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Some results on the Komlòs sets and applications

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ABSTRACT: In this paper, we investigate more about relationships between the Komlòs (resp. pre-Komlòs) sets and other known sets. Also, some new characterizations of Komlòs (resp. pre-Komlòs) property are obtained. Furthermore, we introduce and study new classes of operators that we call Komlòs (resp. pre-Komlòs) operators, we give some characterizations of these classes of operators and we study the relationships between these classes and others classes of operators.

Key Words: Komlòs set, pre-Komlòs set, Komlòs property, pre-Komlòs property, uo-convergence, order continuous norm, KB-space.

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1. Introduction

Recently, N. Gao and et al. [9] introduced the notions of Komlòs set and Komlòs properties. We recall that a subset C of a Banach lattice E is called a Komlòs set if for every sequence (x_n) in C there exist a subsequence (y_n) of (x_n) and $y \in C$ such that the Cesàro means of any subsequence of (y_n) are uo-convergent to y in E ([9, Definition 5.22]). A Banach lattice E is said to have the Komlòs property if for every norm bounded sequence (x_n) in E there exist a subsequence (y_n) of (x_n) and a vector y in E such that the Cesàro means of every subsequence of (y_n) is uo-convergent to y in E ([9, Definition 5.1]). A Banach lattice E is said to have the pre-Komlòs property if for every norm bounded sequence (x_n) in E there exists a subsequence (y_n) of (x_n) such that the Cesàro means of every subsequence of (y_n) is uo-Cauchy in E ([9, Definition 5.1]). A subset C of a Banach lattice E is called a pre-Komlòs set if for every sequence (x_n) in C there is a subsequence (y_n) of (x_n) such that the Cesàro means of any subsequence of (y_n) are uo-Cauchy in E. ([7, Definition 2]).

It can be easily verified that the Komlòs property implies the pre-Komlòs property but the reverse implication is false in general. And that every order continuous Banach lattice E has the pre-Komlòs property. Moreover, E has the Komlòs property if and only if it is a KB-space [9, Corollary 5.14].

In this work, our researches focus about relationships between the Komlòs (resp. pre-Komlòs) sets and others types of sets (compact set, weakly compact set, relatively weakly compact set and Banach-Saks set). Also, some new caracterizations of Komlòs (resp. pre-Komlòs) property are obtained. On the other hand, we introduce and study the class of Komlòs (resp. pre-Komlòs) operators, we characterize Banach lattices on which each operator is a Komlòs (resp. pre-Komlòs) one. After that, we study the relationships between this classes of operators and the class of Banach-Saks (resp. compact) operators.

To state our results, we need to fix some notations and recall some definitions. In a Riesz space, two elements x and y are said to be disjoint whenever $|x| \wedge |y| = 0$ holds ($x \perp y$). A sequence (x_n) in a vector lattice E is said to be disjoint whenever $|x_n| \wedge |x_m| = 0$ holds for $n \neq m$. For each $x, y \in E$ with $x \leq y$, the set $[x, y] := \{z \in E : x \leq z \leq y\}$ is called an order interval. A subset of E is said to be order bounded if it is included in some order interval. A nonzero element x of a vector lattice E is said to be an atom if the order ideal generated by x equals the subspace generated by x. The vector lattice E is said to be atomic if it admits a complete disjoint system of discrete elements. A vector lattice E is

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said to be $\sigma-$laterally complete if the supremum of every disjoint sequence of E^+ exists in E. A vector lattice E is σ-Dedekind complete if every majorized countable nonempty subset of E has a supremum. A Banach lattice is a Banach space $(E, \|.\|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. If E is a Banach lattice, its topological dual E', endowed with the dual norm, is also a Banach lattice. A Banach lattice E is order continuous if for each net (x_α) such that $x_\alpha \downarrow 0$ in E, the net (x_α) converges to 0 for the norm $\|.\|$, where the notation $x_\alpha \downarrow 0$ means that the net (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$. A Banach lattice E is said to be KB-space, if every increasing norm bounded sequence of E^+ is norm convergent. A Banach lattice is said to have weakly sequentially continuous lattice operations whenever $x_n \xrightarrow{w} 0$ implies $|x_n| \xrightarrow{w} 0$.

