega$

$$(\sigma(x,u,\lambda) - \sigma(x,u,Du)) : (\lambda - Du) = 0 \text{ on } \operatorname{supp} v_x$$

5. Proof of Main result

This section is devoted to prove the existence result for (1.1). We consider the following steps
Step I: In this step we try to sow that

$$\int_\Omega (\sigma(x,u_k,Du_k) : Dw - \sigma(x,u,Du) : Dw)\,dx \to 0 \quad \text{as} \quad k \to \infty.$$

By strict monotonicity and remark4.1 implies that supp $v_x = \{Du(x)\}$, then $v_x = \delta_{Du(x)}$ for almost every $x \in \Omega$. By using (3.2) we obtain $Du_k \to Du$ in measure. Furthermore, since $u_k \to u$ in measure, we deduce that $u_k \to u$ and $Du_k \to Du$ for almost every $x \in \Omega$ (for a subsequence). By continuity condition (1.2), we have $\sigma(x,u_k,Du_k) \to \sigma(x,u,Du)$ almost everywhere in Ω. Since $\sigma(x,u_k,Du_k)$ is equiintegrable (see the proof of Step II of section 4), the Vitali theorem give us

$$\int_\Omega (\sigma(x,u_k,Du_k) : Dw - \sigma(x,u,Du) : Dw)\,dx \to 0 \quad \text{as} \quad k \to \infty.$$

Step II: Suppose that v_x is not a Dirac measure on a set $x \in \Omega'$ of positive Lebesgue measure $|\Omega'| > 0$. one has $\bar{\lambda} = \langle v_x, id \rangle = Du(x)$, it follows that

$$\begin{aligned}
&\int_{\mathbb{M}^{m\times n}} \sigma(x,u,\bar{\lambda}) : (\lambda - \bar{\lambda}) dv_x(\lambda) \\
&= \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\bar{\lambda}) : \lambda dv_x(\lambda) - \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\bar{\lambda}) : \bar{\lambda} dv_x(\lambda) \\
&= \sigma(x,u,\bar{\lambda}) : \int_{\mathbb{M}^{m\times n}} \lambda dv_x(\lambda) - \sigma(x,u,\bar{\lambda}) : \bar{\lambda} \int_{\mathbb{M}^{m\times n}} dv_x(\lambda) = 0
\end{aligned}$$

We get by using the strict M-quasimonotonicity of σ that

$$\int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) : \lambda dv_x(\lambda) > \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) : \bar{\lambda} dv_x(\lambda) \tag{5.1}$$

Now by using Lemma 4.2 and integrating (5.1) over Ω, we can have

$$\begin{aligned}
\int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) : \lambda dv_x(\lambda) dx &> \int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) : \bar{\lambda} dv_x(\lambda) dx \\
&\geq \int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) : \lambda dv_x(\lambda) dx,
\end{aligned}$$

which is a contradiction.

Hence v_x is a Dirac measure and we can assume that $v_x = \delta_{h(x)}$.

Then

$$h(x) = \int_{\mathbb{M}^{m\times n}} \lambda d\delta_{h(x)}(\lambda) = \int_{\mathbb{M}^{m\times n}} \lambda dv_x(\lambda) = Du(x)$$

Thus

$$v_x = \delta_{Du(x)}$$

.

By (3.2) It follows that $Du_k \to Du$ in measure for $k \to \infty$. The remainder of the proof in this step is similar as in **step I**.

Step III:

We claim that for almost $x \in \Omega$ and all $F \in \mathbb{M}^{m\times n}$

$$\sigma(x,u,\lambda) : F = \sigma(x,u,Du) : F + (\nabla\sigma(x,u,Du)F) : (Du - \lambda)$$

holds on $\operatorname{supp} v_x$. By the monotonicity of σ we have for all $\tau \in \mathbb{R}$

$$(\sigma(x,u,\lambda) - \sigma(x,u,Du + \tau F)) : (\lambda - Du - \tau F) \geq 0 \tag{5.2}$$

By vertue of remark 4.1 together with the following equality

$$\sigma(x,u,Du + \tau F) = \sigma(x,u,Du) + \nabla\sigma(x,u,Du)\tau F + o(\tau)$$

we conclude from (5.2),

$$\begin{aligned}
-\sigma(x,u,\lambda) : \tau F &\geq -\sigma(x,u,Du) : (\lambda - Du) + \sigma(x,u,Du + \tau F) : (\lambda - Du - \tau F) \\
&= \tau((\nabla\sigma(x,u,Du)F) : (\lambda - Du) - \sigma(x,u,Du) : F) + o(\tau).
\end{aligned}$$

