Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–11.
ISSN-0037-8712
doi:10.5269/bspm.64814

Quotients of Bounded Linear Operators on Non-Archimedean Banach Spaces

J. Ettayb*

ABSTRACT: Let X, Y, Z be non-Archimedean Banach spaces over $\mathbb{K}$ and let $A \in B(X, Y)$ and $B \in B(X, Z)$. In this paper, we define the quotient of bounded linear operators A and B on non-Archimedean Banach spaces with $N(A) \subseteq N(B)$ as the mapping $Ax \mapsto Bx$ for all $x \in X$. We show some results about it. Majorization, range inclusion and factorization are studied, open mapping theorem for quotients of bounded linear operators is investigated and examples are given on non-Archimedean Banach spaces.

Key Words: Non-Archimedean Banach spaces, quotients of bounded linear operators, majorization, range inclusion.

Contents

1. Introduction and Preliminaries

In the classical operators theory, the proof of range inclusion theorems need to use Hahn-Banach theorem. Moreover, in general non-Archimedean normed spaces, Hahn-Banach theorem does not hold [19]. Ingleton [8] showed that a non-Archimedean normed space has the extension property if and only if it is spherically complete.

In the classical setting, Izumino [10] introduced the notion of a quotient of bounded linear operators on a complex Hilbert space and he showed explicit formulae for computing the quotients which correspond to the sum, product, closure, adjoint and weak adjoint of given quotients. For more details on quotients of bounded linear operators on complex Hilbert spaces, we refer to [9], [10], [11], [12] and [13]. Recently, Barnes [1] studied the majorization, range inclusion and factorization for bounded linear operators on complex Banach spaces.

Throughout this paper, X, Y and Z are non-Archimedean Banach spaces over a non-Archimedean complete valued field $\mathbb{K}$ with a nontrivial valuation $|\cdot|$ and $B(X, Y)$ denotes the set of all bounded linear operators from X into Y. If $X = Y$, we write $B(X, X) = B(X)$. For $A \in B(X, Y)$, $R(A)$ and $N(A)$ denote the range and the null space of A, respectively. $\mathbb{Q}_p$ is the field of p-adic numbers ($p \geq 2$ being a prime) equipped with p-adic valuation $|.|_p$. For more details, we refer to [2] and [15]. We begin with some preliminaries.

Definition 1.1 *[2] Let X be a vector space over $\mathbb{K}$. A non-negative real valued function $\|\cdot\| : X \to \mathbb{R}_+$ is called a non-Archimedean norm if:*

(i) For all $x \in X$, $\|x\| = 0$ if and only if $x = 0$;

(ii) For any $x \in X$ and $\alpha \in \mathbb{K}$, $\|\alpha x\| = |\alpha|\|x\|$;

(iii) For each $x, y \in X$, $\|x + y\| \leq \max(\|x\|, \|y\|)$.

Definition 1.2 *[2] A non-Archimedean normed space is a pair $(X, \|\cdot\|)$ where X is a vector space over $\mathbb{K}$ and $\|\cdot\|$ is a non-Archimedean norm on X.*

Definition 1.3 *[2] A non-Archimedean Banach space is a vector space endowed with a non-Archimedean norm, which is complete.*

---

* Corresponding author.
2010 *Mathematics Subject Classification:* 47A10, 47S10.
Submitted August 27, 2022. Published December 04, 2025

For more details on non-Archimedean Banach spaces, see for example [2].

Proposition 1.1 *[2] A closed subspace of a non-Archimedean Banach space is a non-Archimedean Banach space.*

Example 1.1 *[17] The spaces*

$$c_0(\mathbb{K}) = \{(x_1, x_2, \cdots)|x_i \in \mathbb{K}, \lim_{i\to\infty} |x_i| = 0\}$$

and

$$l^\infty(\mathbb{K}) = \{(x_1, x_2, \cdots)|x_i \in \mathbb{K}, \ \sup_i |x_i| < \infty\},$$

equipped with the norm $\|(x_1, x_2, \cdots)\| = \sup_i |x_i|$ *are non-Archimedean Banach spaces over* $\mathbb{K}$.

For more details on non-Archimedean operators theory, we refer to [2], [18] and [19].

2. Quotients of bounded linear operators on non-Archimedean Banach spaces

We start with the following definition.

Definition 2.1 *[17] Let* X, Y, Z *be non-Archimedean Banach spaces over* $\mathbb{K}$, *let* $A \in B(X, Y)$ *and* $B \in B(X, Z)$. *Then* A *majorizes* B, *if there exists a constant* $M > 0$ *such that*

$$\text{for all } x \in X, \ \|Bx\| \le M\|Ax\|.$$

We have the following example.

