Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–17.
ISSN-0037-8712
doi:10.5269/bspm.64812

Common fixed-point for a pair of Hardy-Rogers F-contractions in Quasi partial b-Metric space Style

Santosh Kumar* and Jonasi Chilongola

ABSTRACT: This paper aims to prove common fixed point theorems for a pair of Hardy-Rogers F-contractions in the setting of quasi-partial b- metric space. The main theorem generalizes the results due to Wardowski and many others in the literature. We also provide an illustrative example and an application to boundary value problem to validate the results.

Key Words: F-contraction, Hardy-Rogers F-contractions, quasi partial b-metric space.

Contents

1. Introduction and preliminaries

In 1922, Banach [20] introduced a significant and an outstanding principle called the Banach contraction principle that ensures existence and uniqueness of fixed point for a contractive self map in metric space.

Theorem 1.1 *[20] Let (M, d) be a complete metric space and let $f : M \to M$ be a contraction, that is, there exists a number $k \in [0, 1)$ such that for all $u, v \in M$,*

$$d(fu, fv) \leq kd(u, v).$$

Then f has a unique fixed point w in M.

Since then, there has been a significant development in metric fixed point theory. In 1968, Browder [6] gave a brief and transparent proof of the generalization of the classical Picard-Banach contraction principle in its quantitative form through the following theorem:

Theorem 1.2 *[6] Let M be a complete metric space, D a bounded subset of M , f a mapping of D into D. Suppose that there exists a monotone non decreasing function $\psi(t)$ for $t > 0$, with ψ continuous on the right, such that $\psi(t) < t$ for all $t > 0$, while for all u and v in A,*

$$d(fu, fv) \leq \psi(d(u, v))$$

(where d is the distance function on M). Then, for each $u_0 \in A$, $\{f^n u_0\}$ converges to an element ξ of M, independent of u_0, and

$$d(f^n u_0, \xi) \leq d(u, v)$$

where d_0 is the diameter of M, ψ^n is the n^{th} iterate of ψ, and

$$d_n = \psi_n(d_0) \to 0, (n \to +\infty).$$

* Corresponding author

Submitted August 27, 2022. Published May 23, 2025
2010 *Mathematics Subject Classification:* 47H10, 54H25.

In 1997, Suzuki [30] introduced the notion of ω-distance as follows:

Definition 1.1 *[30] Let (M,d) be a metric space, then a function $p : M \times M \to [0,\infty)$ is called ω-distance on M if the following are satisfied:*

(1) $p(u,w) \leq p(u,v) + p(u,w)$ for all $u,v,w \in M$,

(2) For any $u \in M$, $p : M \to [0,\infty)$, is lower semi-continuous,

(3) For any $\epsilon > 0$, there exists $\delta > 0$ such that $p(w,u) \leq \delta$ and $p(w,v) \leq \delta$ imply $p(u,v) \leq \epsilon$.

Recently, Gautam *et al.* [19] investigated on some ω-interpolative contractions of Suzuki-type mappings in Quasi-Partial b-metric space.

In 1969, Nadler [21] introduced the study of fixed points for multivalued mappings on complete metric spaces by combining the ideas of multi valued mappings and contraction:

Theorem 1.3 *[21] Let (M,d) be a complete metric space and let $f : M \to CB(M)$ be a multi valued mapping satisfying*

$$G(fu, fv) \leq k(d(u,v)),$$

for all $u,v \in M$, where $k \in (0,1)$ and $CB(M)$ denotes the collection of non-empty closed and bounded subsets of M. Then f has a fixed point $v \in M$ such that $v = fv$.

Recently, Wardowski [3] introduced the concept of F-contraction which is defined as follows:

Definition 1.2 *[3] Let (M,d) be a metric space. A mapping $f : M \to M$ is called an F-contraction if there exist $\tau > 0$ and $F \in \mathcal{F}$ such that $\tau + F(d(fu,fv)) \leq F(d(u,v))$ holds for any $u,v \in M$ with $d(fu,fv) > 0$, where $\mathcal{F}$ is the set of all functions $F : R^+ \to R$ satisfying the following conditions:*

(F_1) F is strictly increasing: $u < v \Rightarrow F(u) < F(v)$,

(F_2) For each sequence $\{\alpha_n\}_{n\in N}$ in R^+, $\lim_{n\to\infty} F(\alpha_n) = -\infty$,

(F_3) There exists $k \in (0,1)$ such that $\lim_{\alpha\to\infty} \alpha^k F(\alpha) = 0$.

Here, we denote by $\mathcal{F}$ the set of all functions satisfying the conditions (F_1) and (F_2).

Example 1.1 *[3] It is obvious that the functions, $F_1(a) = lna$, $F_2(a) = -\frac{1}{a}$, $a > 0$, $F_3(a) = \frac{1}{e^{-a}-e^{a}}$ belong to $\mathcal{F}$.*

Example 1.2 *[3] Let $F_1 : (0,\infty) \to \mathbb{R}$ be given by $F_1(\mu) = ln\mu$. It is clear that $F_1 \in \mathcal{F}$. Then each self mapping f on metric space (M,d) satisfy Definition 1.2 is an F_1-contraction such that*

$$d(fu,fv) \leq e^{-\tau} d(u,v),$$

for all $u,v \in M$, $fu \neq fv$.

Example 1.3 *(Wardowski 2012) The following $F : (0,+\infty)$ are the elements of $\mathcal{F}$*

(1) $F\alpha = \alpha$,

(2) $F\mu = ln\mu + \mu$,

(3) $F\mu = -\frac{1}{\sqrt{\mu}}$,

(4) $F\mu = ln(\mu^2 + \mu)$.

Lemma 1.1 *[16] It is well known that if $F \in \mathcal{F}$, then for each sequence $\{\alpha_n\} \subset (0,\infty)$, $\alpha_n \to 0$ if and only if $F(\alpha_n) \to -\infty$.*

Remark 1.1 *([3]) Let f be an F-contraction. Then*

$$d(fu,fv) < d(u,v)$$

for all $u,v \in M$ such that $fu \neq fu$. Also, f is a continuous map.

In 2014, Wardowski and Dung [5] investigated existence and uniqueness of fixed point for F-contractions as follows:

Theorem 1.4 *[5] Let (M, d) be a complete metric space and let $f : M \to M$ be an F-contraction for $F \in \mathcal{F}$. Then f has a unique fixed point u^* in M and for each $u \in M$, the sequence $\{f^n u\}$ converges to u^*. It is obvious that if $F(a) = lna$, then from Theorem 1.1, we have*

$$d(fu, fv) \leq e^{-\tau} d(u, v),$$

for all $u, v \in M$.

Definition 1.3 *[2] Let (M, d) be a metric space and $\alpha : M \times M \to (0, +\infty) \cup \{-\infty\}$ be a symmetric function. The mapping $f : M \to M$ is said to be an μ-type F-Suzuki contraction if there exist $F \in D$ and $\tau > 0$ such that for all $u, v \in M$ with $fu \neq fv$*

$$d(u, fu) \leq d(u, v) \Rightarrow \tau + \alpha(u, v) F(d(fu, fu)) \leq F(d(u, v)).$$

Remark 1.2 *[2] Every μ-type F-Suzuki contraction is an F-Suzuki contraction with $\mu(u, v) = 1$, for all $u, v \in M$. But the converse is not necessarily true. For example, see (Budhia* et al. *2016, Example* 3.2*).*

In 2016, Chang *et al.* [22] introduced the notion of a generalized F-contraction as follows;

Definition 1.4 *[22] Let (M, d) be a complete metric space.*
(1) A mapping $f : M \to M$ is said to be a generalized F-contraction of type (I), if there exists $\tau > 0$ such that for all $u, v \in M$,

$$d(fu, fv) > 0 \Rightarrow \tau + F(d(fu, fv)) \leq F(D(u, v)),$$

where $F \in \mathcal{F}$ and

$$D(u, v) = max\{d(u, v), d(u, fu), d(v, fv), d(u, fv), d(v, fu)\}.$$

(2) A mapping $f : M \to M$ is said to be a generalized F-contraction of type (II), if there exists $\tau > 0$ such that for all $u, v \in M$,

$$d(fu, fv) > 0 \Rightarrow \tau + F(d(fu, fv)) \leq F(E(u, v)),$$

where $F \in \mathcal{F}$ and $E(u, v) = max\{d(u, v), d(u, fu), d(v, fv), \frac{1}{2}d(u, fv), d(v, fu)\}$.

