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Approximation of Signals Belonging to $W'(L^p, \xi(t))$ Class by Generalized $(C^{\alpha,\eta}.E^1.E^1)$ Means

Smita Sonker, Paramjeet Sangwan, Bidu Bhusan Jena and Susanta Kumar Paikray*

ABSTRACT: Approximation of signals has always been of great importance in the field of science and engineering due basically to the fact that it has a wide range of applications in signal analysis, system design in modern telecommunications, radar and image processing systems. In this paper, we introduce and study the notion of $(C^{\alpha,\eta}.E^1.E^1)$ product means of conjugate Fourier series for approximation of signals. Based on this potential notion, we establish and prove various new theorems under certain weaker conditions. Finally, we present the concluding remarks which exhibit the effectiveness of our findings.

Key Words: Signal approximation, Hölder's inequality, Weighted $W'(L^p, \xi(t))$-class, Euler mean, Abel's Lemma, Conjugate Fourier series.

Contents

1. Introduction and Motivation

The subject Theory of Approximation is a vast field of Mathematics as regards to the study of Operator Theory and Fourier Analysis. It has been started from the famous Weierstrass theorem in the year 1885.The study of approximation theory involving approximation of functions and the theory of orthogonal series is having great practical significance. Initially this theory was used to approximate the continuous functions by polynomials, later trigonometric polynomials were used to approximate the functions. Nowadays, trigonometric Fourier polynomials are used to obtain better approximations. Moreover, for the accuracy of the estimations to a certain degree, different summability methods for functions of various classes have been used. Many researchers used different summability methods to approximate Lipschitz and Zygmund classes of functions. However, the product summability means are stronger than the individual summability means and can be used for approximation for the broader class of functions. Furthermore, the stability of the system can be enhanced by adopting appropriate conditions for approximating functions. For some recent research works in this direction, [1], [2], [3], [4], [5], [6], [9], [10], [11] and [12].

Let $\sum u_m$ be a given infinite series with the sequence of its m^{th} partial sums $\{s_m\}$. The Cesàro-means of order (α, η) with

$$A_m^{\alpha+\eta} = O(m^{\alpha+\eta})\ (\alpha+\eta > -1),$$

and for $A_0^{\alpha+\eta} = 1$ is defined as

$$C_m^{\alpha,\eta} = t_m^{C^{\alpha,\eta}} = \frac{1}{A_m^{\alpha+\eta}} \sum_{h=0}^{m} A_{m-h}^{\alpha-1} A_h^{\eta} s_h.$$

* Corresponding Author.

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If $C_m^{\alpha,\eta} \to s$ as $m \to \infty$, then the series $\sum u_m$ is Cesàro ($C^{\alpha,\eta}$)-summable to s.

The $(E,1)$ transformed mean of $\{s_m\}$ is defined as

$$E_m^1 = t_m^{E^1} = \frac{1}{2^m} \sum_{h=0}^{m} \binom{m}{h} s_h.$$

If $E_m^1 \to s$ as $m \to \infty$, then $\sum u_m$ is Euler (E^1) summable to s. If

$$\begin{aligned}(CE)_m^{\alpha,\eta;1} &= t_m^{(CE)^{\alpha,\eta;1}} \\ &= \frac{1}{A_m^{\alpha+\eta}} \sum_{h=0}^{m} A_{m-h}^{\alpha-1} A_h^{\eta} E_h^1 \quad \to s \quad \text{as} \quad m \to \infty \\ &= \frac{1}{A_m^{\alpha+\eta}} \sum_{h=0}^{m} A_{m-h}^{\alpha-1} A_h^{\eta} \frac{1}{2^h} \sum_{j=0}^{h} \binom{h}{j} \quad \to s \quad \text{as} \quad m \to \infty,\end{aligned}$$

then the series $\sum u_m$ is Cesàro-Euler ($C^{\alpha,\eta}.E^1$) product summable to s.

Moreover, if

$$\begin{aligned}(CEE)_m^{\alpha,\eta;1} &= t_m^{(CEE)^{\alpha,\eta;1}} \\ &= \frac{1}{A_m^{\alpha+\eta}} \sum_{h=0}^{m} A_{m-h}^{\alpha-1} A_h^{\eta} \frac{1}{2^h} \sum_{j=0}^{h} \binom{h}{j} E_j^1 \quad \to s \quad \text{as} \quad m \to \infty \\ &= \frac{1}{A_m^{\alpha+\eta}} \sum_{h=0}^{m} A_{m-h}^{\alpha-1} A_h^{\eta} \frac{1}{2^h} \sum_{j=0}^{h} \binom{h}{j} \frac{1}{2^j} \sum_{i=0}^{j} \binom{j}{i} \quad \to s \quad \text{as} \quad m \to \infty,\end{aligned}$$

then the series $\sum u_m$ is Cesàro-Euler-Euler ($C^{\alpha,\eta}.E^1.E^1$) product summable to s.

Let g be a signal (function) with 2π as periodic time and is integrable in the same way as of Lebesgue under the limit $(-\pi, \pi)$.

Let

$$s_m(g;x) = \frac{a_0}{2} + \sum_{h=1}^{m} a_h \cos hx + \sum_{h=1}^{m} b_h \sin hx \tag{1.1}$$

be the $(m+1)^{th}$ partial sum of the Fourier series

$$\frac{a_0}{2} + \sum_{h=1}^{\infty} (a_h \cos hx + b_h \sin hx), \tag{1.2}$$

and

$$\overline{s_m}(g;x) = \sum_{h=1}^{m} (b_h \cos hx) - \sum_{h=1}^{m} (a_h \sin hx) \tag{1.3}$$

be its m^{th} partial sum of the conjugate Fourier series

$$\sum_{h=1}^{\infty} (b_h \cos hx) - \sum_{h=1}^{\infty} (a_h \sin hx). \tag{1.4}$$