A net (x_α) of a vector lattice E is said to be unbounded order convergent (abb. uo-convergent) to x if $(|x_\alpha - x| \wedge u)$ converges in order to zero for every $u \in E^+$; we write $x_\alpha \xrightarrow{uo} x$ ([9]).

A net (x_α) of a Banach lattice E is said to be unbounded norm convergent (abb. un-convergent) to x if $(|x_\alpha - x| \wedge u)$ converges in norm to zero for every $u \in E^+$; we write $x_\alpha \xrightarrow{un} x$ ([6]).

We will use the term operator $T : E \longrightarrow F$ between two Banach lattices to mean a bounded linear mapping. It is positive if $T(x) \geq 0$ in F whenever $x \geq 0$ in E. The operator T is regular if $T = T_1 - T_2$ where T_1 and T_2 are positive operators from E into F. Note that each positive linear mapping on a Banach lattice is continuous. If an operator $T : E \longrightarrow F$ between two Banach lattices is positive, then its adjoint $T' : F' \longrightarrow E'$ is likewise positive, where T' is defined by $T'(f)(x) = f(T(x))$ for each $f \in F'$ and for each $x \in E$.

A linear mapping $T : E \longrightarrow F$ between two Riesz spaces is said to be σ-order continuous if for every $x_n \xrightarrow{o} 0$ in E, we have $T(x_n) \xrightarrow{o} 0$ in F (Definition 1.53 [1]).

A linear mapping $T : E \longrightarrow F$ between two Riesz spaces is said to preserve the disjointness whenever $x \perp y$ in E implies $T(x) \perp T(y)$ in F ([1, page 79]).

A linear mapping $T : E \longrightarrow F$ between two Riesz spaces is said to be a lattice (or Riesz) homomorphism whenever $T(x \wedge y) = T(x) \wedge T(y)$ holds for all $x, y \in E$ ([1, Definition 2.13]).

A linear mapping $T : E \longrightarrow F$ between two Riesz spaces is said to be σ-uo-continuous if for every $x_n \xrightarrow{uo} 0$ in E then $T(x_n) \xrightarrow{uo} 0$ in F ([3]).

2. Preliminaries results

Definition 2.1 *A sequence (x_n) in a Riesz space E is said to be* ***Komlòs sequence*** *if there exist a subsequence (y_n) of (x_n) and a vector y in E such that the Cesàro means of every subsequence of (y_n) are uo-converge to y in E.*

Definition 2.2 *A sequence (x_n) in a Riesz space E is said to be* ***pre-Komlòs sequence*** *if there exists a subsequence (y_n) of (x_n) such that the Cesàro means of every subsequence of (y_n) are uo-Cauchy in E.*

A Komlòs (resp. pre-Komlòs) sequence is not necessarily norm bounded ([5]).

A Banach lattice E has the Komlòs (resp. pre-Komlòs) property, if every norm bounded sequence (x_n) of E is a Komlòs (resp. pre-Komlòs) sequence in E.

A Komlòs sequence is pre-Komlòs, but the converse is not true in generale. In fact, from the example 5.4 [9] there exists a pre-Komlòs sequence in c_0 which is not Komlòs.

Recall from Theorem 3.10 [9] that a sequence (x_n) in a vector lattice E which is σ-Dedekind complete and σ-laterally complete is uo-Cauchy if and only if it is o-convergent. So, under this condition every pre-Komlòs sequence in E is Komlòs. Also, if E is sequentially boundedly uo-complete, then every norm bounded pre-Komlòs sequence in E is a Komlòs sequence.

Proposition 2.1 *If every principal band in a Banach lattice E admits a strictly positive order continuous functional, then a norm bounded sequence (x_n) in E is Komlòs if and only if (x_n) has a Cesàro uo-convergent subsequence in E.*

Proof: $(\Longrightarrow)$ Obvious.