Since τ is arbitrary in $\mathbb{R}$, our claim follows. The equiintegrability of $\sigma(x, u_k, Du_k)$ implies that its weak L^1-limit is given by

$$\begin{aligned}
\bar{\sigma}(x) &:= \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) dv_x(\lambda) \\
&= \int_{\operatorname{supp} v_x} \sigma(x,u,Du) dv_x(\lambda) + (\nabla\sigma(x,u,Du))^t \int_{\operatorname{supp} v_x} (Du - \lambda) dv_x(\lambda) \\
&= \sigma(x,u,Du)
\end{aligned}$$

where we have used our claim and $Du(x) = \int_{\mathbb{M}^{m\times n}} \lambda d v_x(\lambda)$. Since $L_{\bar{\varphi}}(\Omega, \mathbb{M}^{m\times n})$ is reflexive, thus $\sigma(x, u_k, Du_k)$ is weakly convergent in $L_{\varphi}(\Omega, \mathbb{M}^{m\times n})$ and its weak $L_{\bar{\varphi}}$-limit is also $\sigma(x, u, Du)$. For arbitrary $w \in W_0^1 L_{\varphi}(\Omega; \mathbb{R}^m)$, we deduce that

$$\int_{\Omega} (\sigma(x, u_k, Du_k) : Dw - \sigma(x, u, Du) : Dw)\, dx \to 0 \text{ as } k \to \infty.$$

Step IV :

We start by showing that for almost every $x \in \Omega$, $\operatorname{supp} v_x \subset K_x$, where

$$K_x = \left\{\lambda \in \mathbb{M}^{m\times n} : W(x, u, \lambda) = W(x, u, Du) + \sigma(x, u, Du) : (\lambda - Du)\right\}. \tag{5.3}$$

If $\lambda \in \operatorname{supp} v_x$ then by remark 4.1

$$(1-\tau)(\sigma(x, u, \lambda) - \sigma(x, u, Du)) : (\lambda - Du) = 0 \quad \text{for all} \quad \tau \in [0,1] \tag{5.4}$$

Now, by thanking to 5.4 and using monotonicity condition, we have for $\tau \in [0,1]$ that

$$\begin{aligned} 0 &\le (1-\tau)(\sigma(x, u, Du + \tau(\lambda - Du)) - \sigma(x, u, \lambda)) : (Du - \lambda) \\ &= (1-\tau)(\sigma(x, u, Du + \tau(\lambda - Du)) - \sigma(x, u, Du)) : (Du - \lambda). \end{aligned} \tag{5.5}$$

On the other hand, monotonicity condition implies that

$$(\sigma(x, u, Du + \tau(\lambda - Du)) - \sigma(x, u, Du)) : \tau(\lambda - Du) \ge 0$$

thus,

$$(\sigma(x, u, Du + \tau(\lambda - Du)) - \sigma(x, u, Du)) : (1-\tau)(\lambda - Du) \ge 0$$

since $\tau \in [0,1]$. The reverse inequality on the right hand in 5.5 holds and we can deduce that

$$(\sigma(x, u, Du + \tau(\lambda - Du)) - \sigma(x, u, Du)) : (\lambda - Du) = 0,$$

i.e.

$$\sigma(x, u, Du + \tau(\lambda - Du)) : (\lambda - Du) = \sigma(x, u, Du) : (\lambda - Du) \tag{5.6}$$

for all $\tau \in [0,1]$. Integrating 5.6 over $[0,1]$ and using the following equality

$$\sigma(x, u, Du + \tau(\lambda - Du)) : (\lambda - Du) = \frac{\partial W}{\partial \tau}(x, u, Du + \tau(\lambda - Du)) : (\lambda - Du),$$

we may have

$$\begin{aligned} W(x, u, \lambda) &= W(x, u, Du) + \int_0^1 \sigma(x, u, Du + \tau(\lambda - Du)) : (\lambda - Du) d\tau \\ &= W(x, u, Du) + \sigma(x, u, Du) : (\lambda - Du) \end{aligned}$$