Definition 1.1 *For the signal g, the L_∞-norm usually denoted as $\|g\|_\infty$ is defined as*

$$\|g\|_\infty = sup\{|g(x)| : x \in \mathbb{R}\},$$

and the L_p-norm denoted as $\|g\|_p$, described over $[0, 2\pi]$ as

$$\|g\|_p = \left\{ \int_0^{2\pi} |g(x)|^p dx \right\}^{1/p} \quad (p \geq 1).$$

Definition 1.2 *The degree of approximation of the signal g by trigonometric polynomial $\tau_m(x)$ of order m under $\|.\|_\infty$ is given by Zygmund [29] with*

$$\|\tau_m - g\|_\infty = sup\{|\tau_m(x) - g(x)| : x \in \mathbb{R}\},$$

and the degree of approximation $E_m(g)$ of $g \in L_p$ is given by

$$E_m(g) = \min_{\tau_m} \|\tau_m(g; x) - g(x)\|_p.$$

Definition 1.3 *A real valued signal g is of Lipschitz class usually denoted as $g \in Lip\beta$, if*

$$|g(x+t) - g(x)| = \mathcal{O}(|t|^\beta),\ 0 < \beta \leq 1,\ t > 0,$$

and $g \in Lip(\beta, p)$, if

$$\left\{ \int_0^{2\pi} |g(x+t) - g(x)|^p dx \right\}^{1/p} = O(|t|^\beta) \ \text{for}\ 0 < \beta \leq 1,\ p \geq 1,\ \ t > 0.$$

Definition 1.4 *For a positive increasing function $\xi(t)$, $g \in Lip(\xi(t), p)$ if*

$$\left\{ \int_0^{2\pi} |g(x+t) - g(x)|^p dx \right\}^{1/p} = O(\xi(t)) \ \text{for}\ p \geq 1, \quad t \geq 0,$$

and $g \in W'(L^p, \xi(t))$, if

$$\left\{ \int_0^{2\pi} |g(x+t) - g(x)|^p \sin^{\gamma p}(x) dx \right\}^{1/p} = O(\xi(t)) \ \text{for}\ \gamma \geq 0,\ p \geq 1, \quad t \geq 0.$$

We redefine the weighted class for our comfort to evaluate $I^{(2)}$ without error as given below:

$$\left\{ \int_0^{2\pi} |g(x+t) - g(x)|^p \sin^{\gamma p}\left(\frac{x}{2}\right) dx \right\}^{1/p} = O(\xi(t)) \ \text{for}\ \gamma \geq 0,\ p \geq 1 \text{ and } t > 0.$$

Remark 1.1 It should be noted here that, if $\gamma = 0$, then $W'(L^p, \xi(t))$ coincides with $Lip\,(\xi(t), p)$, and so also the other generalizations including this are as follows:

$$W'(L^p, \xi(t)) \xrightarrow{\gamma=0} Lip(\xi(t), p) \xrightarrow{\xi(t)=t^\beta} Lip(\beta, p) \xrightarrow{p\to\infty} Lip\beta \ \text{for}\ 0 < \beta \leq 1,\ p \geq 1, \quad t \geq 0.$$

Notations. We use the notations as follows:

$$\Psi(t) = g(x+t) - g(x-t),$$

$$\overline{(CEE)}_m^{\alpha,\eta;1} = \frac{1}{A_m^{\alpha+\eta}} \sum_{h=0}^{m} \left[A_{m-h}^{\alpha-1} A_h^{\eta} \frac{1}{2^h} \left\{ \sum_{j=0}^{h} \binom{h}{j} \frac{1}{2^j} \left(\sum_{i=0}^{j} \binom{j}{i} \frac{\cos\left(i+\frac{1}{2}\right)t}{2\sin\frac{t}{2}} \right) \right\} \right],$$

and $\tau = \left[\frac{1}{t}\right]$, the integer part of $\frac{1}{t}$.

The approximation of trigonometric periodic signals was developed by Zygmund [29]. Subsequently, under various conditions and criteria, many researchers worked in the this direction for approximation of different functions (signals) associated with various summability means. Krasniqi [7] worked on product $(C,1)(E,q)$ summability means of Fourier-Laguerre series. Sonker [19] also worked on approximation of functions of Fourier-Laguerre series. Mittal and Prasad [13] worked on summability of conjugate series. Later Mittal et al. [16] also worked on the approximation of functions belonging to weighted class. Mursaleen and Alotaibi [14] studied generalized matrix summability of conjugate series. Saxena and Verma [28] worked on $(H,1)(E,q)$ product summability means. Krasniqi and Deepamala [8] worked on approximating signals by $(N,p_n,q_n)(E,\theta)$ means. Recently, Qureshi [18] and Sonker and Sangwan [27] worked on triple product summability means. Many other cases of product summability for various functions belonging to different classes can be seen in [15], [17] and [20]- [26].

Motivated by the above mentioned works, in this paper, $(C^{\alpha,\eta}.E^1.E^1)$ product summability is introduced and some more generalized theorems on $(C^{\alpha,\eta}.E^1.E^1)$ summability of conjugate Fourier series of $g \in W'(L^p,\xi(t))$ are established under certain weaker condition.

2. Known Results

Sonker and Sangwan [27] proved the following theorem for approximating the function belongs to $W'(L^p,\xi(t))$ class by $(C^{\alpha,\eta}.E^\theta)$ product means of the series (1.4).

Theorem 2.1 *If a signal $\overline{g}$ with time period of 2π, integrable in Lebesgue sense in the interval $(-\pi,\pi)$ and of class $W'(L^p,\xi(t))$, $(p \geq 1, t>0)$, then its degree of approximation by $\overline{(CE)}^{\alpha,\eta;\theta}$ product means of series (1.4) is given by*

$$\|t_m^{(CE)^{\alpha,\eta;\theta}} - \overline{g}\|_p = \mathcal{O}\left((1+m)^{\gamma+\frac{1}{p}}\xi\left((1+m)^{-1}\right)\right) \tag{2.1}$$

provided $\{\xi(t).t^{-1}\}$ is a non-increasing sequence with

$$\left(\int_0^{\frac{\pi}{(1+m)}}\left(\frac{|\Psi(t)|}{\xi(t)}\sin^\gamma\left(\frac{t}{2}\right)\right)^p dt\right)^{\frac{1}{p}} = O(1) \tag{2.2}$$

and

$$\left(\int_{\frac{\pi}{(1+m)}}^{\pi}\left(\frac{|\Psi(t)|}{\xi(t)t^\delta}\right)^p dt\right)^{\frac{1}{p}} = O\left(\frac{1}{(1+m)^{-\delta}}\right), \tag{2.3}$$

where δ is an arbitrary number such that $(1-\delta)q-1>0$, $\frac{1}{p}+\frac{1}{q}=1$, $1 \leq p \leq \infty$. The conditions (2.1) and (2.2) hold uniformly in x and $\overline{(CE)}_m^{\alpha,\eta;\theta}$ is $\overline{(C,\alpha,\eta)(E,\theta)}$-summable of the series (1.4) and $\overline{g}(x)$ is defined for x by

$$2\pi\overline{g}(x) = -\lim_{\epsilon\to 0}\int_\epsilon^\pi \Psi(t)\cot\left(\frac{t}{2}\right)dt.$$

The motive of this research article is to determine the degree of approximation of the function $\overline{g}$ by $t_m^{(CEE)^{\alpha,\eta;1}}$ means, for the signal g belongs to $W'(L^p,\xi(t))$, $(p \geq 1)$, $(t>0)$. Our results will significantly extend the above mentioned result.