$(\Longleftarrow)$ Let (x_n) be a norm bounded sequence in E with a Cesàro uo-convergent subsequence in E. Since E is a Banach lattice in which every principal band admits a strictly positive order continuous functional, it follows from Remark 4.6 [9] that a sequence $y_n \xrightarrow{uo} 0$ in E if and only if $y_n \xrightarrow{a.e} 0$ in $L_1(\mu)$, where μ is a probability measure. Moreover, from Remark 4.3 [9], $E \subset L_1(\mu)$. Then, the natural inclusion $i : E \longrightarrow L_1(\mu)$ is continuous ([8]). Since (x_n) is a bounded sequence in E and since the natural inclusion $i : E \longrightarrow L_1(\mu)$ is continuous ([8]), then (x_n) is a bounded sequence in $L_1(\mu)$ such that (x_n) has a subsequence whose Cesàro means are uo-convergent in $L_1(\mu)$. As $L_1(\mu)$ is order continuous with the komlos property, then by Corollary 5.15 [9] (x_n) is a Komlòs sequence in $L_1(\mu)$. So, (x_n) is a Komlòs sequence in E. □

Proposition 2.2 *If E is a sequentially boundedly uo-complete Banach lattice. Then, every norm bounded uo-closed pre-Komlòs set is a Komlòs set.*

Proof: Let A be a norm bounded uo-closed pre-Komlòs subset of E and (x_n) be a sequence in A. It follows that (x_n) is a norm bounded pre-Komlòs sequence in E. Since E is a sequentially boundedly uo-complete Banach lattice, then (x_n) is a Komlòs sequence in E. That is, there exist a subsequence (y_n) of (x_n) and a vector y in E such that the Cesàro means of every subsequence of (y_n) is uo-converge to y in E. As A is a uo-closed subset of E, then $y \in A$. So, A is a Komlòs set. □

A subset A of a vector lattice X is said to be uo-closed in X, if for any net $(x_\alpha) \subset A$ and $x \in X$ with $x_\alpha \xrightarrow{uo} x$ in X, we have $x \in A$ ([9]).

Proposition 2.3 *Every compact (resp. uo-closed relatively compact) subset of a Banach lattice E is Komlòs.*

Proof: Let A be a compact (resp. uo-closed relatively compact) subset of E and (x_n) be a sequence in A. Then (x_n) has a subsequence (x_{n_k}) converges in norm to some vector x of A (resp. converges in norm to some vector x of E). It follows from Proposition 3.4 [6] that (x_{n_k}) has a subsequence $(x_{n_{\phi(k)}})$ converges in order to x in A (resp. converges in order to x in E). That is, $(x_{n_{\phi(k)}})$ is uo-convergent to x in A (resp. uo-convergent to x in E and since A is uo-closed then $x \in A$), and hence it follows from Corollary 3.13 [9] that the Cesàro means of (x_{n_k}) are uo-convergent to x in A. That is, A is a Komlòs set. □

Proposition 2.4 *Every weakly compact (resp. uo-closed relatively weakly compact) subset A of a Banach lattice E is a Komlòs set, if one of the following conditions is valid:*

1. *The linear span of the minimal ideals in E is order dense in E.*
2. *E is an atomic order continuous Banach lattice.*

Proof: (1) Let A be a weakly compact (resp. uo-closed relatively weakly compact) subset of E and (x_n) be a sequence in A. Then, (x_n) has a subsequence (x_{n_k}) which is weakly convergent to some vector x of A (resp. weakly convergent to some vector x of E). Since the linear span of the minimal ideals in E is order dense in E, then it follows from Theorem 1 [12] that (x_{n_k}) is uo-convergent to x in A (resp. is uo-convergent to x in E and since A is uo-closed then $x \in A$). So, it follows from Corollary 3.13 [9] that the Cesàro means of (x_{n_k}) are uo-convergent to x in A. That is A is a Komlòs set.
(2) Let A be a weakly compact (resp. uo-closed relatively weakly compact) subset of E and (x_n) be a sequence in A. Then, (x_n) has a subsequence (x_{n_k}) which is weakly convergent to some vector x of A (resp. weakly convergent to some vector x of E). Since E is order continuous and atomic, then it follows from Proposition 6.2 [6] that (x_{n_k}) is un-convergent to x in A (resp. is un-convergent to x of E) and by Theorem 5.3 [6] we infer that (x_{n_k}) uo-convergent to x in A (resp. is uo-convergent to x of E and since A is uo-closed then $x \in A$). So, it follows from Corollary 3.13 [9] that the Cesàro means of (x_{n_k}) are uo-convergent to x in A. That is A is a Komlòs set. □