Example 2.1 *Let* $(\alpha_n)_n \subset \mathbb{K}$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and let* $A, B \in B(l^\infty(\mathbb{K}))$ *be given respectively by*

$$A(x_1, x_2, x_3, \cdots) = (\alpha_1 x_1, \alpha_2 x_2, \alpha_3 x_3, \cdots) \text{ for all } (x_1, x_2, x_3, \cdots) \in l^\infty(\mathbb{K}),$$

and

$$B(x_1, x_2, x_3, \cdots) = \Big(\sum_{i=1}^{\infty} \alpha_i x_i, 0, 0 \cdots\Big) \text{ for all } (x_1, x_2, x_3, \cdots) \in l^\infty(\mathbb{K}).$$

It is easy to see that for all $(x_1, x_2, x_3, \cdots) \in l^\infty(\mathbb{K})$,

$$\|B(x_1, x_2, x_3, \cdots)\| \le \|A(x_1, x_2, x_3, \cdots)\|,$$

thus A *majorizes* B.

We obtain the following results.

Proposition 2.1 *Let* X *be a non-Archimedean Banach space over* $\mathbb{K}$ *and let* $A, B, D \in B(X)$, *we have:*

(i) If A *majorizes* B *and* D, *then* A *majorizes* $B + D$;

(ii) If A *majorizes* B, *then* A *majorizes* DB.

Proof:

(i) If A majorizes B, then there exists a constant $M > 0$ such that

$$\text{for all } x \in X, \ \|Bx\| \le M\|Ax\|. \tag{2.1}$$

Also, A majorizes D, then there exists a constant $C > 0$ such that

$$\text{for all } x \in X, \ \|Dx\| \le C\|Ax\|. \tag{2.2}$$

Hence, there exists a constant $L > 0$ ($L = \max\{M, C\}$) such that

$$\text{for all } x \in X,\ \|(B+D)x\| \leq \max\{\|Bx\|, \|Dx\|\} \leq \max\{M\|Ax\|, C\|Ax\|\} \leq \max\{M, C\}\|Ax\| = L\|Ax\|.$$

Consequently, A majorizes $B + D$.

(ii) If A majorizes B, then there exists a constant $M > 0$ such that

$$\text{for all } x \in X,\ \|Bx\| \leq M\|Ax\|. \tag{2.3}$$

Hence there exists a constant $K > 0$ ($K = M\|D\|$) such that

$$\text{for all } x \in X,\ \|DBx\| \leq M\|D\|\|Ax\| = K\|Ax\|. \tag{2.4}$$

Then A majorizes DB.

□

Lemma 2.1 *Let X be a non-Archimedean Banach space over $\mathbb{K}$ and let $A \in B(X)$. If A majorizes B, then $N(A) \subseteq N(B)$.*

Proof: Assume that A majorizes B, then there exists a constant $M > 0$ such that

$$\text{for all } x \in X,\ \|Bx\| \leq M\|Ax\|. \tag{2.5}$$

By (2.5), it follows that if $x \in N(A)$, then $x \in N(B)$. □

Proposition 2.2 *Let X, Y be two non-Archimedean Banach spaces over $\mathbb{K}$ and Z be a spherically complete Banach space over $\mathbb{K}$. Let $A \in B(X, Y)$ and $B \in B(X, Z)$, then the following statements are equivalent:*

(i) A majorizes B;

(ii) There exists $V \in B(Y, Z)$ such that $B = VA$.

Proof:

(i) Assume that A majorizes B. Define $U : R(A) \to Z$ by $U(Ax) = Bx$. The map U is well-defined since $N(A) \subseteq N(B)$. Hence $\|U(Ax)\| = \|Bx\| \leq M\|Ax\|$. Thus U has a bounded extension, which denoted as V from Y onto Z. From the definition of U, we get $B = VA$.

(ii) Assume that there exists $V \in B(Y, Z)$ such that $B = VA$. It is easy to check that A majorizes B.