Thus we may deduce that $\lambda \in K_x$, i.e. $\operatorname{supp} v_x \subset K_x$. Using the convexity of W we can get

$$\underbrace{W(x, u, \lambda)}_{=:A(\lambda)} \ge \underbrace{W(x, u, Du) + \sigma(x, u, Du) : (\lambda - Du)}_{=:B(\lambda)}$$

for all $\lambda \in \mathbb{M}^{m\times n}$. Since the mapping $\lambda \mapsto A(\lambda)$ is continuously differentiable, then for every $F \in \mathbb{M}^{m\times n}, \tau \in \mathbb{R}$

$$\begin{aligned} \frac{A(\lambda + \tau F) - A(\lambda)}{\tau} &\ge \frac{B(\lambda + \tau F) - B(\lambda)}{\tau} \text{ if } \tau > 0 \\ \frac{A(\lambda + \tau F) - A(\lambda)}{\tau} &\le \frac{B(\lambda + \tau F) - B(\lambda)}{\tau} \text{ if } \tau < 0. \end{aligned}$$

Hence $DA = DB$ and we obtain

$$\sigma(x,u,\lambda)=\sigma(x,u,Du) \text{ for all } \lambda\in K_x\supset \operatorname{supp} v_x$$

and thus

$$\bar{\sigma}(x)=\int_{\mathbb{M}^{m\times n}}\sigma(x,u,\lambda)dv_x(\lambda)=\sigma(x,u,Du).$$

Now we choose the following Carathéodory function

$$h(x,\rho,\lambda)=|\sigma(x,\rho,\lambda)-\bar{\sigma}(x)|,\quad \rho\in\mathbb{R}^m,\quad \lambda\in\mathbb{M}^{m\times n}.$$

Since $\sigma\left(x,u_k,Du_k\right)$ is equiintegrable, then $h_k(x):=h\left(x,u_k(x),Du_k(x)\right)$ is equiintegrable and its weak L^1-limit is given by

$$h_k\rightarrow\bar{h} \text{ in } L^1(\Omega)$$

where

$$\begin{aligned}\bar{h}(x)&=\int_{\mathbb{R}^m\times\mathbb{M}^{m\times n}}|\sigma(x,\rho,\lambda)-\bar{\sigma}(x)|d\delta_{u(x)}(\rho)\otimes dv_x(\lambda)\\&=\int_{\operatorname{supp} v_x}|\sigma(x,u,\lambda)-\bar{\sigma}(x)|dv_x(\lambda)=0\quad \text{(by (13))}.\end{aligned}$$

Since $h_k\geq 0$ it follows that

$$h_k\rightarrow 0 \text{ strongly in } L^1(\Omega).$$

We can conclude from the previous steps that

$$\lim_{k\rightarrow\infty}\int_\Omega\sigma\left(x,u_k,Du_k\right):Dw(x)=\int_\Omega\sigma(x,u,Du):Dw(x)dx\quad\forall w\in\bigcup_{k=1}^{\infty}V_k.$$

Since $u_k\rightarrow u$ and $Du_k\rightarrow Du$ almost everywhere for $k\rightarrow\infty$.

Furthermore, by continuity condition 1.2 we may infer that

$f\left(x,u_k,Du_k\right)\cdot w(x)\rightarrow f(x,u,Du)\cdot w(x)$ for arbitrary $w\in W_0^1L_\varphi\left(\Omega;\mathbb{R}^m\right)$. Thanking to the growth condition 1.4, we get$\left(f\left(x,u_k,Du_k\right)\cdot w(x)\right)$ is equi-integrable,

thus by using the Vitali Convergence Theorem we obtain that $f\left(x,u_k,Du_k\right)\cdot w(x)\rightarrow f(x,u,Du)\cdot w(x)$ in $L^1(\Omega)$.

Consequently, we have

$$\lim_{k\rightarrow\infty}\int_\Omega f\left(x,u_k,Du_k\right)\cdot w(x)=\int_\Omega f(x,u,Du)\cdot w(x)dx\quad\forall w\in\bigcup_{k=1}^{\infty}V_k.$$

Now, if $F\mapsto f(x,u,F)$ is linear, we argue as follows:

$$\begin{aligned}f\left(x,u_k,Du_k\right)\rightarrow\int_{\mathbb{M}^{m\times n}}f(x,u,\lambda)dv_x(\lambda)&=\sigma(x,u,.)o\int_{\mathbb{M}^{m\times n}}\lambda dv_x(\lambda)\\&=\sigma(x,u,Du)\end{aligned}$$

where we have used $Du(x)=\langle v_x,id\rangle$ and the equii-ntegrability of $f\left(x,u_k,Du_k\right)$.

Finally, since $\cup_{k=1}^{\infty}V_k$ is dense in $W_0^1L_\varphi\left(\Omega;\mathbb{R}^m\right)$, the Dirichlet problem (1.1) admit a weak solution $u\in W_0^1L_\varphi\left(\Omega;\mathbb{R}^m\right)$.

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