Recall from Proposition 2.3 [10] that every Banach-Saks set is relatively weakly-compact. On the other hand, a Komlòs (resp. pre-Komlòs) set is not necessarily relatively weakly-compact, and then a Komlòs (resp. pre-Komlòs) set is not necessarily a Banach-Saks one.

Proposition 2.5 *If E is an order continuous Banach lattice with weak unit, then every order bounded Komlòs subset A of E is a Banach-Saks set.*

Proof: Let A be an order bounded Komlòs subset of E and (x_n) be a sequence in A. Since E is order continuous with weak unit. It follows from Proposition 4.5 [9] that E admits a strictly positive functional and hence the Proposition 2.1 imply that (x_n) has a subsequence (x_{n_k}) whose Cesàro means are uo-convergent in X. As E is order continuous, then from Proposition 2.5 [6] we infer that the Cesàro means of (x_{n_k}) are un-convergent in E. On the other hand, since A is an order bounded Komlòs subset of E, then (x_n) is order bounded and hence the Cesàro means of (x_{n_k}) are norm convergent in E. So, A is a Banach-Saks set. □

3. Main results

We start this section by giving the following definitions.

Definition 3.1 *An operator T from a Banach space X into a Banach lattice F is said to be Komlòs (resp. pre-Komlòs) if $T(B_X)$ is a Komlòs (resp. pre-Komlòs) set in F.*

It is clear that E has the Komlòs (resp. pre-Komlòs) property if and only if the identity operator Id_E of E is Komlòs (resp. pre-Komlòs).

We can obtain easily the following result.

Proposition 3.1 *Let T be an operator from a Banach space X into a Banach lattice F. Then, we have:*

1. *T is a Komlòs operator if and only if $(T(x_n))$ is a Komlòs sequence of F, for every norm bounded sequence (x_n) of X.*
2. *T is a pre-Komlòs operator if and only if $(T(x_n))$ is a pre-Komlòs sequence of F, for every norm bounded sequence (x_n) of X.*

Proposition 3.2 *Let E, F be Banach lattices and X, Y be Banach spaces. We have the following assertions:*

1. *If $T : Y \longrightarrow F$ is a Komlòs (resp. pre-Komlòs) operator, then $T \circ S$ is a Komlòs (resp. pre-Komlòs) operator, for every operator $S : X \longrightarrow Y$.*
2. *If $T : X \longrightarrow E$ is a Komlòs (resp. pre-Komlòs) operator and $S : E \longrightarrow F$ is an onto σ-order continuous lattice homomorphism (resp. σ-uo-continuous) operator, then $S \circ T$ is a Komlòs (resp. pre-Komlòs) operator.*

Proof:

1. The proof is straightforward.
2. Let (x_n) be a norm bounded sequence of X. Since $T : X \longrightarrow E$ is a Komlòs operator, then there exist $(T(x_{n_k}))$ a subsequence of $(T(x_n))$ and a vector y in E such that the Cesàro means of every subsequence of $(T(x_{n_k}))$ is uo-converge to y.
   (a) If S is σ-uo-continuous, then the Cesàro means of every subsequence of $(S \circ T(x_{n_k}))$ is uo-converge to $S(y)$ in F.
   (b) Suppose that $S : E \longrightarrow F$ is an onto σ-order continuous lattice homomorphism operator. We put:
   $$z_m = \frac{1}{m} \sum_{\phi(k)=1}^{m} x_{n_{\phi(k)}}.$$
   We have $(T(z_m))$ is uo-converge to y in E, then there exists $(y_m) \downarrow 0$ in E such that:
   $$|T(z_m) - y| \wedge u \leq y_m \downarrow 0 \text{ for all } u \in E^+.$$