□

Proposition 2.3 *Let X be a non-Archimedean Banach space over $\mathbb{K}$ and let $A, B \in B(X)$ such that $AB = BA$. If A majorizes B, then for all $n \geq 1$, A^n majorizes B^n.*

Proof: We show by induction that for all $n \geq 1$ and for all $x \in X$, $\|B^n x\| \leq M^n\|A^n x\|$. By hypothesis, this holds for $n = 1$. If for all $x \in X$, $\|B^n x\| \leq M^n\|A^n x\|$. Then for all $x \in X$, we have

$$\|B^{n+1}x\| = \|B^n(Bx)\| \leq M^n\|A^n(Bx)\| \leq M^n\|B(A^n x)\| \leq M^n M\|A(A^n x)\| \leq M^{n+1}\|A^{n+1}x\|.$$

□

Let $A, B \in B(X)$. The set $G(A,B) = \{(Ax, Bx);\ x \in X\}$ is a subset of $X \oplus X$, then $G(A,B)$ is the graph of an operator F defined on X, if for all $x \in X$, $Ax = 0 \Longrightarrow Bx = 0$. It follows that $G(A,B)$ is a graph, if and only if $N(A) \subseteq N(B)$. In this case, F is the mapping defined as follows:

$$\text{for all } x \in X,\ Ax \longmapsto Bx.$$

Thus for all $x \in X,\ Bx = FAx$. We set $F = B/A$. We have the following definition.

Definition 2.2 *Let X, Y, Z be non-Archimedean Banach spaces over $\mathbb{K}$, let $A \in B(X,Y)$ and $B \in B(X,Z)$ such that $N(A) \subseteq N(B)$. The quotient B/A of two bounded linear operators A and B is the mapping $Ax \longmapsto Bx$ for all $x \in X$.*

We obtain the following remark.

Remark 2.1

(i) It is clear that $R(A)$ and $R(B)$ are, respectively, the domain and the range of B/A.

(ii) Let $A \in B(X)$, we have $A = A/I$ where I denotes the identity operator on X.

(iii) Let $A \in B(X)$. If A is invertible, then $A^{-1} = I/A$.

(iv) The quotient of bounded linear operators on non-Archimedean Banach spaces over $\mathbb{K}$ can be bounded or unbounded.

(v) One can see that the quotient of bounded linear operators is a generalization of linear operators. We have the following examples.

Example 2.2 *Let $A \in B(c_0(\mathbb{Q}_p))$ be defined by*

$$A(x_1, x_2, x_3, \cdots) = (px_1, p^2x_2, p^3x_3, \cdots) \quad \text{for all } (x_1, x_2, x_3 \cdots) \in c_0(\mathbb{Q}_p).$$

One can see that $A = A/I$ is a bounded linear operator on $c_0(\mathbb{Q}_p)$ and I/A is an unbounded linear operator on $c_0(\mathbb{Q}_p)$.

Example 2.3 *Let $A, B \in B(\mathbb{Q}_p^2)$, set*

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}.$$

It is easy to see that $N(A) \subseteq N(B)$ and $B/A = \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}$.

Example 2.4 *Let $A, B \in B(c_0(\mathbb{Q}_p))$ be two operators given by for all $x = (x_1, x_2, x_3, x_4, x_5, \cdots) \in c_0(\mathbb{Q}_p)$,*

$$Ax = (px_1, p^2x_2, p^3x_3, p^4x_4, p^5x_5, \cdots, p^nx_n, \cdots)$$

and

$$Bx = (p^{1!}x_1, p^{2!}x_2, p^{3!}x_3, p^{4!}x_4, p^{5!}x_5, \cdots, p^{n!}x_n, \cdots).$$

It is easy to see that $N(A) \subseteq N(B)$ and B/A is a bounded linear operator on $R(A)$.

Remark 2.2 Let $A, B \in B(X)$ such that $N(A) \subseteq N(B)$, the quotient B/A is the solution of the equation $XA = B$.

We obtain the following proposition.

Proposition 2.4 *Let X, Y be two non-Archimedean Banach spaces over $\mathbb{K}$ and Z be a spherically complete Banach space over $\mathbb{K}$. Let $A \in B(X,Y)$ and $B \in B(X,Z)$, then the following conditions are equivalent:*

(i) *A majorizes B*;

(ii) *There exists* $V \in B(Y, Z)$ *such that* $B = VA$;

(iii) *For all* $(x_n)_n \subset X$ *such that* $\|Ax_n\| \longrightarrow 0$, *then* $\|Bx_n\| \longrightarrow 0$.

Proof: Obvious. □

Proposition 2.5 *Let* X *be a spherically complete non-Archimedean Banach space over* $\mathbb{K}$. *Let* $A, B \in B(X)$ *such that* A *majorizes* B. *If* A *is completely continuous, then* B *is completely continuous.*

Proof: Since A majorizes B. From Proposition 2.2, there is $V \in B(X)$ such that $B = VA$. Since the product of a completely continuous operator and a bounded linear operator is completely continuous. This completes the proof. □

Remark 2.3 The Proposition 2.5 is valid for any non-Archimedean Banach space over a spherically complete field with a nontrivial valuation.