In 2015, Kumam *et al.* [18] generalized the contraction condition by adding four new values $d(f^2u, u)$, $d(f^2u, fu)$, $d(f^2u, v)$, $d(f^2u, fv)$ and introduced F-Suzuki contraction mappings in complete metric space while Luambano *et al.* [11] established a fixed point theorem for F-contraction mappings in partial metric spaces. The Suzuki-type generalization may have many applications, as in computer science, game theory, bio sciences, and in other areas of mathematical sciences such as in dynamic programming, integral equations, and data dependence. Wangwe and Kumar [12] proved fixed point results for interpolative ψ-Hardy-Rogers type contraction mappings in quasi-partial b-metric spaces.

Definition 1.5 *[4]. Let (M, d), be a metric space, $F : (0, \infty) \to \mathbb{R}$ and $\phi : (0, \infty) \to (0, \infty)$ satisfy the following;*

(1) F *is strictly increasing, i.e $u < v$ implies $F(u) < F(v)$ for all $u, v \in (0, \infty)$,*

(2) $\lim_{\alpha \to +0} F(\alpha) = -\infty$,

(3) $\lim_{\mu \to s^+} inf \phi(\mu) > 0$ *for all $s > 0$.*

A mapping $f : M \to M$ is called an (ϕ, F)-contraction on (M, d) if

$$\phi(d(u, v)) + F(d(fu, fv)) \leq F(d(u, v)),$$

for all $u, v \in M$ for which $fu \neq fv$.

In 2014, Piri and Kumam, [9] introduced the notion of F-Suzuki contraction as follows:

Definition 1.6 *[9] Let (M, d) be a complete metric space. A mapping $f : M \to M$ is said to be an F-Suzuki contraction, if there exists $\tau > 0$ such that for all $u, v \in M$ with $fu \neq fv$,*

$$\frac{1}{2}d(u, fu) < d(u, v) \Rightarrow \tau + F(d(fu, fv)) \leq F(d(u, v)),$$

where $F \in \mathcal{F}$.

The following theorem, which is a modified version of the Suzuki F-contraction in the Definition 1.6, was also proven by Piri and Kumam [9].

Theorem 1.5 *[9] Let (M, d) be a complete metric space and let $f : M \to M$ be an F-Suzuki contraction. Then f has a unique fixed point $u^* \in M$ and for each $u \in M$, the sequence $\{f^n u\}$ converges to u^*.*

Wardowski [3] introduced Hardy-Rogers-type F-contraction as follows:

Definition 1.7 *[3] Let (M, d) be a metric space. A self-mapping f on M is called an F-contraction of Hardy-Rogers-type if there exist $F \in \mathcal{F}$ and $\tau \in \mathbb{R}^+$ such that*

$$\tau + F(d(fu, fv)) \leq F(\alpha d(u, v) + \beta d(u, fu) + \gamma d(v, fv) + \delta d(u, fv) + Ld(v, fu)),$$

for all $u, v \in M$ with $d(fu, fv) > 0$, where $\alpha + \beta + \gamma + 2\delta = 1$, $\gamma \neq 1$ and $L \geq 0$.

Wardowski [3] introduced Hardy-Rogers-type F-contraction that is restricted to some conditions as follows:

Definition 1.8 *[3] Let (M, d) be a metric space and $f : M \to M$ an F-contraction of Hardy-Rogers type. Assume that f satisfies with $\beta = \gamma = L = 0$ and $F : \mathbb{R}^+ \to \mathbb{R}$ given by $F(u) = lnu$. It is clear that F satisfies $(F_1), (F_2)$. From*

$$\tau + F(d(fu, fv))) \leq F(d(u, v))$$

for all $u, v \in M$, $fu \neq fv$, we obtain

$$d(fu, fv) \leq e^{-\tau} d(u, v)$$

for all $u, v \in M$, $fu \neq fv$.

The following theorem gives a condition for existence and uniqueness of fixed point for Hardy-Rogers F-contraction in a complete metric space as follows:

Theorem 1.6 *[14] Let (M, d) be a complete metric space, and let f be a self-mapping on M. Assume that there exist a continuous function $F \in \mathcal{F}$ and $\tau \in S$ such that f is an F-contraction of Hardy-Rogers type, that is, Definition 1.7 holds for all $u, v \in M$ with $fu \neq fv$. Then f has a unique fixed point.*

Corollary 1.1 *[14] Let (M, d) be a complete metric space, and let T be a self-mapping on M. Assume that there exist an upper semi continuous $F \in \mathcal{F}$ and $\tau \in S$ such that*

$$\tau + F(d(fu, fv)) \leq F(\beta d(u, fu) + \gamma d(v, fv)).$$

Also, a version of the Chatterjea [24] fixed point is obtained from the Theorem 1.6 by putting $\alpha = \beta = \gamma = 0$ and $\delta = L = \frac{1}{2}$.

Wangwe and Kumar [13] studied the common fixed point for generalized F-Kannan contraction in metric space as follows:

Definition 1.9 *[13] Let F be a mapping satisfying $(F1) - (F3)$. A pair of two self-mappings $f, g : M \to M$ is said to be an F-Kannan -mapping if the following holds:*

(FK1)$fgu \neq fgy \Rightarrow fgu \neq u$ or$fgv \neq v$,

(FK2) There exists $\gamma > 0$ such that $\gamma + F(d(fgu, fgv)) \leq F[\frac{d(fu, fgu) + d(fv, fgv)}{2}]$,
$\forall u, v \in M$, with $fgu \neq fgv$.

The following example illustrates how to choose an appropriate mapping F which allows us to obtain specific classes of contractions that are well-known in the literature.

Example 1.4 *[3] Let $F : R^+ \to R$ be given by $F(u) = lnu$. It is clear that F satisfies $(F1) - (F2)$ and $(F3)$ for any $k \in (0, 1)$. Each mapping $T : M \to M$ satisfying Definition (1.7) is an F-contraction such that*

$$d(fu, fv) \leq e^{-\tau} d(u, v),$$

for all $u, v \in M$, $fu \neq fv$. It is clear that for $u, v \in M$ such that $fu = fv$ the previous inequality also holds and hence f is a contraction.

Proposition 1.1 *[14] Let (M, d) be a complete metric space and let f be a self-mapping on M. Assume that there exist $F \in \mathcal{F}$ and $\tau \in S$ such that*

$$\tau d(u, fu) + Fd(fu, fu^2) \leq F((\alpha + \beta)d(u, fu) + \gamma d(fu, f^2u) + \delta d(u, f^2u))$$

for all $u \in M$ with $fu \neq f^2u$, where $\alpha, \beta, \gamma, \delta \in [0, +\infty), \alpha + \beta + \gamma + 2\delta = 1$ and $\gamma \neq 1$. Then the sequence $\{d(f^{n-1}u_0, f^n u_0)\}$ is decreasing and $d(f^{n-1}u_0, f^n u_0) \to 0$ as $n \to +\infty$ for all $u_0 \in M$.