3. Axillary Lemmas

In order to establish our main results, we need the following auxiliary statements (lemmas).

Lemma 3.1 $\overline{(CEE)}_m^{\alpha,\eta;1}(t) = \mathcal{O}\left(\frac{1}{t}\right)$, *for* $0<t \leq \frac{\pi}{m+1}$; $t \leq \pi\sin\left(\frac{t}{2}\right)$ *and* $|\cos(mt)| \leq 1$.

Proof: If $0 \leq t \leq \frac{\pi}{m+1}$, then according to Young's inequality $\sin\frac{t}{2} \geq \frac{t}{\pi}$.
We have,

$$\left|\overline{(CEE)}_m^{\alpha,\eta;1}(t)\right| \leq \frac{1}{2\pi A_m^{\alpha+\eta}}\left|\sum_{h=0}^{m}\left[A_{m-h}^{\alpha-1}A_h^\eta\frac{1}{2^h}\left\{\sum_{j=0}^{h}\binom{h}{j}\frac{1}{2^j}\right.\right.\right.$$

$$
\begin{aligned}
&\cdot\left(\sum_{i=0}^{j}\binom{j}{i}\frac{\cos\left(i+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right)\Bigg\}\Bigg]\Bigg|\\
&\le \frac{1}{2\pi A_m^{\alpha+\eta}}\sum_{h=0}^{m}\left[A_{m-h}^{\alpha-1}A_h^{\eta}\frac{1}{2^h}\left\{\sum_{j=0}^{h}\binom{h}{j}\frac{1}{2^j}\right.\right.\\
&\left.\left.\cdot\left(\sum_{i=0}^{j}\binom{j}{i}\frac{|\cos\left(i+\frac{1}{2}\right)t|}{|\sin\frac{t}{2}|}\right)\right\}\right]\\
&\le \frac{1}{2\pi A_m^{\alpha+\eta}}\sum_{h=0}^{m}\left[A_{m-h}^{\alpha-1}A_h^{\eta}\frac{1}{2^h}\left\{\sum_{j=0}^{h}\binom{h}{j}\frac{1}{2^j}\left(\sum_{i=0}^{j}\binom{j}{i}\frac{1}{\frac{t}{\pi}}\right)\right\}\right]\\
&\le \frac{1}{2tA_m^{\alpha+\eta}}\sum_{h=0}^{m}\left[A_{m-h}^{\alpha-1}A_h^{\eta}\frac{1}{2^h}\left\{\sum_{j=0}^{h}\binom{h}{j}\frac{1}{2^j}2^j\right\}\right]\quad\left\{\because\sum_{i=0}^{j}\binom{j}{i}=2^j\right\}\\
&\le \frac{1}{2tA_m^{\alpha+\eta}}\sum_{h=0}^{m}\left[A_{m-h}^{\alpha-1}A_h^{\eta}\frac{1}{2^h}\left\{\sum_{j=0}^{h}\binom{h}{j}\right\}\right]\\
&\le \frac{1}{2tA_m^{\alpha+\eta}}\sum_{h=0}^{m}\left[A_{m-h}^{\alpha-1}A_h^{\eta}\frac{1}{2^h}.2^h\right]\quad\left\{\because\sum_{j=0}^{h}\binom{h}{j}=2^h\right\}\\
&\le \frac{1}{2tA_m^{\alpha+\eta}}\sum_{h=0}^{m}A_{m-h}^{\alpha-1}A_h^{\eta}\\
&=\mathcal{O}\left(\frac{1}{t}\right)\quad\left\{\because\sum_{h=0}^{m}A_{m-h}^{\alpha-1}A_h^{\eta}=A_m^{\alpha+\eta}\right\}.
\end{aligned}
$$

□

Lemma 3.2 $\overline{(CEE)}_m^{\alpha,\eta;1}(t)=\mathcal{O}\left(\frac{1}{t}\right)$, *for* $0<\frac{\pi}{m+1}\le t\le\pi;\ \ t\le\pi\sin\left(\frac{t}{2}\right)$.

Proof:

$$
\begin{aligned}
\left|\overline{(CEE)}_m^{\alpha,\eta;1}(t)\right| &\le \frac{1}{2\pi A_m^{\alpha+\eta}}\left|\sum_{h=0}^{m}\left[A_{m-h}^{\alpha-1}A_h^{\eta}\frac{1}{2^h}\left\{\sum_{j=0}^{h}\binom{h}{j}\frac{1}{2^j}\right.\right.\right.\\
&\left.\left.\left.\cdot\left(\sum_{i=0}^{j}\binom{j}{i}\frac{\cos\left(i+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right)\right\}\right]\right|\\
&\le \frac{1}{2\pi A_m^{\alpha+\eta}}\left|\sum_{h=0}^{m}\left[A_{m-h}^{\alpha-1}A_h^{\eta}\frac{1}{2^h}\left\{\sum_{j=0}^{h}\binom{h}{j}\frac{1}{2^j}\right.\right.\right.\\
&\left.\left.\left.\cdot\left(\sum_{i=0}^{j}\binom{j}{i}\frac{\cos\left(i+\frac{1}{2}\right)t}{\frac{t}{\pi}}\right)\right\}\right]\right|\\
&\le \frac{1}{2tA_m^{\alpha+\eta}}\left|\sum_{h=0}^{m}\left[A_{m-h}^{\alpha-1}A_h^{\eta}\frac{1}{2^h}\left\{\sum_{j=0}^{h}\binom{h}{j}\frac{1}{2^j}\right.\right.\right.\\
&\left.\left.\left..\,Re\left(\sum_{i=0}^{j}\binom{j}{i}e^{\iota\left(i+\frac{1}{2}\right)t}\right)\right\}\right]\right|\\
&\le \frac{1}{2tA_m^{\alpha+\eta}}\left|\sum_{h=0}^{m}\left[A_{m-h}^{\alpha-1}A_h^{\eta}\frac{1}{2^h}\left\{\sum_{j=0}^{h}\binom{h}{j}\frac{1}{2^j}\right.\right.\right.
\end{aligned}
$$