If X and Z are non-Archimedean normed spaces over $\mathbb{K}$. Let $A \in B(X, Z)$ be fixed. Setting:

$$B_A(X, Z) = \{B \in B(X, Z) : N(A) \subseteq N(B)\}.$$

Lemma 2.2 *The set* $B_A(X, Z)$ *is a subspace of* $B(X, Z)$.

Proof: Obvious. □

Remark 2.4 It is easy to see that $B_A(X, Z)$ is a closed subspace of $B(X, Z)$.

We define the set of quotient operators as follows:

$$\mathcal{Q}_A(R(A), Z) = \{B/A : N(A) \subseteq N(B)\}.$$

Let B/A, $C/A \in \mathcal{Q}_A(R(A), Z)$ and $\alpha \in \mathbb{K}$, the addition and the scalar multiplication in $\mathcal{Q}_A(R(A), Z)$ are defined by

$$\begin{aligned} B/A + C/A &= (B + C)/A, \\ \alpha\Big(B/A\Big) &= \alpha B/A. \end{aligned}$$

Then $\mathcal{Q}_A(R(A), Z)$ is a linear space by satisfying the null space inclusions. Let $A \in B(X, Z)$ be fixed, set: $\mathcal{Q}^b_A(R(A), Z) = \{B/A \in \mathcal{Q}_A(R(A), Z) : B, B/A \text{ are bounded}\}$.

Definition 2.3 *Let* $B/A \in \mathcal{Q}^b_A(R(A), Z)$, *we define two norms of* B/A *on* $\mathcal{Q}^b_A(R(A), Z)$ *as follows:*

$$\|B/A\| = \sup_{x \in X, Ax \neq 0} \frac{\|Bx\|}{\|Ax\|}$$

and

$$\|B/A\|_0 = \sup_{y \in R(A), \|y\| \leq 1} \|(B/A)y\| = \sup_{x \in X, \|Ax\| \leq 1} \|Bx\|.$$

We continue with the following theorem.

Theorem 2.1 *Suppose that* X *and* Z *are non-Archimedean Banach spaces over* $\mathbb{K}$. *Then the set* $B_A(X, Z)$ *of bounded linear operators* B *with* $N(A) \subseteq N(B)$ *is a non-Archimedean Banach space.*

Proof: Let $(B_n)_n$ be a Cauchy sequence in $B_A(X,Z)$. Since $B(X,Z)$ is complete, there exists $B \in B(X,Z)$ such that $B_n \to B$ in the norm of $B(X,Z)$ as $n \to \infty$. It suffices to show that if $x \in N(A)$, then $x \in N(B)$. Let $x \in N(A)$, since for all $n \in \mathbb{N}$, $B_n \in B_A(X,Z)$, we get $B_n x = 0$. By

$$\|Bx\| \leq \max\{\|B_n x - Bx\|, \|B_n x\|\}$$

and $\lim\limits_{n\to\infty} \|B_n x - Bx\| = 0$, we conclude that $x \in N(B)$. □

Set $B_A^b(X,Z) = \{B \in B_A(X,Z) : B/A \text{ is bounded}\}$, since $0/A$ is bounded, the set $B_A^b(X,Z)$ is nonempty.

Remark 2.5 If $B \in (B_A^b(X,Z), \|\cdot\|_0)$, then

$$\|B\|_0 = \sup_{x\in X, \|x\|\leq 1} \|Bx\|$$

and

$$\|B/A\|_0 = \sup_{y\in R(A), \|y\|\leq 1} \|(B/A)y\| = \sup_{x\in X, \|Ax\|\leq 1} \|Bx\| \quad \text{are finite.}$$

We have the following theorem.