Arshad [15] introduced the concept of F-contraction of rational-type:

Definition 1.10 *[15] Let (M, d) be a metric space and $f : M \to M$ be a self mapping. Then f is said to be generalized F-contraction of rational-type A if there exists $\tau > 0$ such that for all $u, v \in M$*

$$d(fu, fv) > 0 \Rightarrow \tau + F(d(fu, fv)) \leq F(D(u, v)),$$

where

$$D(u, v) = max\{d(u, v), d(u, fu), d(v, fv), (\frac{d(u, fu) + d(v, fu)}{d(u, fu) + d(v, fv) + 1}), d(u, v)\},$$

where $F : \mathbb{R}^+ \to \mathbb{R}$ is a mapping satisfying the following conditions:

(F_1) F is strictly increasing i.e, $\forall u, v \in \mathbb{R}^+$ such that $u < v$ implies $F(u) < F(v)$;

(F_2) For each sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n\to\infty} \mu_n = 0$ if and only if $\lim_{n\to\infty} F(\mu_n) = -\infty$;

(F_3) F is continuous on $(0, \infty)$.

We denote by $\mathcal{F}$, the set of all functions satisfying the conditions $(F_1) - (F_3)$.

The concept of an F-contraction has been generalized by Several authors, e.g. [5,7,28].

A revolutionary idea centered on searching for ways to prove existing theorems in spaces that are generalized from the normal metric space emerged along with the research of contraction mappings.

The idea of b-metric space was first suggested by Bakhtin in 1989. This idea generalized the axiom of triangle inequality by adding a variable $s \geq 1$ to the definition of metric space. Kumar [26] studied on coincidence points for a pair of ordered F-contraction mappings in ordered partial metric spaces as well as [27] in metric space.

Definition 1.11 *[10] A b-metric space on a nonempty set M is a function $d : M \times M \to [0, \infty)$, such that for all $u, v, w \in M$ and for some real number $s \geq 1$, it satisfies the following;*

(i) If $d(u, v) = 0$, then $u = v$,

(ii) $d(u, v) = d(v, u)$,

(iii) $d(u, v) \leq s[d(u, w) + d(w, v)]$, then a pair (M, d) is called b-metric space.

Example 1.5 *[31] Let $M = \{0, 1, 2\}$ and $d : M \times M \to \mathbb{R}_+$ defined by*

$$d(0, 0) = d(1, 1) = d(2, 2) = 0,$$
$$d(1, 0) = d(0, 1) = d(2, 1) = d(1, 2) = 1,$$
$$d(0, 2) = d(2, 0) = h,$$

where, m is given real number such that $h \geq 2$. *It is easy to check that for all* $u, v, w \in M$

$$d(u,v) \leq \frac{h}{2}(d(u,w) + d(w,v)).$$

Therefore, (M, d) *is a b-metric space with a coefficient* $s = \frac{h}{2}$. *The ordinary triangle inequality does not hold if* $h > 2$ *and so* (M, d) *is not a metric space.*

Example 1.6 *[17] Let* (M, d) *be a metric space and*

$$\rho(u,v) = (d(u,v))^k,$$

where $k > 1$ *is a real number. Then* ρ *is a b-metric with* $s = 2^{k-1}$.

Definition 1.12 *[10] Let* M *be a b-metric space and* $u \in M$, *then*

(i) A sequence $\{u_n\}$ *is convergent and converges to* u *in* M *if, for every* $\epsilon > 0$, *there exists* $n_0 \in N$ *such that* $d_b(u_n, u) < \epsilon$, *for all* $n > n_0$ *is represented as* $\lim_{n\to\infty} u_n = u$ *or* $u_n \to u$ *as* $n \to \infty$.

(ii) A sequence $\{u_n\}$ *is Cauchy sequence in* M, *if for every* $\epsilon > 0$, *there exists* $n_0 \in N$, *such that* $d_b(u_n, u_m) < \epsilon$, *for all* $n, m > n_0$ *or equivalently, if* $\lim_{n,m\to\infty} db(u_n, u_m) = 0$.

(iii) A b-metric space M *is a complete b-metric space if every Cauchy sequence in* (M, d_b) *is convergent in* M.

It was further asserted that the self distance of a point is no longer required to be zero as stated in the definition of metric space in the work of Matthews [23].

Definition 1.13 *[23] Let* $M \neq \emptyset$. *A partial metric is a function* $p : M \times M \to R^+$ *satisfying*

(i) $p(u,v) = p(v,u)$,

(ii) If $0 \leq p(u,u) = p(u,v) = p(v,v)$, *then* $u = v$,

(iii) $p(u,v) - p(w,w) \leq p(u,w) + p(w,v)$ *for all* $u, v, w \in M$.

Then a pair (M, p) *is called partial metric space. It is clear that, If* $p(u,v) = 0$, *then* $u = v$ *however, if* $u = v$, *then* $p(u,v)$ *may not be zero.*

Example 1.7 *[17] Let* $M = R^+$ *and* $p : M \times M \to R^+$ *given by* $p(u,v) = max\{u,v\}$ *for all* $u, v \in R^+$. *Then* (R^+, p) *is a partial metric space.*

Example 1.8 *[17] Let* $M = \{[\alpha,\beta] : \alpha,\beta \in \mathbb{R}, \alpha \leq \beta\}$. *Then* $p([\alpha,\beta],[\delta,\gamma]) = max\{\beta,\delta\} - min\{\alpha,\delta\}$ *defines a partial metric* p *on* M.

In 2013, Shukla [25] introduced the concept of a partial b-metric space as follows:

Definition 1.14 *[32] Let* M *be a nonempty set and* $s \geq 1$ *be a given real number. A function* $pb : M \times M \to [0,1)$ *is called a partial b-metric if for all* $u, v, w \in M$ *the following conditions are satisfied:*

(i) $u = v$ *if and only if* $pb(u,u) = pb(u,v) = pb(v,v)$,

(ii) $pb(u,u) \leq pb(u,v)$,

(iii) $pb(u,v) = pb(v,u)$,

(iv) $pb(u,v) \leq s[pb(u,w) + pb(w,v)] - pb(w,w)$.

The pair (M, pb) *is called a partial b-metric space. The number* $s \geq 1$ *is called the coefficient of* (M, pb).

Remark 1.3 *[32] Since a partial metric space is a special case of a partial b-metric space, the class of partial b-metric space* (M, pb) *is actually bigger than the class of partial metric space since it is a partial metric space when s = 1. Also, the class of partial b-metric space* (M, pb) *is effectively larger than the class of b-metric space, since a b-metric space is a special case of a partial b-metric space* (M, pb) *when the self distance* $p(u,u) = 0$.

The following example shows that a partial b-metric on M need not be a partial metric, nor a b-metric on M see also [25,32].

Example 1.9 *[25] Let $M = [0, 1)$. Define a function $pb : M \times M \to [0, 1)$ such that*

$$pb(u, v) = [max\{u, v\}]^2 + |u - v|^2$$

for all $u, v \in M$. Then (M, pb) is a partial b-metric space on M with the coefficient $s = 2 > 1$. But, pb is not a b-metric nor a partial metric on M.

Proposition 1.2 *[25] Let M be a nonempty set, and let p be a partial metric and d be a b-metric with the coefficient $s \geq 1$ on M. Then the function $pb : M \times M \to [0, 1)$ defined by*

$$pb(u, v) = p(u, v) + d(u, v)$$

for all $u, v \in M$, pb is a partial b-metric on M with the coefficient s.

Proposition 1.3 *[32] Let (M, p) be a partial metric space and $k \geq 1$. Then (M, pb) is a partial b-metric space with coefficient $s = 2^{k-1}$, where pb is defined by*

$$pb(u, v) = [p(u, v)]^k.$$

On the other hand, Mustafa [32] improved the Definition 1.14 in order that each partial b-metric pb generates a b-metric d as follows:

Definition 1.15 *[32] Let M be a nonempty set and $s \geq 1$ be a given real number. A function $pb : M \times M \to [0, 1)$ is called a partial b-metric if for all $u, v, w \in M$ the following conditions are satisfied:*

(i) $u = v$ *if and only if* $pb(u, u) = pb(u, v) = pb(v, v)$,

(ii) $pb(u, u) \leq pb(u, v)$,

(iii) $pb(u, v) = pb(v, u)$;

(iv) $pb(u, v) \leq s[pb(u, w) + pb(w, v)] - pb(w, w) + (\frac{1-s}{2})(pb(u, u) + pb(v, v))$.