$$
\begin{aligned}
& . \, Re\left(\sum_{i=0}^{j}\binom{j}{i}e^{\iota(ti)}\right)\Bigg\}\Bigg]\Bigg| |e^{\iota\frac{t}{2}}| \\
& \leq \frac{1}{2tA_m^{\alpha+\eta}}\Bigg|\sum_{h=0}^{\tau-1}\Bigg[A_{m-h}^{\alpha-1}A_h^{\eta}\frac{1}{2^h}\Bigg\{\sum_{j=0}^{h}\binom{h}{j}\frac{1}{2^j} \\
& . \, Re\left(\sum_{i=0}^{j}\binom{j}{i}e^{\iota(ti)}\right)\Bigg\}\Bigg]\Bigg| \\
& + \frac{1}{2tA_m^{\alpha+\eta}}\Bigg|\sum_{h=\tau}^{m}\Bigg[A_{m-h}^{\alpha-1}A_h^{\eta}\frac{1}{2^h}\Bigg\{\sum_{j=0}^{h}\binom{h}{j}\frac{1}{2^j} \\
& . \, Re\left(\sum_{i=0}^{j}\binom{j}{i}e^{\iota(ti)}\right)\Bigg\}\Bigg]\Bigg|.
\end{aligned} \tag{3.1}
$$

Now, considering the first term of (3.1), we have

$$
\begin{aligned}
& \frac{1}{2tA_m^{\alpha+\eta}}\Bigg|\sum_{h=0}^{\tau-1}\Bigg[A_{m-h}^{\alpha-1}A_h^{\eta}\frac{1}{2^h}\Bigg\{\sum_{j=0}^{h}\binom{h}{j}\frac{1}{2^j}Re\left(\sum_{i=0}^{j}\binom{j}{i}e^{\iota(ti)}\right)\Bigg\}\Bigg]\Bigg| \\
& \leq \frac{1}{2tA_m^{\alpha+\eta}}\Bigg|\sum_{h=0}^{\tau-1}\Bigg[A_{m-h}^{\alpha-1}A_h^{\eta}\frac{1}{2^h}\Bigg\{\sum_{j=0}^{h}\binom{h}{j}\frac{1}{2^j}\left(\sum_{i=0}^{j}\binom{j}{i}\right)\Bigg\}\Bigg]\Bigg| \\
& \leq \frac{1}{2tA_m^{\alpha+\eta}}\Bigg|\sum_{h=0}^{\tau-1}\Bigg[A_{m-h}^{\alpha-1}A_h^{\eta}\frac{1}{2^h}\Bigg\{\sum_{j=0}^{h}\binom{h}{j}\Bigg\}\Bigg]\Bigg| \\
& \leq \frac{1}{2tA_m^{\alpha+\eta}}\Bigg|\sum_{h=0}^{\tau-1}\Bigg[A_{m-h}^{\alpha-1}A_h^{\eta}\frac{1}{2^h}.2^h\Bigg]\Bigg| \quad \Bigg\{\because \sum_{j=0}^{h}\binom{h}{j} = 2^h\Bigg\} \\
& \leq \frac{1}{2tA_m^{\alpha+\eta}}\Bigg|\sum_{h=0}^{\tau-1}A_{m-h}^{\alpha-1}A_h^{\eta}\Bigg| \\
& = \mathcal{O}\left(\frac{1}{t}\right) \quad \Bigg\{\because \sum_{h=0}^{\tau-1}A_{m-h}^{\alpha-1}A_h^{\eta} = A_m^{\alpha+\eta}\Bigg\}.
\end{aligned} \tag{3.2}
$$

Now, considering the 2^{nd} term of (3.1), we have

$$
\begin{aligned}
& \frac{1}{2tA_m^{\alpha+\eta}}\Bigg|\sum_{h=\tau}^{m}\Bigg[A_{m-h}^{\alpha-1}A_h^{\eta}\frac{1}{2^h}\Bigg\{\sum_{j=0}^{h}\binom{h}{j}\frac{1}{2^j}Re\left(\sum_{i=0}^{j}\binom{j}{i}e^{\iota(ti)}\right)\Bigg\}\Bigg]\Bigg| \\
& \leq \frac{1}{2tA_m^{\alpha+\eta}}\sum_{h=\tau}^{m}A_{m-h}^{\alpha-1}A_h^{\eta}\frac{1}{2^h}\Bigg\{\sum_{j=0}^{h}\binom{h}{j}\frac{1}{2^j}\max_{0\leq k\leq j}\Bigg|\sum_{i=0}^{k}\binom{j}{i}e^{\iota(ti)}\Bigg|\Bigg\} \\
& \leq \frac{1}{2tA_m^{\alpha+\eta}}\sum_{h=\tau}^{m}A_{m-h}^{\alpha-1}A_h^{\eta}\frac{1}{2^h}\Bigg\{\sum_{j=0}^{h}\binom{h}{j}\frac{1}{2^j}\max_{0\leq k\leq j}\sum_{i=0}^{k}\binom{j}{i}|e^{\iota(ti)}|\Bigg\} \\
& \leq \frac{1}{2tA_m^{\alpha+\eta}}\sum_{h=\tau}^{m}A_{m-h}^{\alpha-1}A_h^{\eta}\frac{1}{2^h}\sum_{j=0}^{h}\binom{h}{j}\frac{1}{2^j}\max_{0\leq k\leq j}\sum_{i=0}^{k}\binom{j}{i} \\
& \leq \frac{1}{2tA_m^{\alpha+\eta}}\sum_{h=\tau}^{m}A_{m-h}^{\alpha-1}A_h^{\eta}
\end{aligned}
$$

$$= \mathcal{O}\left(\frac{1}{t}\right) \quad \left\{\because \sum_{h=0}^{\tau-1} A_{m-h}^{\alpha-1} A_h^{\eta} = A_m^{\alpha+\eta}\right\}. \tag{3.3}$$

Collecting (3.1), (3.2) and (3.3), we get

$$\left|\overline{(CEE)}_m^{\alpha,\eta;1}(t)\right| = \mathcal{O}\left[\frac{1}{t}\right].$$

□

Lemma 3.3 *$\overline{(CEE)}_m^{\alpha;1}(t) = \mathcal{O}\left(\frac{1}{t}\right)$, for $0<t \leq \frac{\pi}{m+1}$; $t \leq \pi \sin\left(\frac{t}{2}\right)$ and $|\cos(mt)| \leq 1$.*