Theorem 2.2 *The space $(B_A^b(X,Z), \|\cdot\|_0)$ is a non-Archimedean normed subspace of $(B_A(X,Z), \|\cdot\|_0)$ (where $\|B\|_0 = \sup_{x\in X, \|x\|\leq 1} \|Bx\|$ if $B \in (B(X,Z), \|\cdot\|_0)$).*

Proof: It is easy to see that $(B_A^b(X,Z), \|\cdot\|_0)$ is closed under linear space operations. Let $B, C \in (B_A^b(X,Z), \|\cdot\|_0)$, $\alpha \in \mathbb{K}$ and $x \in X$ such that $\|Ax\| \leq 1$, we have

$$\begin{aligned} \|(B+C)x\| &\leq \max\{\|Bx\|, \|Cx\|\} \\ &\leq \max\{\|B/A\|_0, \|C/A\|_0\}, \end{aligned}$$

and

$$\begin{aligned} \|(\alpha B)x\| &= |\alpha|\|Bx\| \\ &\leq |\alpha|\|B/A\|_0. \end{aligned}$$

Hence

$$\|B/A + C/A\|_0 \leq \max\{\|B/A\|_0, \|C/A\|_0\} < \infty,$$

and

$$\|\alpha(B/A)\|_0 \leq |\alpha|\|B/A\|_0 < \infty.$$

□

Remark 2.6 Let $B \in (B_A^b(X,Z), \|\cdot\|)$, then

$$\|B\| = \sup_{x\in X, x\neq 0} \frac{\|Bx\|}{\|x\|}$$

and

$$\|B/A\| = \sup_{x\in X, Ax\neq 0} \frac{\|Bx\|}{\|Ax\|} \quad \text{are finite.}$$

We have the following proposition.

Proposition 2.6 *Let X, Z be two non-Archimedean normed spaces over $\mathbb{K}$. Then the space $\Big(B_A^b(X,Z), \|\cdot\|\Big)$ is a non-Archimedean normed subspace of $(B_A(X,Z), \|\cdot\|)$ (where $\|B\| = \sup_{x\in X\setminus\{0\}} \frac{\|Bx\|}{\|x\|}$ if $B \in (B(X,Z), \|\cdot\|)$).*

Proof: Let $A \in (B(X,Z), \|\cdot\|)$ be fixed, $B, C \in (B_A^b(X,Z), \|\cdot\|)$ and $\alpha \in \mathbb{K}$, we have

$$\begin{aligned}
\|B/A + C/A\| &= \sup_{x\in X, Ax\neq 0} \left(\frac{\|Bx+Cx\|}{\|Ax\|}\right) \\
&\leq \sup_{x\in X, Ax\neq 0} \max\left(\frac{\|Bx\|}{\|Ax\|}, \frac{\|Cx\|}{\|Ax\|}\right) \\
&= \max \sup_{x\in X, Ax\neq 0} \left(\frac{\|Bx\|}{\|Ax\|}, \frac{\|Cx\|}{\|Ax\|}\right) \\
&= \max\Bigg(\|B/A\|, \|C/A\|\Bigg),
\end{aligned}$$

and

$$\begin{aligned}
\|\alpha B/A\| &= \sup_{x\in X, Ax\neq 0} \left(\frac{\|\alpha Bx\|}{\|Ax\|}\right) \\
&= |\alpha| \sup_{x\in X, Ax\neq 0} \left(\frac{\| Bx\|}{\|Ax\|}\right) \\
&= |\alpha|\|B/A\|.
\end{aligned}$$

□

We have the following theorem.

Theorem 2.3 *Suppose that Z is a non-Archimedean Banach space over $\mathbb{K}$, if the set $\{x \in X : \|Ax\| \leq 1\}$ is bounded in X, then $(B_A^b(X,Z), \|\cdot\|_0)$ is a non-Archimedean Banach space (where $\|B\|_0 = \sup_{x\in X, \|x\|\leq 1} \|Bx\|$ if $B \in (B(X,Z), \|\cdot\|_0)$).*

Proof: Setting: $M = \{\|x\| : x \in X, \|Ax\| \leq 1\}$. Let $(B_n)_n$ be a Cauchy sequence in $(B_A^b(X,Z), \|\cdot\|_0)$, since $(B_A(X,Z), \|\cdot\|_0)$ is complete, there exists $B \in (B_A(X,Z), \|\cdot\|_0)$ such that $B_n \to B$ in the norm of $(B_A(X,Z), \|\cdot\|_0)$. Hence it suffices to show that B/A is bounded. Since $(\|B_n\|_0)_{n\in\mathbb{N}}$ converges, there exists a constant $C > 0$ such that for all $n \in \mathbb{N}$, $\|B_n\|_0 \leq C$. Let $N \in \mathbb{N}$ such that for all $n \geq N$, $\|(B_n - B)x\| \leq 1$, hence for each $x \in X$ such that $\|Ax\| \leq 1$,

$$\begin{aligned}
\|Bx\| &\leq \max\{\|B_n x - Bx\|, \|B_n x\|\} \\
&\leq \max\{1, CM\},
\end{aligned}$$

for all $n \geq N$. Then $\|B/A\|_0 = \sup_{x\in X, \|Ax\|\leq 1} \|Bx\|$ is finite. □