The pair (M, pb) is called a partial b-metric space. The number $s \geq 1$ is called the coefficient of (M, pb).

Example 1.10 *[32] Let $M = \mathbb{R}$ is the set of real numbers. Consider the metric space (M, d) where d is the Euclidean distance metric*

$$d(u, v) = |u - v|$$

for all $u, v \in M$. Define

$$pb(u, v) = (u - v)^2 + 5$$

for all $u, v \in M$. Then pb is a partial b-metric on M with $s = 2$, but it is not a partial metric on M. To see this,
Let $u = 1, v = 4$ and $w = 2$. Then

$$pb(1, 4) = (1 - 4)^2 + 5 = 14 \neq pb(1, 2) + pb(2, 4) - pb(2, 2) = 6 + 9 - 5 = 10.$$

Also, pb is not a b-metric since $pb(u, u) \neq 0$ for all $u \in M$.

In 2018 K̈unzi *et al.* [8] dropped symmetry condition in the definition of a partial metric in order to obtain another type of partial metrics, namely partial quasi-metrics. Partial quasi metric was defined as follows:

Definition 1.16 *[8] A quasi partial metric on a nonempty set M is a function $qp : M \times M \to [0, \infty)$ such that;-*

1. *$qp(u, u) \leq qp(u, v)$ whenever $u, v \in M$,*

2. $qp(u,u) \leq qp(v,u)$ *whenever* $u,v \in M$,

3. $qp(u,w) + qp(v,v) \leq (qp(u,v) + qp(v,w))$, *whenever* $u,v,w \in M$,

4. $u = v$ *if and only if* $qp(u,u) = qp(u,v) = qp(v,v)$ *whenever* $u,v \in M$.

A pair (M, qp) *is called a quasi partial metric space .*

In 2015 Gupta *et al.* [1] introduced the concept of quasi partial $b-$metric space as follows:

Definition 1.17 *[1] A quasi-partial b-metric on a nonempty set* M *is a function* $qp_b : M \times M \to [0,\infty)$ *such that for some real number* $s \geq 1$, *it satisfies the following:*

(i) *If* $qp_b(u,u) = qp_b(u,v) = qp_b(v,v)$ *then* $u = v$ *(indistancy implies equality),*

(ii) $qp_b(u,u) \leq qp_b(u,v)$ *(small self-distances),*

(iii) $qp_b(u,u) \leq qp_b(v,u)$ *(small self-distances),*

(iv) $qp_b(u,v) + qp_b(w,w) \leq s[qp_b(u,w) + qp_b(w,v)]$ *(triangularity).*
Then a pair (M, qp_b) *is called a quasi partial b-metric space.*

The purpose of this paper is to introduce the concept of F-contraction of Hardy-Rogers type and study the existence and uniqueness problems of common fixed points for a pair of self-contraction mappings in the setting of complete quasi partial b-metric space. As an application, we use our results to study the existence of solutions for boundary value problems. The results presented in the paper extend and improve the Banach contraction principle [20], Suzuki contraction theorem [29], and the corresponding main results in Wardowski [3] and Piri and Kuman [9].

2. Main Results

To establish our first main results, we start by extending Definition 1.7 using quasi partial b-metric space concept.

Definition 2.1 A pair (f, g) of self commuting maps on complete quasi partial b-metric space M are called F-contractions of Hardy-Rogers type if there exists $\tau > 0$ such that for all $u, v \in M$,

$$\begin{aligned} qp_b(fg(u), fg(v))) > 0 \quad \Rightarrow \quad & \tau + F(qp_b(fg(u), fg(v))) \\ \leq \quad & F(\alpha qp_b(gu, gv) + \beta qp_b(gu, fu) + \gamma qp_b(gv, fv) + \delta qp_b(gu, y) + L qp_b(gv, fu)), \end{aligned}$$

where $F : \mathbb{R}^+ \to \mathbb{R}$ is an increasing mapping, $\alpha, \beta, \gamma,$ and δ are non-negative numbers, $\delta < \frac{1}{2}, \gamma < 1, \alpha + \beta + \gamma + 2\delta = 1, 0 < \alpha + \delta + L \leq 1$.

Now, we introduce and prove our main theorem.

Theorem 2.1 *Let* (f, g) *be a pair of self commuting maps of a complete quasi partial b-metric space* M. *Suppose there exists* $\tau > 0$ *such that for all* $u, v \in M$, *if*

$$\begin{aligned} qp_b(fg(u), fg(v)) > 0 \quad \Rightarrow \quad & \tau + F(qp_b(fg(u), fg(v))) \\ \leq \quad & F(\alpha qp_b(gu, gu) + \beta qp_b(gu, fu) + \gamma qp_b(gv, fv) + \delta qp_b(gu, fv) + L qp_b(gv, fu)), \end{aligned}$$

where $F : \mathbb{R}^+ \to \mathbb{R}$ *is an increasing mapping,* $\alpha, \beta, \gamma,$ *and* δ *are non-negative numbers,* $\delta < \frac{1}{2}, \gamma < 1, \alpha + \beta + \gamma + 2\delta = 1, 0 < \alpha + \delta + L \leq 1$. *Then* f *and* g *have a unique common fixed point* $u^* \in M$.

Proof: Let $u_0 \in M$ be an arbitrary point and construct a sequence $\{u_n\}_{n\in\mathbb{N}} \in M$, that satisfies the following:

$$gu_1 = fg(u_0), gu_2 = fg(u_1) = fg^2(u_0), ..., gu_n = fg(u_{n-1}) = fg^n(u_0), \forall n \in N.$$

Then, if there exists $n \in \mathbb{N} \cup 0$ such that

$$qp_b(gu_n, fgu_n) = 0. \tag{2.-1}$$

Thus, u_n is a common fixed point of f and g and the proof is complete. Hence, we assume that

$$0 \ < qp_b(gu_n, fgu_n) = qp_b(fgu_{n-1}, fgu_n), \tag{2.0}$$

$\forall n \in \mathbb{N}$. By the hypothesis and the monotony of F, we have for all $n \in \mathbb{N}$.

Now, let

$$(qp_b)_n = qp_b(gu_n, gu_{n+1}) \tag{2.1}$$

$\forall n \in \mathbb{N}$. Now, By the hypothesis and the monotony of F, we have for all $n \in \mathbb{N}$

$$\begin{aligned}
\tau + F(qp_b(fu_n, gu_{n+1})) &= \tau + F(qp_b(fgu_{n-1}, fgu_n)) \\
&\leq F(\alpha qp_b(gu_{n-1}, gu_n) \\
&\quad +\beta qp_b(gu_{n-1}, fgu_{n-1}) + \gamma qp_b(gu_n, fgu_n) + \delta qp_b(gu_{n-1}, fgu_n) \\
&\quad +Lqp_b(gu_n, fgu_{n-1}) \\
&= F(\alpha qp_b(gu_{n-1}, gu_n) + \beta qp_b(gu_{n-1}, gu_n) + \gamma qp_b(gu_n, gu_{n+1}) + \\
&\quad \delta qp_b(gu_{n-1}, gu_{n+1}) + Lqp_b(gu_n, gu_n)).
\end{aligned}$$