Proof: If $0 \leq t \leq \frac{\pi}{m+1}$, then according to Young's inequality $\sin\frac{t}{2} \geq \frac{t}{\pi}$.
We have,

$$\begin{aligned}
\left|\overline{(CEE)}_m^{\alpha;1}(t)\right| &\leq \frac{1}{2\pi A_m^{\alpha}}\left|\sum_{h=0}^{m}\left[A_{m-h}^{\alpha-1}\frac{1}{2^h}\left\{\sum_{j=0}^{h}\binom{h}{j}\frac{1}{2^j}\left(\sum_{i=0}^{j}\binom{j}{i}\frac{\cos\left(i+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right)\right\}\right]\right| \\
&\leq \frac{1}{2\pi A_m^{\alpha}}\sum_{h=0}^{m}\left[A_{m-h}^{\alpha-1}\frac{1}{2^h}\left\{\sum_{j=0}^{h}\binom{h}{j}\frac{1}{2^j}\left(\sum_{i=0}^{j}\binom{j}{i}\frac{|\cos\left(i+\frac{1}{2}\right)t|}{|\sin\frac{t}{2}|}\right)\right\}\right] \\
&\leq \frac{1}{2\pi A_m^{\alpha}}\sum_{h=0}^{m}\left[A_{m-h}^{\alpha-1}\frac{1}{2^h}\left\{\sum_{j=0}^{h}\binom{h}{j}\frac{1}{2^j}\left(\sum_{i=0}^{j}\binom{j}{i}\frac{1}{\frac{t}{\pi}}\right)\right\}\right] \\
&\leq \frac{1}{2tA_m^{\alpha}}\sum_{h=0}^{m}\left[A_{m-h}^{\alpha-1}\frac{1}{2^h}\left\{\sum_{j=0}^{h}\binom{h}{j}\frac{1}{2^j}2^j\right\}\right] \quad \left\{\because \sum_{i=0}^{j}\binom{j}{i} = 2^j\right\} \\
&\leq \frac{1}{2tA_m^{\alpha}}\sum_{h=0}^{m}\left[A_{m-h}^{\alpha-1}\frac{1}{2^h}\left\{\sum_{j=0}^{h}\binom{h}{j}\right\}\right] \\
&\leq \frac{1}{2tA_m^{\alpha}}\sum_{h=0}^{m}\left[A_{m-h}^{\alpha-1}\frac{1}{2^h}.2^h\right] \quad \left\{\because \sum_{j=0}^{h}\binom{h}{j} = 2^h\right\} \\
&\leq \frac{1}{2tA_m^{\alpha}}\sum_{h=0}^{m}A_{m-h}^{\alpha-1} \\
&= \mathcal{O}\left(\frac{1}{t}\right) \quad \left\{\because \sum_{h=0}^{m}A_{m-h}^{\alpha-1} = A_m^{\alpha}\right\}.
\end{aligned}$$

□

Lemma 3.4 *$\overline{(CEE)}_m^{\alpha;1}(t) = \mathcal{O}\left(\frac{1}{t}\right)$, for $0<\frac{\pi}{m+1} \leq t \leq \pi$; $t \leq \pi \sin\left(\frac{t}{2}\right)$.*

Proof:

$$\begin{aligned}
\left|\overline{(CEE)}_m^{\alpha;1}(t)\right| &\leq \frac{1}{2\pi A_m^{\alpha}}\left|\sum_{h=0}^{m}\left[A_{m-h}^{\alpha-1}\frac{1}{2^h}\left\{\sum_{j=0}^{h}\binom{h}{j}\frac{1}{2^j}\left(\sum_{i=0}^{j}\binom{j}{i}\frac{\cos\left(i+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right)\right\}\right]\right| \\
&\leq \frac{1}{2\pi A_m^{\alpha}}\left|\sum_{h=0}^{m}\left[A_{m-h}^{\alpha-1}\frac{1}{2^h}\left\{\sum_{j=0}^{h}\binom{h}{j}\frac{1}{2^j}\left(\sum_{i=0}^{j}\binom{j}{i}\frac{\cos\left(i+\frac{1}{2}\right)t}{\frac{t}{\pi}}\right)\right\}\right]\right| \\
&\leq \frac{1}{2tA_m^{\alpha}}\left|\sum_{h=0}^{m}\left[A_{m-h}^{\alpha-1}\frac{1}{2^h}\left\{\sum_{j=0}^{h}\binom{h}{j}\frac{1}{2^j}\, Re\left(\sum_{i=0}^{j}\binom{j}{i}e^{\iota\left(i+\frac{1}{2}\right)t}\right)\right\}\right]\right|
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2tA_m^{\alpha}}\left|\sum_{h=0}^{m}\left[A_{m-h}^{\alpha-1}\frac{1}{2^h}\left\{\sum_{j=0}^{h}\binom{h}{j}\frac{1}{2^j}\,Re\left(\sum_{i=0}^{j}\binom{j}{i}e^{\iota(ti)}\right)\right\}\right]\right||e^{\iota\frac{t}{2}}| \\
&\leq \frac{1}{2tA_m^{\alpha}}\left|\sum_{h=0}^{\tau-1}\left[A_{m-h}^{\alpha-1}\frac{1}{2^h}\left\{\sum_{j=0}^{h}\binom{h}{j}\frac{1}{2^j}\,Re\left(\sum_{i=0}^{j}\binom{j}{i}e^{\iota(ti)}\right)\right\}\right]\right| \\
&+ \frac{1}{2tA_m^{\alpha}}\left|\sum_{h=\tau}^{m}\left[A_{m-h}^{\alpha-1}\frac{1}{2^h}\left\{\sum_{j=0}^{h}\binom{h}{j}\frac{1}{2^j}\,Re\left(\sum_{i=0}^{j}\binom{j}{i}e^{\iota(ti)}\right)\right\}\right]\right|. \qquad (3.4)
\end{aligned}$$