We have the following theorem:

Theorem 2.4 $(\mathcal{Q}_A^b(R(A), Z), \|\cdot\|)$ *is a non-Archimedean normed space.*

Proof: Let $B/A \in (\mathcal{Q}_A^b(R(A), Z), \|\cdot\|)$ such that $\|B/A\| = 0$, then for all $x \in X$ such that $Ax \neq 0$, $Bx = 0$, whenever $Ax = 0$, due to $N(A) \subseteq N(B)$, we have $Bx = 0$, hence B is the zero operator, thus $B/A = 0/A$.

Let $B/A, C/A \in (\mathcal{Q}_A^b(R(A), Z), \|\cdot\|)$ and $\alpha \in \mathbb{K}$, we have

$$\begin{aligned}\|B/A + C/A\| &= \sup_{x\in X, Ax\neq 0}\left(\frac{\|Bx + Cx\|}{\|Ax\|}\right)\\ &\leq \sup_{x\in X, Ax\neq 0} \max\left(\frac{\|Bx\|}{\|Ax\|}, \frac{\|Cx\|}{\|Ax\|}\right)\\ &= \max \sup_{x\in X, Ax\neq 0}\left(\frac{\|Bx\|}{\|Ax\|}, \frac{\|Cx\|}{\|Ax\|}\right)\\ &= \max\left(\|B/A\|, \|C/A\|\right),\end{aligned}$$

and

$$\begin{aligned}\|\alpha B/A\| &= \sup_{x\in X, Ax\neq 0}\left(\frac{\|\alpha Bx\|}{\|Ax\|}\right)\\ &= |\alpha| \sup_{x\in X, Ax\neq 0}\left(\frac{\|\alpha Bx\|}{\|Ax\|}\right)\\ &= |\alpha|\|B/A\|.\end{aligned}$$

□

In the next theorem, we will work with the space $(B(X, Z), \|\cdot\|_0)$ where $\|B\|_0 = \sup_{x\in X, \|x\|\leq 1}\|Bx\|$ if $B \in (B(X, Z), \|\cdot\|_0)$.

Theorem 2.5 *Suppose that Z is a non-Archimedean Banach space over $\mathbb{K}$ and $\{x : \|Ax\| \leq 1\}$ is a bounded set in X, then $(\mathcal{Q}_A^b(R(A), Z), \|\cdot\|_0)$ is a non-Archimedean Banach space.*

Proof: Let $(B_n/A)_n$ be a Cauchy sequence in $(\mathcal{Q}_A^b(R(A), Z), \|\cdot\|_0)$. We show that $(B_n)_n$ is a Cauchy sequence in $(B(X, Z), \|\cdot\|_0)$. Let $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}$, $m, n \geq N$, $\|B_n/A - B_m/A\|_0 < \frac{\varepsilon}{\|A\|_0+1}$. That is, for all $n, m \geq N$, $\sup_{x\in X, \|Ax\|\leq 1}\|(B_n - B)x\| < \frac{\varepsilon}{\|A\|_0+1}$. Let $x \in X$ such that $\|x\| \leq 1$, hence $\|Ax\| \leq \|A\|_0$, then for all $n, m \geq N$,

$$\begin{aligned}\|(B_n - B_m)x\| &= \|(B_n - B_m)/A)Ax\|\\ &\leq \|(B_n - B_m)/A)\|_0\|Ax\|\\ &\leq \|(B_n - B_m)/A)\|_0\|A\|_0\\ &\leq \frac{\varepsilon}{1 + \|A\|_0}\|A\|_0\\ &< \varepsilon.\end{aligned}$$

Hence, for all $n, m \geq N$, $\|(B_n - B_m)\|_0 = \sup_{x\in X, \|x\|\leq 1}\|B_n x - B_m x\| < \varepsilon$, then $(B_n)_n$ converges to B in $(B(X, Z), \|\cdot\|_0)$ as $n \to \infty$. Set $M = \sup\{\|x\| : \|Ax\| \leq 1\}$. Using Theorem 2.3, we have B/A is bounded. For a given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}, n \geq N$, $\|B_n - B\|_0 < \frac{\varepsilon}{M+1}$, hence for all $n \in \mathbb{N}$, $n \geq N$, we have

$$\begin{aligned}\|B_n/A - B/A\|_0 &= \|(B_n - B)/A\|_0\\ &= \sup_{x\in X, \|Ax\|\leq 1}\{\|(B_n - B)x\|\}\\ &\leq \sup_{x\in X, \|Ax\|\leq 1}\{\|(B_n - B)\|_0\|x\|\}\\ &= \|B_n - B\|_0 \sup_{x\in X, \|Ax\|\leq 1}\|x\|\\ &= \|B_n - B\|_0 M\\ &< \varepsilon.\end{aligned}$$