Since , $qp_b(gu_n, gu_n)) \geq 0$ we consider two cases:
Case(i) When $qp_b(gu_n, gu_n)) = 0$ and $s = 1$,

$$\begin{aligned}
&F(\alpha qp_b(gu_{n-1}, gu_n) + \beta qp_b(gu_{n-1}, fgu_{n-1}) + \gamma qp_b(gu_n, fgu_n) \\
&+ \delta qp_b(gu_{n-1}, fgu_n) + Lqp_b(gu_n, fgu_{n-1}) \\
&= F(\alpha(qp_b)_{n-1} + \beta(qp_b)_{n-1} + \gamma(qp_b)_n + \delta qp_b(gu_{n-1}, gu_{n+1})). \\
&\leq F(\alpha(qp_b)_{n-1} + \beta(qp_b)_{n-1} + \gamma(qp_b)_n \\
&\quad +\delta[qp_b(gu_{n-1}, gu_n) + qp_b(gu_n, gu_{n+1})]) \\
&= F(\alpha(qp_b)_{n-1} + \beta(qp_b)_{n-1} + \gamma(qp_b)_n \\
&\quad +\delta[(qp_b)_{n-1} + (qp_b)_n]) \\
&= F((\alpha + \beta + \delta)(qp_b)_{n-1} + (\gamma + \delta)(qp_b)_n).
\end{aligned}$$

It follows that ,

$$\begin{aligned}
F((qp_b)_n) &\leq F((\alpha + \beta + \delta)(qp_b)_{n-1} + (\gamma + \delta)(qp_b)_n) - \tau \\
&< F((\alpha + \beta + \delta)(qp_b)_{n-1} + (\gamma + \delta)(qp_b)_n)
\end{aligned}$$

so from the monotony of F, we get

$$(qp_b)_n < (\alpha + \beta + \delta)(qp_b)_{n-1} + (\gamma + \delta)(qp_b)_n,$$

for all $n \in \mathbb{N}$. Since $\gamma \neq 1$ and $\alpha + \beta + \gamma + 2\delta = 1$, we deduce that $1 - \gamma - \delta > 0$ and so

$$(qp_b)_n < \frac{\alpha + \beta + \delta}{1 - \gamma - \delta} = (qp_b)_{n-1},$$

for all $n \in N$. Thus, we conclude that the sequence $(qp_b)_n$ is strictly decreasing, so there exists $\lim_{n\to\infty}(qp_b)_n = d$. Suppose that $d > 0$. Since F is an increasing mapping there exists $\lim_{x\to(qp_b)+} F(x) = F(d + 0)$, so taking the limit as $n \to \infty$ in Inequality 2.2 we get $F(d + 0) \leq F(d + 0) - \tau$, which is a contradiction. Therefore,

$$\lim_{n\to\infty}(qp_b)_n = 0.$$

Case (ii) We now consider that, $qp_b(gu_n, gu_n) > 0$ and $s > 1$, then

$$F(\alpha qp_b(gu_{n-1}, gu_n) + \beta qp_b(gu_{n-1}, fgu_{n-1}) + \gamma qp_b(gu_n, fgu_n) + \delta qp_b(gu_{n-1}, fgu_n) + Lqp_b(gu_n, fgu_{n-1})$$

$$\begin{aligned} &= F(\alpha qp_b(gu_{n-1}, gu_n) + \beta qp_b(gu_{n-1}, gu_n) \\ &\quad +\gamma qp_b(gu_n, gu_{n+1}) \\ &\quad +\delta qp_b(gu_{n-1}, gu_{n+1}) + Lqp_b(gu_n, gu_n)) \\ &\leq F(\alpha (qp_b)_{n-1} + \beta (qp_b)n - 1 + \gamma (qp_b)_n \\ &\quad +s\delta[(qp_b)_{n-1} + (qp_b)_n - qp_b(gu_n, gu_n)]), \end{aligned}$$

where $s > 1$.

$$= F((\alpha + \beta + s\delta)(qp_b)_{n-1} + (qp_b)_n(\gamma + s\delta) - s\delta qp_b(gu_n, gu_n)).$$

It follows that,

$$F((qp_b)_n) \leq F((\alpha + \beta + s\delta)(qp_b)_{n-1} + \gamma (qp_b)_n). \tag{2.2}$$

Assuming that, $\gamma = 0$ and $s = 2$, and using monotony of F, we get

$$(qp_b)_n < (\alpha + \beta + 2\delta)(qp_b)_{n-1} + \gamma (qp_b)_n$$

$$(qp_b)_n < \frac{\alpha + \beta + 2\delta}{1 - \gamma}(qp_b)_{n-1} = (qp_b)_{n-1}.$$

To complete, we follow the same steps as in Case 1.
We claim that $\{u_n\}$ is a Cauchy sequence. Arguing by contradiction, we assume that there exists $\epsilon > 0$ and a sequences $p(n)_{n\in\mathbb{N}}$ and $q(n)_{n\in\mathbb{N}}$ of natural numbers such that $p(n) > q(n) > n$,

$$qp_b(u_{p(n)}, u_{q(n)}) > \epsilon, qp_b(u_{p(n)-1}, u_{q(n)}) \leq \epsilon, \forall n \in \mathbb{N}.$$

Then, using triangle inequality, we have

$$\begin{aligned} &\epsilon < qp_b(u_{p(n)}, u_{q(n)}) \\ &\leq s[qp_b(u_{p(n)}, u_{q(n)-1}) + qp_b(u_{q(n)-1}, u_{q(n)}) \\ &- qp_b(u_{p(n)-1}, p(n) - 1)] \\ &\leq s[qp_b(u_{q(n)-1}, u_{q(n)}) + \epsilon] \end{aligned}$$

where $s \geq 1$. It follows from the relation 2.3 and the above inequality that

$$\lim_{n\to\infty} qp_n(u_{p(n)}, u_{q(n)}) = \epsilon. \tag{2.3}$$

Since

$$qp_b(u_{p(n)-1}, u_{q(n)-1}) > \epsilon > 0.$$

By the hypothesis and monotony of F, we have

$$\begin{aligned} \tau + F(qp_b(fgu_{p(n)-1}, fgu_{q(n)-1})) &\leq F(\alpha qp_b(gu_{p(n)-1}, gu_{q(n)-1}) + \beta qp_b(gu_{p(n)-1}, fu_{q(n)-1}) \\ &\quad +\gamma qp_b(gu_{p(n)-1}, fu_{q(n)-1}) + \delta qp_b(gu_{p(n)-1}, fu_{q(n)-1}) \\ &\quad +Lqp_b(fu_{p(n)-1}, gu_{q(n)-1}), \\ &= F(\alpha qp_b(u_{p(n)-1}, u_{q(n)-1}) + \beta qp_b(u_{p(n)-1}, u_{q(n)-1}) \\ &\quad +\gamma qp_b(u_{p(n)-1}, u_{q(n)-1}) \\ &\quad +\delta qp_b(u_{q(n)-1}, u_{p(n)-1}) + Lqp_b(u_{q(n)-1}, u_{p(n)-1})), \\ &\leq F(\alpha (qp_b)_{p(n)-1} + \beta (qp_b)_{p(n)-1} + L(qp_b)_{p(n)-1}) \end{aligned} \tag{2.4}$$

$$\begin{aligned} F(qp_b(fgu_{p(n)-1}, fgu_{q(n)-1})) &\leq F(\alpha qp_b(gu_{p(n)-1}, gu_{q(n)-1}) + \beta qp_b(gu_{p(n)-1}, fu_{q(n)-1}) \\ &\quad +\gamma qp_b(gu_{p(n)-1}, fu_{q(n)-1}) + \delta qp_b(gu_{p(n)-1}, fu_{q(n)-1}) \\ &\quad +Lqp_b(fu_{p(n)-1}, gu_{q(n)}) - \tau. \end{aligned} \tag{2.5}$$

Taking the limit as $n \to \infty$ in the inequality 2.5, we get $F(\epsilon+0) \leq F(\epsilon+0) - \tau$, which is a contradiction. This shows that $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since (M, qp_b) is a complete quasi partial b- metric space, we have that$\{x_n\}_{n\in\mathbb{N}}$ converges to some point u^* in M.