Since $|S\circ T(z_m)-S(y)|\leq |S||T(z_m)-y|$, then $|S\circ T(z_m)-S(y)|\wedge v\leq |S||T(z_m)-y|\wedge v$ for all $v\in F^+$. As S is an onto lattice homomorphism, then $\forall v\in F^+, \exists u\in E^+$ such that $v=S(u)$. So,

$$\begin{aligned}|S\circ T(z_m)-S(y)|\wedge v &\leq S|T(z_m)-y|\wedge v\\ &\leq S|T(z_m)-y|\wedge S(u)\\ &\leq S(|T(z_m)-y|\wedge u)\\ &\leq S(y_m)\end{aligned}$$

As S is σ-order continuous, then $(S(y_m))\downarrow 0$ and hence $(S\circ T(z_m))$ is uo-converge to $S(y)$. That is $S\circ T$ is a Komlòs operator.

□

As consequences of the Proposition 3.2, we have the following results.

Corollary 3.1 *For a Banach lattice F, the following assertions are equivalent:*

1. *F has the Komlòs (resp. pre-Komlòs) property.*
2. *Every operator $T:X\longrightarrow F$ is Komlòs (resp. pre-Komlòs), for any arbitrary Banach space X.*

Corollary 3.2 *Let $T:E\longrightarrow F$ be an operator between two Banach lattices E and F. If T is an onto σ-order continuous lattice homomorphism (resp. σ-uo-continuous) operator and E has the Komlòs (resp. pre-Komlòs) property, then T is a Komlòs (resp. pre-Komlòs) operator.*

Recall that a subset A of a Banach lattice E is **almost order bounded** if for any $\varepsilon>0$ there exists $u\in E^+$ such that $A\subset[-u,u]+\varepsilon B_E$. Every norm convergent sequence is almost order bounded.

Proposition 3.3 *Let $T:X\longrightarrow F$ be an operator defined from a Banach space X to a Banach lattice F such that F is order continuous. If for every bounded sequence (x_n) of X, $(T(x_n))$ has a subsequence whose Cesàro means are almost order bounded in F then T is a Komlós operator.*

Proof: Let (x_n) be a norm bounded sequence of X such that $(T(x_n))$ has a subsequence whose Cesàro means are almost order bounded in F. Since F is order continuous, it follows from Lemma 6.3 [9] that there exist a subsequence $(T(x_{n_k}))$ of $(T(x_n))$ and a vector $y\in F$ such that the Cesàro means of any subsequence of $(T(x_{n_k}))$ is uo-convergent and norm convergent to y. □

In the following result, we give necessary conditions on E and F under which each operator from E into F is Komlòs (resp. pre-Komlòs).

Theorem 3.1 *If each operator from a Banach lattice E into a Banach lattice F is Komlòs (resp. pre-Komlòs), then one of the following assertions is valid:*

1. *E' is order continuous;*
2. *F has the Komlòs (resp. pre-Komlòs) property.*

Proof: Assume that neither the norm of E' is order continuous nor F has the Komlòs property. Then, by Theorem 2.4.14 and Proposition 2.3.11 [11] E contains a complemented copy of ℓ^1 and there exists a positive projection $P:E\longrightarrow\ell^1$. On the other hand, since F does not have the Komlòs (resp. pre-Komlòs) property then, there exists (y_n) a bounded sequence in F which is not Komlòs (resp. pre-Komlòs).

Consider the operator $S:\ell^1\longrightarrow F$ defined by:

$$S(\lambda_n)=\sum_{n=1}^{\infty}\lambda_n y_n.$$

As the sequence $(\lambda_n) \subset \ell^1$ and (y_n) is a norm bounded sequence, then $\sum_{n=1}^{\infty} \lambda_n y_n$ is norm convergent, and hence S is well defined.