Thus $(R(A), Z), \|\cdot\|_0)$ is complete. □

In the next theorem, we will work with the space $(B(X,Z), \|\cdot\|)$ where $\|B\| = \sup_{x\in X, x\neq 0} \frac{\|Bx\|}{\|x\|}$ if $B \in (B(X,Z), \|\cdot\|)$.

Theorem 2.6 *Suppose that the valuation of $\mathbb{K}$ is dense, Z is a non-Archimedean Banach space over $\mathbb{K}$ and $\{x : \|Ax\| \leq 1\}$ is a bounded set in X, then $(\mathcal{Q}_A^b(R(A), Z), \|\cdot\|_0)$ is a non-Archimedean Banach space.*

We have the following remark.

Remark 2.7 The condition $\{x : \|Ax\| \leq 1\}$ is a bounded set in X is sufficient but it is not a necessary condition.

Example 2.5 *Let $\mathbb{K}$ be a non-Archimedean valued field with a nontrivial valuation $|\cdot|$. Consider the zero map on $\mathbb{K}$, then $\mathcal{Q}_A^b(\{0\}, \mathbb{K})$ being singleton is a non-Archimedean Banach space but the set $\{x : |0x| \leq 1\} = \mathbb{K}$ is not bounded with the non-Archimedean metric.*

By inspiration of the proof of (ii) of Lemma 2.1.18 of [18]. We prove the open mapping theorem for quotients of bounded linear operators without using its boundedness, we have the following lemma.

Lemma 2.3 *Let X, Y, Z be non-Archimedean Banach spaces over $\mathbb{K}$, let $A \in B(X,Y)$ injective with closed range, and $B \in B(X,Z)$ with $N(A) \subseteq N(B)$. If $\overline{(B/A)(B_r(0))}$ is open, then $(B/A)(B_r(0)) = \overline{(B/A)(B_r(0))}$.*

Proof: It suffices to prove that if $B_Z(0,s) \subset \overline{(B/A)(B_r(0))}$ for some $s > 0$, then $B_Z(0,s) \subset (B/A)(B_r(0))$. Let $\rho \in \mathbb{K}$ such that $0 < |\rho| < 1$, since $z \in \overline{(B/A)(B_r(0))}$, there is an element $y_0 \in B_r(0)$ such that $\|z - (B/A)y_0\| < |\rho| s$, then $\|\rho^{-1}(z - (B/A)y_0)\| < s$ and again there is an element $y_1 \in B_r(0)$ such that $\|\rho^{-1}(z - (B/A)y_0) - (B/A)y_1\| < |\rho| s$, then $\|z - (B/A)y_0 - (B/A)\rho y_1\| < |\rho|^2 s$, we inductively find, for all $n \in \mathbb{N}$, $y_0, \cdots y_n \in B_r(0)$ such that

$$\|z - \sum_{k=0}^{n} \rho^k (B/A) y_k\| = \|z - \Big(B/A\Big) \sum_{k=0}^{n} \rho^k y_k\| < |\rho|^{n+1} s.$$

Since $(\sum_{k=0}^{n} \rho^k y_k)_{n\in\mathbb{N}}$ is Cauchy sequence and $R(A)$ is complete, then $\sum_{k=0}^{\infty} \rho^k y_k \in R(A)$ and $\| \sum_{k=0}^{\infty} \rho^k y_k\| < r$. Set, for all $k \in \mathbb{N}$, $x_k = \rho^k A^{-1} y_k$. Using the open mapping theorem, A as a homeomorphism from X onto $R(A)$. From

$$\begin{aligned}
\|x_0 + \cdots + x_n\| &= \|A^{-1}\Big(\sum_{k=0}^{n} \rho^k y_k\Big)\| \\
&\leq \|A^{-1}\| \| \sum_{k=0}^{n} \rho^k y_k\|,
\end{aligned}$$