If there exists a sequence $\{p(n)\}_{n\in\mathbb{N}}$ of natural numbers such that

$$gu_{p(n)+1} = fgu_{p(n)} = fgu^*,$$

then

$$\lim_{n\to\infty} gu_{p(n)+1} = fu^*.$$

Otherwise, there exists $N \in \mathbb{N}$ such that

$$gu_{p(n)+1} = fgu_{p(n)} = fgu_n \neq fgx^*. \tag{2.6}$$

Assume that

$$fgu^* \neq gu^*.$$

By the hypothesis, we have

$$\begin{aligned} \tau + F(qp_b(fgu_n, fgu^*)) &\leq F(\alpha qp_b(gu_n, gu^*) + \beta qp_b(gu_n, fu^n) \\ &\quad +\gamma qp_b(gu^*, fu^*) + \delta qp_b(gu_n, fu^*) + Lqp_b(fu_n, gu^*)), \end{aligned}$$

so

$$\begin{aligned} \tau + F(qp_b(fgu_n, fgu^*)) &\leq F(\alpha qp_b(gu_n, gu^*) + \beta qp_b(gu_n, fu_{n+1}) \\ &\quad +\gamma qp_b(gu^*, fu^*) + \delta qp_b(gu_n, fu^*) + Lqp_b(gu_{n+1}, fu^*)). \end{aligned}$$

Since F is an increasing function, we deduce that

$$\begin{aligned} qp_b(fgx_n, fgx^*) &< \alpha qp_b(gu_n, gu^*) + \beta qp_b(gu_n, fu_{n+1}) + \gamma qp_b(gu^*, fu^*) \\ &\quad +\delta qp_b(gu_n, fu^*) + Lqp_b(gu_{n+1}, fu^*) \end{aligned}$$

so letting $n \to \infty$, we get

$$qp_b(fgu^*, fgu^*) \leq \gamma qp_b(gu^*, fu^*) + \delta qp_b(gx^*, fx^*) < qp_b(fgx^*, fgx^*).$$

This is a contradiction. Therefore,

$$fgu^* = gu^*.$$

Now, we will show that a pair (f, g) have a unique common fixed point. Let $u, v \in M$ be two distinct common fixed points of (f, g). Thus

$$fgu = gu \neq gu = fgu.$$

Hence ,

$$qp_b(fgu, fgv) = qp_b(gu, gv) > 0.$$

By hypothesis, since

$$0 < \alpha + \delta + L \leq 1$$

we have

$$\begin{aligned} \tau + F(qp_b(gu, gv)) &= \tau + F(qp_b(fgu, fgv)) \\ &\leq F(\alpha qp_b(gu, gv) + \beta qp_b(gu, fu) + \gamma qp_b(gv, fv) + \delta qp_b(gu, fv)) \\ &\quad +Lqp_b(fv, fu)) \\ &= F(\alpha qp_b(gu, gv) + \delta qp_b(gu, gv) + Lqp_b(gu, gv)) \\ &\leq F((\alpha + \beta + L) qp_b(gu, gv)) \\ &\leq F(qp_b(gu, gv)). \end{aligned}$$

This is a contradiction. Therefore, f and g have a unique common fixed point. □

We obtain the following Corollary by inserting $L = 1$ in Theorem 2.1 as follows:

Corollary 2.1 *Let (M, qp_b) be a complete quasi partial b metric space and let (f, g) be a pair of self-commuting maps on space M. Assume that there exists $F : \mathbb{R}^+ \to \mathbb{R}$ an increasing mapping and $\tau > 0$ such that*

$$\tau + F(qp_b(fgu, fgv)) \leq F(\alpha qp_b(gu, gv) + \beta qp_b(gu, fv) + \gamma qp_b(gv, fv))$$

$\forall u, v \in M, fu \neq fv$ and $gu \neq gv$, where $\alpha + \beta + \gamma = 1, \alpha > 0$. Then f and g have a unique fixed point in X.

We obtain the following Corollary by inserting $\alpha = 1$, $\beta = \gamma = \delta = L = 0$ in Theorem 2.1 as follows:

Corollary 2.2 *Let (M, qp_b) be a quasi partial b metric space and let (f, S) be a pair of self-commuting maps on space M. Assume that there exist $F : \mathbb{R}^+ \to \mathbb{R}$ an increasing mapping and that $\tau > 0$ such that*

$$\tau + F(qp_b(fgu, fgv)) \leq F(qp_b(gu, gv))$$

$\forall u, v \in M$, $fu \neq fv$ and $gu \neq gv$. Then f and g have a unique fixed point in M.

Theorem 2.2 *Let (f, g) be a pair of commuting-mappings from M to itself. Suppose there exists $\tau > 0$ such that $\forall u, v \in M$,*

$$qp_b(fgu, fgv) > 0 \Rightarrow \tau + F(qp_b(fgu, fgv)) \leq F(qp_b(gu, gv)),$$

where $F : \mathbb{R}^+ \to \mathbb{R}$ is a mapping satisfying conditions (F_2) and (F_3) in the Definition 1.2 and F is continuous on $(0, \alpha)$, with α a real number. Then, f and g have a unique fixed point u^* and for every $u \in M$ the sequence $\{fu_n\} = \{gu_{n+1}\}$ converges to u^*.

Proof: We choose $u_0 \in M$ and construct a sequence

$$gu_1 = fgu_0, gu_2 = fgu_1 = fg^2u_0, ..., gu_n = fgu_{n-1} = fg^n u_0, \forall n \in N.$$

$\forall n \in \mathbb{N}$.
If there exists $n \in N \cup \{0\}$ such that

$$qp_b(gu_n, fgu_n) = 0,$$

then u_n is a fixed point of f and g the proof is complete. Hence, we assume that $0 < qp_b(gu_n, fgu_n) = qp_b(fgu_n, fgu_n), \forall n \in \mathbb{N}$.
By the hypothesis we have

$$\tau + F(qp_b(fgu_{n-1}, fgu_n)) \leq F(qp_b(gu_{n-1}, gu_n)), \forall n \in \mathbb{N}$$

i.e

$$\begin{aligned}
F(qp_b(fgu_{n-1}, fgu_n)) &\leq F(qp_b(gu_{_{n-1}}, gu_n)) - \tau \\
&= F(qp_b(fgu_{n-2}, fgu_{n-1})) - \tau \\
&\leq F(qp_b(gu_{n-2}, gu_{n-1}) - 2\tau \\
&= F(qp_b(fg_{u_{n-2}}, fgu_{n-1})) - 2\tau \\
&\leq F(qp_b(gu_{n-3}, gu_{n-2})) - 3\tau \\
&= F(qp_b(fgu_{n-3}, fgu_{n-2})) - 3\tau \\
&\leq \text{..............................} \\
&\leq F(qp_b(gu_0, gx_1)) - n\tau.
\end{aligned}$$

This implies that

$$\lim_{n\to\infty} F(qp_b(gu_n, gu_{n+1})) = \lim_{n\to\infty} F(qp_b(fgu_{n-1}, fgu_n)) = -\infty,$$

by (F_2) we obtain that

$$\lim_{n\to\infty} qp_b(gu_n, gu_{n+1}) = 0.$$

Now, we claim that $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Arguing by contradiction, we assume that there exists $0 < \epsilon < \alpha$ such that for any $k \in \mathbb{N}$ there exist $m_k > n_k \geq k$ such that

$$qp_b(u_{n_k}, u_{m_k}) \geq \epsilon,$$

Now, let us assume that:

$$qp_b(u_{n_k-1}, u_{m_k}) < \epsilon,$$

$\forall n \in \mathbb{N}$. like in the proof of the Theorem 2.1 we obtain

$$\lim_{n\to\infty} qp_b(u_{n_k}, u_{m_k}) = \lim_{n\to\infty} qp_b(u_{n_k-1}, u_{m_k-1}) = \epsilon.$$