Now, we consider the composed operator $T = S \circ P$. To end the proof we have to claim that T is not a Komlòs (resp. pre-Komlòs) operator. Otherwise, the operator $T \circ i$ will be a Komlòs (resp. pre-Komlòs) operator, where $i : \ell^1 \longrightarrow E$ is the injection from ℓ^1 into E. But by taking (e_n) the standard basis of ℓ^1 as a bounded sequence, we have $(T \circ i(e_n)) = (y_n)$ which is not a Komlòs (resp. pre-Komlòs) sequence, and this is a contradiction. □

Proposition 3.4 *If F is an order continuous Banach lattice and X is a Banach space, then the following statements are equivalent:*

1. *An operator $T : X \longrightarrow F$ is Komlòs.*
2. *$(T(x_n))$ has a Cesáro uo-convergent subsequence in F, for every norm bounded sequence (x_n) of X.*

Proof: (1) $\Longrightarrow$ (2) Let $T : X \longrightarrow F$ be a Komlòs operator and (x_n) be a norm bounded sequence of X. It follows that $(T(x_n))$ is a Komlòs sequence in F. Since F is an order continuous Banach lattice, then by Corollary 5.15 [9] $(T(x_n))$ has a Cesáro uo-convergent subsequence in F.

(2) $\Longleftarrow$ (1) Let $T : X \longrightarrow F$ be an operator and (x_n) be a norm bounded sequence of X such that $(T(x_n))$ has a Cesáro uo-convergent subsequence in F. Since F is an order continuous Banach lattice, then it follows from Corollary 5.15 [9] that $(T(x_n))$ is a Komlòs sequence in F, and hence, $T : X \longrightarrow F$ is Komlòs. □

Note that every Komlòs operator is a pre-Komlòs but the converse is not true in general. Indeed, the identity operator of the Banach lattice c_0 is pre-Komlòs (because c_0 has the pre-Komlòs property) but it is not a Komlòs operator (because c_0 does not have the Komlòs property).

In the following result, we give sufficient conditions on the Banach lattice F under which each pre-Komlòs operator from a Banach space X into F is Komlòs.

Theorem 3.2 *Each pre-Komlòs operator $T : X \longrightarrow F$ from a Banach space X into a Banach lattice F is Komlòs, if one of the following assertions is valid:*

1. *F is σ-Dedekind complete and σ-laterally complete;*
2. *F is sequentially boundedly uo-complete.*

Proof:

1. Let (x_n) be a norm bounded sequence of X and $T : X \longrightarrow F$ be a pre-Komlòs operator. It follows that $(T(x_n))$ is a pre-Komlòs sequence in F. Since F is σ-Dedekinde complete and σ-laterally complete, then by Theorem 3.10 [9] $(T(x_n))$ is a Komlòs sequence of F.
2. Let (x_n) be a norm bounded sequence of X and $T : X \longrightarrow F$ be a pre-Komlòs operator. It follows that $(T(x_n))$ is a bounded pre-Komlòs sequence of F. Since F is sequentially boundedly uo-complete, then by Corollary 3 [7] $(T(x_n))$ is a Komlòs sequence of F.

□

In the following result, we give necessary and sufficient conditions on the Banach lattice F under which each pre-Komlòs operator from a Banach space X into F is Komlòs.

We note that an order continuous Banach lattice is not necessarily a sequentially boundedly uo-complete. Indeed, c_0 is order continuous but is not sequentially boundedly uo-complete (if not c_0 will be a $KB-$space).

Proposition 3.5 *Let F be an order continuous Banach lattice. Then, the following statements are equivalent:*

1. *F is sequentially boundedly uo-complete.*
2. *Each pre-Komlòs operator from X into F is Komlòs, for any arbitrary Banach space X.*

Proof: (1) $\Longrightarrow$ (2) It's obvious.

(2) $\Longrightarrow$ (1) By absurd we assume that F is not sequentially boundedly uo-complete. Then, by Proposition 5.8 [9] the Banach lattice F does not have the Komlòs property. It follows from Corollary 5.14 [9] that F is not a KB-space.