Now, considering the first term of (3.4), we have

$$\begin{aligned}
&\frac{1}{2tA_m^{\alpha}}\left|\sum_{h=0}^{\tau-1}\left[A_{m-h}^{\alpha-1}\frac{1}{2^h}\left\{\sum_{j=0}^{h}\binom{h}{j}\frac{1}{2^j}\,Re\left(\sum_{i=0}^{j}\binom{j}{i}e^{\iota(ti)}\right)\right\}\right]\right| \\
&\leq \frac{1}{2tA_m^{\alpha}}\left|\sum_{h=0}^{\tau-1}\left[A_{m-h}^{\alpha-1}\frac{1}{2^h}\left\{\sum_{j=0}^{h}\binom{h}{j}\frac{1}{2^j}\left(\sum_{i=0}^{j}\binom{j}{i}\right)\right\}\right]\right| \\
&\leq \frac{1}{2tA_m^{\alpha}}\left|\sum_{h=0}^{\tau-1}\left[A_{m-h}^{\alpha-1}\frac{1}{2^h}\left\{\sum_{j=0}^{h}\binom{h}{j}\right\}\right]\right| \\
&\leq \frac{1}{2tA_m^{\alpha}}\left|\sum_{h=0}^{\tau-1}\left[A_{m-h}^{\alpha-1}\frac{1}{2^h}.2^h\right]\right| \quad \left\{\because \sum_{j=0}^{h}\binom{h}{j}=2^h\right\} \\
&\leq \frac{1}{2tA_m^{\alpha}}\left|\sum_{h=0}^{\tau-1}A_{m-h}^{\alpha-1}\right| \\
&= \mathcal{O}\left(\frac{1}{t}\right). \quad \left\{\because \sum_{h=0}^{\tau-1}A_{m-h}^{\alpha-1}=A_m^{\alpha}\right\}. \qquad (3.5)
\end{aligned}$$

Now, considering the 2^{nd} term of (3.4), we have

$$\begin{aligned}
&\frac{1}{2tA_m^{\alpha}}\left|\sum_{h=\tau}^{m}\left[A_{m-h}^{\alpha-1}\frac{1}{2^h}\left\{\sum_{j=0}^{h}\binom{h}{j}\frac{1}{2^j}\,Re\left(\sum_{i=0}^{j}\binom{j}{i}e^{\iota(ti)}\right)\right\}\right]\right| \\
&\leq \frac{1}{2tA_m^{\alpha}}\sum_{h=\tau}^{m}A_{m-h}^{\alpha-1}\frac{1}{2^h}\left\{\sum_{j=0}^{h}\binom{h}{j}\frac{1}{2^j}\max_{0\leq k\leq j}\left|\sum_{i=0}^{k}\binom{j}{i}e^{\iota(ti)}\right|\right\} \\
&\leq \frac{1}{2tA_m^{\alpha}}\sum_{h=\tau}^{m}A_{m-h}^{\alpha-1}\frac{1}{2^h}\left\{\sum_{j=0}^{h}\binom{h}{j}\frac{1}{2^j}\max_{0\leq k\leq j}\sum_{i=0}^{k}\binom{j}{i}|e^{\iota(ti)}|\right\} \\
&\leq \frac{1}{2tA_m^{\alpha}}\sum_{h=\tau}^{m}A_{m-h}^{\alpha-1}\frac{1}{2^h}\sum_{j=0}^{h}\binom{h}{j}\frac{1}{2^j}\max_{0\leq k\leq j}\sum_{i=0}^{k}\binom{j}{i} \\
&\leq \frac{1}{2tA_m^{\alpha}}\sum_{h=\tau}^{m}A_{m-h}^{\alpha-1} \\
&= \mathcal{O}\left(\frac{1}{t}\right) \quad \left\{\because \sum_{h=0}^{\tau-1}A_{m-h}^{\alpha-1}=A_m^{\alpha}\right\}. \qquad (3.6)
\end{aligned}$$

Collecting (3.4), (3.5) and (3.6), we get

$$\left|\overline{(CEE)}_m^{\alpha;1}(t)\right| = \mathcal{O}\left[\frac{1}{t}\right].$$

□

4. Main Result

Theorem 4.1 *If a signal $\overline{g}$ with time period of 2π, integrable as Lebesgue for $(-\pi,\pi)$ and of class $W'(L^p,\xi(t))$, $p\geq 1, t>0$, then its degree of approximation by $\overline{(CEE)}^{\alpha,\eta;1}$ product means of series (1.4) is given by*

$$\|t_m^{\overline{(CEE)}^{\alpha,\eta;1}}-\overline{g}\|_p=\mathcal{O}\left((1+m)^{\gamma+\frac{1}{p}}\xi\left((1+m)^{-1}\right)\right) \tag{4.1}$$

provided $\{\xi(t).t^{-1}\}$ is a non-increasing sequence

$$\left(\int_0^{\frac{\pi}{(1+m)}}\left(\frac{|\Psi(t)|}{\xi(t)}\sin^{\gamma}\left(\frac{t}{2}\right)\right)^p dt\right)^{\frac{1}{p}}=O(1) \tag{4.2}$$

$$\left(\int_{\frac{\pi}{(1+m)}}^{\pi}\left(\frac{|\Psi(t)|}{\xi(t)t^{\delta}}\right)^p dt\right)^{\frac{1}{p}}=O\left(\frac{1}{(1+m)^{-\delta}}\right) \tag{4.3}$$

where δ is an arbitrary number such that $(1-\delta)q-1>0$, $\frac{1}{p}+\frac{1}{q}=1$, $1\leq p\leq\infty$. The conditions (4.2) and (4.3) hold uniformly in x and $\overline{(CEE)}_m^{\alpha,\eta;1}$ is $\overline{(C,\alpha,\eta)(E,1)(E,1)}$-summable, and $\overline{g}(x)$ is defined by

$$2\pi\overline{g}(x)=-\lim_{\epsilon\to 0}\int_{\epsilon}^{\pi}\Psi(t)\cot\left(\frac{t}{2}\right)dt.$$

Proof: Follwing Zygmund [29], $\overline{s_m}(g;x)$ is written as

$$\overline{s_m}(g,x)=\overline{g}(x)+\frac{1}{2\pi}\int_0^{\pi}\Psi(t)\frac{\cos(j+\frac{1}{2})t}{\sin\frac{t}{2}}dt.$$

The Euler (E^1) means of the series (1.3) is

$$\begin{aligned}E_m^1(x) &= \frac{1}{2^m}\sum_{h=0}^{m}\binom{m}{h}\overline{s_m}(g;x)\\ &= \overline{g}(x)+\frac{1}{2\pi\,.\,2^m}\int_0^{\pi}\frac{\Psi(x,t)}{\sin\frac{t}{2}}\sum_{h=0}^{m}\binom{m}{h}\cos\left(h+\frac{1}{2}\right)t\,dt\end{aligned}$$