By the hypothesis, we have

$$\tau + F(qp_b(fu_{n_k-1}, fu_{m_k})) \leq F(qp_b(gu_{n_k-1}, gu_{m_k-1})),$$

$\forall n \in \mathbb{N}$. This implies

$$\tau + F(qp_b(gu_{n_k}, gu_{m_k})) \leq F(qp_b(gu_{n_k-1}, gu_{m_k-1})),$$

$\forall n \in \mathbb{N}$.
By (F_3) taking the limit as $n \to \infty$, we get $\tau + F(\epsilon) \leq F(\epsilon)$, which is a contradiction. Therefore the $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. By completeness of (M, qp_b), it follows that $\{u_n\}_{n\mathbb{N}}$ converges to some point $u^* \in M$.
Now , we claim that u^* is a common fixed point for f and g. Arguing by contradiction, we assume that

$$fu^* = gu^* = u^*$$

if there exists a sequence of natural $\{u_{n_k}\}$ numbers such that

$$\lim_{n\to\infty} gu_{n_k} = \lim_{n\to\infty} fu_{n_k} = fu^*$$

then

$$\lim_{n\to\infty} fu_{n_k} = \lim_{n\to\infty} gu_{n_k+1} = u^*.$$

This is a contradiction. Then, by hypothesis, we get

$$\tau + F(qp_b(gu_{n+1}, Tu^*)) \leq F(qp_b(gu_n, gu^*)),$$

$\forall n \geq \mathbb{N}$. By (F_3) , taking the limit as $n \to \infty$ in the above inequality, we have

$$\lim_{n\to\infty} F(qp_b(gu_{n+1}, fu^*) = -\infty.$$

Hence, by (F_2) we get

$$\lim_{n\to\infty} qp_b(gu_{n+1}, fu^*) = 0$$

which implies that

$$qp_b(gu^*, fu^*) = 0.$$

This is a contradiction therefore, u^* is a common fixed point of f and g. The uniqueness yield the same results as in the proof of Theorem 2.1. $\square$

Example 2.1 Let $M = \{A_n : n \in \mathbb{N}\} \cup \{B\}$ and $qp_b : M \times M \to \mathbb{R}^+$ for all $s \geq 1$, such that $qp_b(A_n, A_n) \leq qp_b(A_n, B)$ and $qp_b(B, A_n) \geq qp_b(A_n, A_n)$. Further more, $qp_b(A_n, A_n) = qp_b(B, B) = qp_b(A_n, B) = \frac{1}{n} \Rightarrow A_n = B$, it is obviously that (M, qp_b) is a complete quasi partial b-metric space.

Let $f, g : M \to M$ be a pair of commuting maps, such that $fA_n = A_{n+1} = gA_{n+1} = A_{n+1}$ and $fB = gB = B$. Let us suppose there exists $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying the hypothesis of Theorem 2.1.

Taking $u = A_{n+i}$, $v = A_{n+i+1}$, we have $\forall n \geq 1$ and $i \geq 0$

$$qp_b(fA_{n+i}, fA_{n+i+1}) = qp_b(A_{n+i}, A_{n+i+1}) > 0,$$

which implies that,

$$\begin{aligned} &\tau + F(qp_b(fA_{n+i}, fA_{n+i+1}) \\ &= \tau + F(\frac{1}{n+i+1}) \leq F(\frac{1}{n+i}). \end{aligned}$$

Thus, we get,

$$\sum_{i=1}^{n-1}[\tau + F(\frac{1}{n+i+1})] \leq \sum_{i=1}^{n-1} F(\frac{1}{n+i}),$$

where,

$$n\tau + F(\frac{1}{2n}) \leq F(\frac{1}{n}).$$

Hence

$$\tau + \frac{1}{n}F(\frac{1}{2n}) \leq \frac{1}{n}F(\frac{1}{n}) \tag{2.7}$$

Therefore, taking the limit as $n \to \infty$ in inequality 2.7, we obtain $\tau \leq 0$, which is a contradiction. Hence, F cannot satisfy the hypothesis of Theorem 2.1.

But for, $F : \mathbb{R}^+ \to \mathbb{R}$, $F(x) = -\frac{1}{x}$, is increasing and satisfies Corollary 2.2:

$$\begin{aligned} &\tau + F(qp_b(A_{n+i}, A_{n+i+1})) \\ &= \tau + F(\frac{1}{n+i+1}) \\ &= \tau - n - i - 1 \leq -n - i, \end{aligned}$$

which implies that, $\tau \leq 1$

3. An application to boundary value problem

In this section, we apply Theorem 2.1, to obtain the existence of solution to the boundary value problem,

$$\begin{cases} -\frac{d^2v}{du^2} = g(u, v(u)), & x \in [0, 1], \\ v(0) = v(1) = 0. \end{cases} \tag{3.1}$$

where $g : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous mapping. The Green-function associated with the boundary value problem 3.1 is defined by

$$G(u, v) = \begin{cases} u(1 - v), & 0 \leq u \leq v \leq 1, \\ v(1 - u), & 0 \leq v \leq u \leq 1. \end{cases} \tag{3.2}$$

Let $C[0, 1]$ be the space of all continuous mappings defined on $[0, 1]$ and

$$\mathcal{C} = (C[0, 1], \mathbb{R}^+),$$

we can define $a, h \in \mathcal{C}$. If we define the mapping $qp_b : \mathcal{C} \times \mathcal{C} \to \mathbb{R}^+$, defined by

$$qp_b(a, h) = Sup\{|g(u, a) - g(u, h)| + |g(u, a))|\}, \tag{3.3}$$

the pair $(\mathcal{C}, qp_b)$ is a complete quasi partial b-metric space.

If $f, g : \mathcal{C} \to \mathcal{C}$ is a pair of maps, the boundary value problem 3.1 associated with this pair has the following form;

$$f(v(u)) = g(v(u)) = \int_0^1 G(u,s)g(s,u(s))ds = v(u), \tag{3.4}$$

for $u \in [0,1]$, note that the boundary value problem has a unique solution if and only if f and g have a common fixed point.

Theorem 3.1 *Let $\mathcal{C} = C([0,1], \mathbb{R})$ and the mapping $g : [0,1] \times \mathcal{C} \to \mathbb{R}$. Suppose that we have the functions $a, h \in \mathcal{C}$ such that g satisfies*

$$|g(u,a) - g(u,h)| + |g(u,a)| \leq 8 \ln(\frac{e^{e^{-\tau}}\omega}{\rho})$$

where, $u \in [0,1]$, $t, h \in \mathcal{C}$ and $\omega = |qp_b(f(a), f(h))| = |qp_b(g(a), g(h))|, \rho > 0$ and $\tau > 0$. Then the boundary value problem 3.1 has a unique solution.

Proof: Note that $\mu(u) \in (C^2[0,1], \mathbb{R})$ (say) is a solution of Equation 3.1 is given by the common fixed point of f and g. i.e,

$$\mu(u) = f(\mu(u)) = g(\mu(u)).$$

Let $t, h \in \mathcal{C}$ and $x \in [0,1]$ by using equation 3.4 , we get

$$\begin{aligned} |f(a(u)) - f(h(u))| + |f(a(u))| &= [|\int_0^1 G(u,s)[g(s,a(s)) - g(s,h(s))]ds)|] \\ &\quad + |\int_0^1 G(u,s)g(s,a(s))ds| \end{aligned}$$

$$\leq [\int_0^1 G(u,s)[|g(u,a(s)) - g(s,h(s))| + g(s,a(s))]ds \leq [8 \int_0^1 G(u,s) \ln(\frac{e^{e^{-\tau}}\omega}{\rho})ds]$$

$$= 8 \ln(\frac{e^{e^{-\tau}}\omega}{\rho})(sup_{u\in[0,1]}[\int_0^1 G(u,s))ds]).$$

Since

$$\int_0^1 G(u,s)ds = \frac{-u^2}{2} + \frac{u}{2}, \forall u \in [0,1],$$

we have

$$sup\{\int_0^1 G(u,s)ds\} = \frac{1}{8},$$

which is obtained at $u = \frac{1}{2}$.