Since F is order continuous, it follows from Lemma 2.1 [2] that F contains a complemented copy of c_0 and there exists a positive projection $P : F \longrightarrow c_0$. We note by $i : c_0 \longrightarrow F$ the canonical injection of c_0 on F. On the other hand, as F is order continuous then F has the pre-Komlòs property, and hence by the Corollary 3.1 i is a pre-Komlòs operator. As P is a positive operator, then it follows from the Proposition 3.2 that $i \circ P = Id_F$ is a pre-Komlòs operator. By our hypothesis Id_F would be a Komlós operator, that is F has the Komlós property, but this is a contradiction. □

An operator $T : X \longrightarrow Y$ between two Banach spaces X and Y is said to be Banach-Saks if for every bounded sequence in X, $(T(x_n))$ has a Cesàro convergent subsequence in Y.

Note that a Komlòs operator from a Banach space into a Banach lattice is not necessarily a Banach-Saks operator. In fact, from Example 5.11 [9] the Banach lattice ℓ_∞ has the Komlòs property. Then, the identity $Id_{\ell_\infty} : \ell_\infty \longrightarrow \ell_\infty$ is a Komlòs operator but it is not Banach-Saks.

In the following result, we give sufficient conditions on the Banach lattices E and F under which each Komlòs operator from E into F is Banach-Saks.

Theorem 3.3 *If the Banach lattice E has an order unit and the Banach lattice F is order continuous with weak unit, then each order bounded Komlòs operator T from E into F is Banach-Saks.*

Proof: Let $T : E \longrightarrow F$ be an order bounded Komlòs operator and B_E the unit ball of E. Then, $T(B_E)$ is a Komlòs set in F. Since E has an order unit e, then by [1, page 194] the unit ball B_E is the order interval $[-e, e]$, and we have T is order bounded. So, $T(B_E)$ is an order bounded Komlòs set in F which is order continuous with weak unit. Then, it follows from the Proposition 2.5 that $T(B_E)$ is a Banach-Saks set in F. Hence, the operator T is a Banach-Saks. □

In the following result, we give sufficient conditions on the Banach lattice F under which each Banach-Saks operator from X into F is Komlòs, for each Banach space X.

Theorem 3.4 *If F is an order continuous Banach lattice, then each Banach-Saks operator T from X into F is Komlòs, for each Banach space X.*

Proof: Let $T : X \longrightarrow F$ be a Banach-Saks operator and (x_n) be a norm bounded sequence in X. Since F is order continuous, then by Theorem 6.17 [9] $(T(x_n))$ has a subsequence whose Cesàro means are almost order bounded. It follows from the Proposition 3.3 that T is Komlós. □

In the following theorem, we show that every compact operator is Komlòs.

Theorem 3.5 *Every compact operator T from a Banach space X into a Banach lattice F is Komlòs.*

Proof: Let (x_n) be a norm bounded sequence in X. Since T is compact, then $(T(x_n))$ has a norm convergent subsequence $(T(y_n))$ in F. It follows from Proposition 3.4 [6] that $(T(y_n))$ has a subsequence $(T(y_{n_k}))$ such that $T(y_{n_k}) \xrightarrow{o} y$ in F. So, $(T(y_{n_k}))$ is uo-converges to y in F, and hence it follows from Corollary 3.13 [9] that the Cesàro means of $(T(y_{n_k}))$ are uo-converge to y in F. So, $(T(x_n))$ is a Komlòs sequence, and hence T is a Komlòs operator. □

The converse of the Theorem 3.5 is not holds in generale. Indeed, we consider the operator defined by:

$$\begin{array}{llll} T : & L_1[0,1] & \longrightarrow \ell^\infty & \\ & f & \longmapsto & T(f) = \left(\int_0^1 f(x) r_1 dx, \int_0^1 f(x) r_2 dx, \ldots.\right) \end{array},$$

where (r_n) is the sequence of Rademacher functions on $[0,1]$.

Since ℓ^∞ has the Komlòs (resp. pre- Komlòs) property, then T is a Komlòs (resp. pre-Komlòs) operator. But from Example 5.17 [1] the operator T is not compact.

In the following result, we give necessary conditions on Banach lattices E and F under which each Komlòs (resp. pre-Komlòs) operator from E into F is compact.