The $\overline{(CEE)}^{\alpha,\eta;1}$ transform of $\overline{s_m}(g;x)$ is

$$\begin{aligned}\left|t_m^{\overline{(CEE)}_m^{\alpha,\eta;1}}-\overline{g}\right| &= \frac{1}{2\pi.A_m^{\alpha+\eta}}\left|\sum_{h=0}^{m}\left[A_{m-h}^{\alpha-1}A_h^{\eta}\frac{1}{2^h}\int_0^{\pi}\Psi(t)\left\{\sum_{j=0}^{h}\binom{h}{j}\frac{1}{2^j}\right.\right.\right.\\ &\quad\left.\left.\left.\cdot\left(\sum_{i=0}^{j}\binom{j}{i}\frac{\cos\left(i+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}\right)\right\}dt\right]\right|\\ &= \int_0^{\pi}|\Psi(t)|.|\overline{(CEE)}_m^{\alpha,\eta;1}(t)|dt.\end{aligned}$$

By using the assumptions of the theorem and on taking $\Psi(t)$ as Ψ, it is to be shown that

$$\int_0^{\pi}|\Psi|.|\overline{(CEE)}_m^{\alpha,\eta;1}(t)|dt=O\left((1+m)^{\gamma+\frac{1}{p}}\xi\left((1+m)^{-1}\right)\right).$$

Now,

$$\left|t_m^{\overline{(CEE)}_m^{\alpha,\eta;1}}-\overline{g}\right| = \int_0^{\pi}|\Psi|.|\overline{(CEE)}_m^{\alpha,\eta;1}(t)|dt$$

$$\begin{aligned} &= \left[\int_0^{\frac{\pi}{(1+m)}} |\Psi| + \int_{\frac{\pi}{(1+m)}}^{\pi} |\Psi|\right] |\overline{(CEE)}_m^{\alpha,\eta;1}(t)|dt \\ &= |J^{(1)}| + |J^{(2)}| \quad (say). \end{aligned} \tag{4.4}$$

Using Lemma 3.1, condition (4.2) and Hölder's inequality, we get

$$\begin{aligned} |J^{(1)}| &\leq \int_0^{\frac{\pi}{(1+m)}} |\Psi||\overline{(CEE)}_m^{\alpha,\eta;1}(t)|dt \\ &\leq \left(\int_0^{\frac{\pi}{(1+m)}} \left(\frac{|\Psi|}{\xi(t)}\sin^{\gamma}\left(\frac{t}{2}\right)\right)^p dt\right)^{\frac{1}{p}} \left[\int_0^{\frac{\pi}{(1+m)}} \left(\frac{\xi(t)|\overline{(CEE)}_m^{\alpha,\eta;1}(t)|}{\sin^{\gamma}(t/2)}\right)^q dt\right]^{\frac{1}{q}} \\ &= O(1) \underset{0<t\leq\frac{\pi}{(1+m)}}{ess\ sup} [\xi(t)^q]^{\frac{1}{q}} \left[\int_0^{\frac{\pi}{(1+m)}} (t^{-1-\gamma})^q dt\right]^{\frac{1}{q}} \\ &= O\left(\xi\left(\frac{\pi}{1+m}\right)\right) \underset{0<t\leq\frac{\pi}{(1+m)}}{ess\ sup} \left[\int_0^{\frac{\pi}{(1+m)}} (t^{-1-\gamma})^q dt\right]^{\frac{1}{q}} \\ &= O\left(\xi\left((1+m)^{-1}\right)\right) \left[\lim_{\epsilon\to 0}\int_0^{\frac{\pi}{(1+m)}} (t^{-1-\gamma})^q dt\right]^{\frac{1}{q}} \\ &= O\left[\xi\left((1+m)^{-1}\right)(1+m)^{\gamma+1-\frac{1}{q}}\right] \\ &= O\left[(1+m)^{\gamma+\frac{1}{p}}\xi\left((1+m)^{-1}\right)\right]. \quad \left\{\because \frac{1}{p}+\frac{1}{q}=1,\ 1\leq p\leq\infty\right\} \end{aligned} \tag{4.5}$$

Now, in view of Lemma 3.2, condition (4.3) and Hölder's inequality, we have

$$\begin{aligned} |J^{(2)}| &\leq \int_{\frac{\pi}{(1+m)}}^{\pi} |\Psi||\overline{(CEE)}_m^{\alpha,\eta;1}(t)|dt \\ &\leq \left(\int_{\frac{\pi}{(1+m)}}^{\pi} \left(\frac{|\Psi|\sin^{\gamma}(t/2)}{\xi(t).t^{\delta}}\right)^p dt\right)^{\frac{1}{p}} \left[\int_{\frac{\pi}{(1+m)}}^{\pi} \left(\frac{\xi(t).t^{\delta}|\overline{(CEE)}_m^{\alpha,\eta;1}(t)|}{\sin^{\gamma}(t/2)}\right)^q dt\right]^{\frac{1}{q}} \\ &\leq \left(\int_{\frac{\pi}{(1+m)}}^{\pi} \left(\frac{|\Psi|}{\xi(t).t^{\delta}}\right)^p dt\right)^{\frac{1}{p}} \left[\int_{\frac{\pi}{(1+m)}}^{\pi} \left(\frac{\xi(t).t^{\delta-1}}{\sin^{\gamma}(t/2)}\right)^q dt\right]^{\frac{1}{q}} \\ &= O\left(\frac{1}{(1+m)^{-\delta}}\right) \left[\int_{\frac{\pi}{(1+m)}}^{\pi} \left(\xi(t).t^{\delta-1-\gamma}\right)^q dt\right]^{\frac{1}{q}} \\ &= O\left(\frac{1}{(1+m)^{-\delta}}\right) \left[\int_{\frac{1}{\pi}}^{\frac{(1+m)}{\pi}} \left(\xi\left(\frac{1}{z}\right).z^{-\delta+1+\gamma}\right)^q \frac{dz}{z^2}\right]^{\frac{1}{q}} \quad \left\{Putting\ t=\frac{1}{z}\right\} \\ &= O\left(\frac{1}{(1+m)^{-\delta}}\xi\left(\frac{\pi}{1+m}\right)\right) \left[\int_{\frac{1}{\pi}}^{\frac{(1+m)}{\pi}} z^{(-\delta+1+\gamma)q-2}\ dz\right]^{\frac{1}{q}} \\ &= O\left(\frac{1}{(1+m)^{-\delta}}\xi\left((1+m)^{-1}\right)\right) \left[\left(\frac{z^{(-\delta+1+\gamma)q-1}}{(-\delta+1+\gamma)q-1}\right)_{\frac{1}{\pi}}^{\frac{(1+m)}{\pi}}\right]^{\frac{1}{q}} \\ &= O\left(\frac{1}{(1+m)^{-\delta}}\xi\left((1+m)^{-1}\right)\right) \left[(1+m)^{(-\delta+1+\gamma)-\frac{1}{q}}\right] \end{aligned}$$

$$= O\left[(1+m)^{\gamma+\frac{1}{p}}\,\xi\left((1+m)^{-1}\right)\right]. \quad \left\{\because \frac{1}{p}+\frac{1}{q}=1,\ 1\le p\le\infty\right\} \tag{4.6}$$