This implies that

$$\begin{aligned} |qp_b(f(a), f(h))| = |sup_{u\in[0,1]}(|f(a(u)) - f(h(u))| + f(a(u))| &= ln(\frac{e^{e^{-\tau}}\omega}{\rho}) \\ &\leq e^{-\tau}\omega = e^{-\tau}|qp_b(f(a), f(h)|, \end{aligned}$$

since $\omega = qp_b(g(a), g(h))$. This implies that

$$qp_b(f(a), f(h)) \leq e^{-\tau}(qp_b(g(a), g(h))).$$

By taking the natural logarithm both sides, we obtain;

$$\ln |qp_b(f(a), f(h))| \leq \ln e^{-\tau}|(qp_b(ga, gh))|.$$

Define $F(a) = \ln(a), \forall a \in \mathcal{C}$ and simplifying the last inequality, we have

$$\tau + F(qp_b(|f(a), f(h)|) \leq F(|qp_b(g(a), g(h))|)$$

in which Theorem 2.1 holds for $\alpha = 1, \beta = \delta = \gamma = L = 0$. Hence, the application of Theorem 2.1 ensures that a pair of maps f and g has at least one fixed point $M(u) \in \mathcal{C}$, that is,

$$f(\mu(u)) = \mu(u) = g(\mu(u))$$

which is a solution of the boundary problem 3.1. □

4. Declarations

Funding: Authors declare that there is no funding available for this article.

Competing interests/Conflict of interest: The authors declare that they have no competing interests.

Author's contributions: All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Availability of data and material: This clause is not applicable to this paper.

Code availability: There is no code required in this paper.

Compliance with ethical standards: Not applicable.

Acknowledgments

We thank the referee for valuable suggestions.

References

1. A. Gupta, and P. Gautam, *Quasi-partial b- metric space and some related fixed point theorems,* Fixed point Theory Appl, *(*18),56-61, (2015). https//doi.org/10.1186/s13663-015-0260-21.

2. A. Taheri and A. P. Farajzadeh, *A new generalization of α-type almost-F-contractions and α-type F-Suzuki contractions in metric spaces and their fixed point theorems,* Carpathian Mathematical Publications, **11**(2), 475-492, (2019).

3. D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces,* Fixed Point Theory and Applications, 94, (2012).

4. D. Wardowski, *Solving existence problems via F-contractions,* Proceedings of the AMS, vol., **146**,1585–1598, (2018).

5. D. Wardowski and N. Van Dung, *Fixed points of F-weak contractions on complete metric spaces,* Demonstratio Mathematica, **47**(1), 146-155, (2014).

6. F. E. Browder, *on convergence of successive approximations for nonlinear functional equations,* Nederl. Akad. Wetensch. Proc.Ser, **30**,27-35, (1968).

7. G. Minak, A. Helvaci and I. Altun, *Ćirić type generalized F-contractions on complete metric spaces and fixed point results,* Filomat, **28**(6), 1143-1151, (2014).

8. H. Künzi, P. Homeira, and P. Michel Schellekens, *Partial quasi-metrics,* Theoretical Computer Science, **365**(3), 237-246, (2006).

9. H. Piri and P. Kumam, *Some fixed point theorems concerning F-contraction in complete metric spaces,* Fixed Point Theory and Applications, 210, (2014).

10. I. A. Bakhtin, *The contraction principle in quasimetric spaces,* Funct. Anal., **30**, 26-37, (1989).

11. L. Scholastica, S. Kumar and G. Kakiko *Fixed Points for F-contraction Mappings in Partial Metric Spaces,* Lobachevskii Journal of Mathematics, 40(2), 183:188, (2019).

12. L. Wangwe, and S. Kumar, *Fixed Points Results for Interpolative ϕ-Hardy-Rogers type Contraction Mappings in Quasi-Partial b-Metric Space with some Applications,* Journal of Analysis, 18 pages, (2022).

13. L. Wangwe, S. Kumar, *Common Fixed Theorems for Generalized F-Kannan mapping in Metric Space with an Applications,* Abstact and Applied Analysis, Vol.2021, Article ID 6619877,12 pages,(2021).

14. M. Abbas, B. Ali, and S. Romaguera, *Fixed and periodic points of generalized contractions in metric spaces,* Fixed Point Theory and Applications, **(1)**, 1-11, (2013).

15. M. Arshad, S. U. Khan, and J. Ahmad, *Fixed point results for F-contractions involving some new rational expressions,* JP Journal of Fixed Point Theory and Applications, **1**, (2016).

16. N. A. Secelean, *Iterated function systems consisting of F-contractions,* Fixed PointTheory Appl, Article **ID 277** .doi:10.1186/1687-1812-2013-277.

17. N. Hussain, V. Parvaneh, J. R. Roshan, and Z. Kadelburg, *Fixed points of cyclic (y, j, L, A, B)-contractive mappings in ordered b-metric spaces with applications*, Fixed Point Theory and Applications, Article No **256**, (2013).

18. P. Kumam, N. V. Dung, and K. Sitthithakerngkiet, *A generalization of Ćirić fixed point theorem,* Filomat, vol. 29, no. **7**,1549–1556, (2015).

19. P. Gautam, S. Kumar, S. Verma, and S. Gailati, *On some ω-Interpolative Contractions of Suzuki-type Mappings in Quasi-Partial b-Metric space,* Journal of Function Spaces, Vol. 2022, Article id.9158199, 12pages, (2022).

20. S. Banach, *Sur les operationes dans les ensembles abstraits et leur application aux equation integrales, Fund. Math, (*3), 133-181, (1922).

21. S. B. Nadler, *Multi-valued contraction mappings,* Pacific J. Math, **30** , no. 2, 475–488, (1969).

22. S. S. Chang, J. Kim, L. Wang and J. Tang, *Existence of fixed points for generalized F-contractions and generalized F-suzuki contractions in metric spaces,* Global Journal of Pure and Applied Mathematics, **12**(6), 4867-4882, (2016).

23. S. G. Metthews, *partial metric topology,* Ann.N.Y. Acad.Sci, *728*, 183–197, (1994).

24. S. K. Chatterjea, *Fixed point theorems, C. R. Acad. Bulg. Sci,* **25**, 727–730, (1972).

25. S. Shukla, *Partial b-metric spaces and fixed point theorems,* Mediterranean Journal of Mathematics, (2013), doi:10.1007/s00009-013-0327-4.

26. S. Kumar, *Coincidence Points for a pair of Ordered F-contraction Mappings in Ordered Partial Metric Spaces,* Malaya Journal of Mathematics, 7(3), 423-428, (2019).

27. S. Kumar, and M. Asim, *Fixed Point Theorems for a pair of F-contraction mappings in Ordered Metric Space* , Advances in Nonlinear Variational Inqualities, Vol.25, 17:28, (2022).

28. S. Pauline and S.Kumar, *Common Fixed Point theorems for T-Hardy-Rogers contraction Mappings in Complete Cone-b-Metric Spaces with an application,* Topol. Algebra Appl., ,9:105-117, (2021).

29. T. C. Suzuki, *A generalized Banach contraction principle that characterizes metric completeness,* Proc. Amer. Math. Soc., **136**, 1861-1869, (2008).

30. T. Suzuki, *Several fixed point theorems in complete metric spaces,* Yokohama Math. Journal Vol. 44, (1997).

31. W. Sintunavarat, S. Plubtieng and P. Katchang, *Fixed point result and applications on b-metric space endowed with an arbitrary binary relation,* Fixed Point Theory and Applications, Article No **296**, (2013).

32. Z. Mustafa, J. R. Roshan, V. Parvaneh and Z. Kadelburg, *Some common fixed point result in ordered partial b-metric spaces,* Journal of Inequalities and Applications, 2013:562, (2013).

Santosh Kumar,

Department of Mathematics,

School of Physical Sciences,

North-Eastern Hill University, Shillong-793022,

Meghalaya, India.

E-mail address: santoshkumar@nehu.ac.in; drsengar2002@gmail.com

and

Jonasi Chilongola,

Department of Mathematics, College of Natural and Applied Sciences,

University of Dar es Salaam,

Tanzania.

E-mail address: chilongolajonasi@gmail.com