we conclude that $(\sum_{k=0}^{n} x_k)_n$ is a Cauchy sequence in X, hence $\sum_{k=0}^{\infty} x_k \in X$. Continuity of A^{-1} and B give $\sum_{k=0}^{\infty} x_k = A^{-1}\Big(\sum_{k=0}^{\infty} \rho^k y_k\Big)$ and $B\Big(\sum_{k=0}^{\infty} x_k\Big) = \sum_{k=0}^{\infty} Bx_k$ respectively, then

$$\begin{aligned} z &= \sum_{k=0}^{\infty} (B/A)\rho^k y_k \\ &= \sum_{k=0}^{\infty} (B/A)Ax_k \\ &= \sum_{k=0}^{\infty} Bx_k \\ &= B\Big(\sum_{k=0}^{\infty} x_k\Big) \\ &= (B/A)\Big(A(\sum_{k=0}^{\infty} x_k)\Big) \\ &= (B/A)\Big(\sum_{k=0}^{\infty} \rho^k y_k\Big), \end{aligned}$$

thus $z \in (B/A)(B_r(0))$. □

The proof of the next theorem follows easily from the Lemma 2.3.

Theorem 2.7 *Let X, Y, Z be non-Archimedean Banach spaces over $\mathbb{K}$, let $A \in B(X, Y)$ be injective and $B \in B(X, Z)$ be surjective. If $N(A) \subseteq N(B)$ and $R(A)$ is closed, then B/A is an open mapping on $R(A)$.*

References

1. B.A. Barnes, *Majorization, range inclusion, and factorisation for bounded linear operators*, Proc. Amer. Math. Soc. **133**(1) (2005), 155-162.
2. T. Diagana and F. Ramaroson, *Non-archimedean Operators Theory*, Springer, 2016.
3. R. Bouldin, *A counterexample in the factorization of Banach space operators*, Proc. Amer. Math. Soc., **68** (1978), 3-27.
4. R. G. Douglas, *On majorization and range inclusion of operators in Hilbert space*, Proc. Amer. Math. Soc., **17** (1966), 413-415.
5. R. G. Douglas, *Addendum for "Majorization, factorization and range inclusion"*. Unpublished manuscript
6. M. R. Embry, *Factorization of operators on Banach space*, Proc. Amer. Math. Soc., **38** (1973), 587-590.
7. G. Hirasawa, *Quotient of bounded operators and Kaufman's theorem*, Math. J. Toyama Univ., **18** (1995), 215-224.
8. A. W. Ingleton, *The Hahn-Banach theorem for non-Archimedean-valued fields, Mathematical Proceedings of the Cambridge Philosophical Society*, Cambridge University Press, **48**(1) (1952), 41-45.
9. S. Izumino, *Decomposition of quotients of bounded operators with respect to closability and Lebesgue-type decomposition of positive operators*, Hokkaido Math. J., **18**(2) (1989), 199-209.
10. S. Izumino, *Quotients of bounded operators*, Proc. Amer. Math. Soc., **106** (1989), 427-435.
11. S. Izumino, *Quotients of bounded operators and their weak adjoints*, J. Operator theory, **29** (1993), 83-96.
12. W. E. Kaufman, *Representing a closed operator as a quotient of continuous operators*, Proc. Amer. Math. Soc., **72** (1978), 531-534.
13. W. E. Kaufman, *Semiclosed operators in Hilbert space*, Proc. Amer. Math. Soc., **76** (1979), 67-73.
14. W. E. Kaufman, *Closed operators and pure contractions in Hilbert space*, Proc. Amer. Math. Soc., **87** (1983), 83-87.
15. N. Koblitz, *p-adic Analysis: a Short Course on Recent Work, Cambridge University Press*, Cambridge, 1980.

16. M. K. Krishna and S. P. Johnson, *Quotient Operators and the Open Mapping Theorem*, Filomat, **32**(18) (2018), 6221-6227.

17. W. PengHui and Z. Xu, *Range inclusion of operators on non-archimedean Banach space*, Science China Mathematics, **53**(12) (2010), 3215-3224.

18. C. Perez-Garcia and W. H. Schikhof, *Locally convex spaces over non-Archimedean valued fields*, Cambridge Studies in Advanced Mathematics, vol. **119**, Cambridge University Press, Cambridge, 2010.

19. A. C. M. van Rooij, *Non-Archimedean functional analysis*, Monographs and Textbooks in Pure and Applied Math., **51**. Marcel Dekker, Inc., New York, 1978.

C. High of Haumman El fetouaki, Had Soualem, Morocco.

E-mail address: jawad.ettayb@gmail.com