Putting equations (4.4), (4.5) in (4.6), we have

$$\left|t_m^{(CEE)^{\alpha,\eta;1}} - \overline{g}\right| = \mathcal{O}\left((1+m)^{\gamma+\frac{1}{p}}\xi\left((1+m)^{-1}\right)\right).$$

Thus

$$\begin{aligned}
\|t_m^{(CEE)^{\alpha,\eta;1}} - \overline{g}\|_p &= \left(\int_0^{2\pi}\left|t_m^{(CEE)^{\alpha,\eta;1}} - \overline{g}\right|^p dx\right)^{\frac{1}{p}}\\
&= O\left(\int_0^{2\pi}\left((1+m)^{\gamma+\frac{1}{p}}\xi\left((1+m)^{-1}\right)\right)^p dx\right)^{\frac{1}{p}}\\
&= \mathcal{O}\left((1+m)^{\gamma+\frac{1}{p}}\xi\left((1+m)^{-1}\right)\right).
\end{aligned}$$

Which completes the proof of the theorem. □

Theorem 4.2 *If a signal $\overline{g}$ with time period of 2π, integrable as Lebesgue for $(-\pi,\pi)$ and of class $W'(L^p,\xi(t))$, $p\ge 1, t>0$, then its degree of approximation by $\overline{(CEE)}^{\alpha;1}$ product means of series (1.3) is given by*

$$\|t_m^{(CEE)^{\alpha,1}} - \overline{g}\|_p = \mathcal{O}\left((1+m)^{\gamma+\frac{1}{p}}\xi\left((1+m)^{-1}\right)\right) \tag{4.7}$$

provided $\{\xi(t).t^{-1}\}$ is a non-increasing sequence with

$$\left(\int_0^{\frac{\pi}{(1+m)}}\left(\frac{|\Psi(t)|}{\xi(t)}\sin^\gamma\left(\frac{t}{2}\right)\right)^p dt\right)^{\frac{1}{p}} = O(1) \tag{4.8}$$

$$\left(\int_{\frac{\pi}{(1+m)}}^{\pi}\left(\frac{|\Psi(t)|}{\xi(t)t^\delta}\right)^p dt\right)^{\frac{1}{p}} = O\left(\frac{1}{(1+m)^{-\delta}}\right), \tag{4.9}$$

where δ is an arbitrary number such that $(1-\delta)q-1>0$, $\frac{1}{p}+\frac{1}{q}=1$, $1\le p\le\infty$. The conditions (4.9) and (4.9) hold uniformly in x and $\overline{(CEE)}_m^{\alpha;1}$ is $\overline{(C,\alpha)(E,1)(E,1)}$-summable, and $\overline{g}(x)$ is defined by

$$2\pi\overline{g}(x) = -\lim_{\epsilon\to 0}\int_\epsilon^\pi \Psi(t)\cot\left(\frac{t}{2}\right)dt.$$

Details of Theorem 4.2. The proof can be omitted as under Lemma 3 and Lemma 4, it is on the same lines as Theorem 4.1.

5. Concluding Remarks and Observations

Several known and previous results can also be derived from the main result. Let us see some of them as follows.

Remark 5.1 In Theorem 4.1, if $\alpha=1$, $\gamma=\eta=0$, then $W'(L^p,\xi(t))$, $p\ge 1, t>0$ reduces to $Lip(\xi(t),p)$ and the approximation of the function $g\in Lip(\xi(t),p)$ is

$$\|t_m^{(CEE)^{1,0;1}} - \overline{g}\|_p = \mathcal{O}\left((1+m)^{\frac{1}{p}}\xi\left((1+m)^{-1}\right)\right).$$

Remark 5.2 In Theorem 4.1, if $\alpha = 1$, $\gamma = \eta = 0$ and $\xi(t) = t^{\beta}$, $0<\beta \leq 1$, then the approximation of the function $g \in Lip(\beta, p)$, $\frac{1}{p} \leq \beta \leq 1$ is

$$\|t_m^{(CEE)^{1,0;1}} - \overline{g}\|_p \;=\; \mathcal{O}\left((1+m)^{-\beta+\frac{1}{p}}\right).$$

Remark 5.3 In Theorem 4.1, if $\alpha = 1$, $\gamma = \eta = 0$ and $\xi(t) = t^{\beta}$, and if in remark 5.2 $p \to \infty$ then $g \in Lip(\beta, p)$ reduces to $Lip\beta$, $0<\beta<1$. Moreover, the approximation of the function $g \in Lip(\beta, p)$, is

$$\|t_m^{(CEE)^{1,0;1}} - \overline{g}\|_\infty \;=\; \mathcal{O}\left((1+m)^{-\beta}\right).$$

Remark 5.4 This paper focuses on the approximation of signal belongs to classes $W'(L^p, \xi(t))$, $p \geq 1, t>0$ by using generalized Cesàro-Euler-Euler ($C^{\alpha,\eta}.E^1.E^1$) means of conjugate Fourier series. To summarize, the purpose of introducing product summability is to minimize the error of approximation. As much as we reduce the error, the result becomes stronger. In doing this, the concept of product summability is very helpful. Since, by using double, triple, or higher product means, the error of approximation decreases. Further, we develop new and well known arbitrary results from the main result by using appropriate conditions as explained in this concluding section.

Conflicts of Interest: The authors declare that they have no conflicts of interest.

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Smita Sonker,
School of Physical Sciences,
Jawaharlal Nehru University,
India.
E-mail address: smitafma@nitkkr.ac.in

and

Paramjeet Sangwan,
Department of Applied Sciences and Humanities,
Ganga Institute of Technology and Management, Jhajjar
India.
E-mail address: appliedsciencepj@gmail.com

and

Bidu Bhusan Jena,
Faculty of Science (Mathematics),
Sri Sri University,
India.
E-mail address: bidumath.05@gmail.com

and

Susanta Kumar Paikray,
Department of Mathematics,
Veer Surendra Sai University of Technology,
India.
E-mail address: skpaikray_math@vssut.ac.in