Theorem 3.6 *If each Komlòs (resp. pre-Komlòs) operator from a Banach lattice E into a Banach lattice F is compact, then one of the following assertions is valid:*

1. *E' is order continuous;*
2. *F' is order continuous.*

Proof: Assume that neither the norm of E' nor that of F' is order continuous, it follows from Theorem 2.4.14 [11] and Proposition 2.3.11 [11] that E contains a complemented copy of ℓ^1 (resp. F contains a complemented copy of ℓ^1) and there exists a positive projection $P_1 : E \longrightarrow \ell^1$ (resp. $P_2 : F \longrightarrow \ell^1$). We consider the positive operator $T = \imath_2 \circ P_1$, where $\imath_2 : \ell^1 \longrightarrow F$ is the canonical injection of ℓ^1 in F. Since ℓ^1 has the Komlòs property, then it follows from the Corollary 3.1 that P_1 is a Komlòs (resp. pre-Komlòs) operator. As $\imath_2 : \ell^1 \longrightarrow F$ is a lattice homomorphism, it follows from the Proposition 3.2 that T is a Komlòs (resp. pre-Komlòs) operator. To end the proof, we have to claim that T is not a compact operator. Otherwise, the operator $P_2 \circ T \circ \imath_1 = Id_{\ell^1}$ will be a compact operator, where $i_1 : \ell^1 \longrightarrow E$ is the canonical injection from ℓ^1 into E, and this is a contradiction. □

Remark 3.1 *Neither the assumption "E' is order continuous" nor the assumption "F' is order continuous" is a sufficient condition in the Theorem 3.6. Indeed, the identity $Id_{\ell^\infty} : \ell^\infty \longrightarrow \ell^\infty$ is a Komlòs (resp. pre-Komlòs) operator and $(\ell^\infty)'$ is order continuous but $Id_{\ell^\infty} : \ell^\infty \longrightarrow \ell^\infty$ is not compact. If we take the operator*

$$\begin{array}{rll} T: & L_1[0,1] & \longrightarrow \ell^\infty \\ & f & \longmapsto \quad T(f) = (\int_0^1 f(x) r_1 dx, \int_0^1 f(x) r_2 dx, ...) \end{array}$$

where (r_n) is the sequence of Rademacher functions on $[0,1]$. We have T is a Komlòs (resp. pre-Komlòs) operator and $(\ell^\infty)'$ is order continuous but it follows from Example 5.17 [1] that T is not compact.

There exist Banach lattices E and F and an operator $T : E \longrightarrow F$ which is pre-Komlòs such that its modulus $|T|$ does not exist. We consider the operator $T : L_1[0,1] \longrightarrow c_0$ defined by:

$$\begin{array}{rll} T: & L_1[0,1] & \longrightarrow c_0 \\ & f & \longmapsto \quad (\int_0^1 f(x) r_1 dx, \int_0^1 f(x) r_2 dx, ...) \end{array}$$

where (r_n) is the sequence of Rademacher functions on $[0,1]$. It is clear that T is pre-Komlòs because c_0 has the pre-Komlòs property, but it follows from the Exercise 2.8.E2 [11] that T is not a regular operator and hence the modulus of T does not exist.

Proposition 3.6 *Let E and F be two Banach lattices such that F is Dedekind-complet and $T : E \longrightarrow F$ be an order bounded preserving disjointness operator, then:*

T is pre-Komlòs if and only if $|T|$ is pre-Komlòs.

Proof: Let $T : E \longrightarrow F$ be an order bounded preserving disjointness operator such that F is Dedekind-complet. Clearly that $|T|$ exists, and by Theorem 2.2 [4] we have

$$(*) \quad |T|(|x|) = |T(x)| = ||T|(x)| \quad \forall x \in E.$$

$\Longrightarrow$) Let (x_n) be a norm bounded sequence of E. Since T is a pre-Komlòs operator, then there exists $(T(x_{n_k}))$ a subsequence of $(T(x_n))$ such that the Cesàro means of every subsequence of $(T(x_{n_k}))$ are uo-Cauchy in F. So, the Cesàro means of every subsequence of $(|T(x_{n_k})|)$ are uo-Cauchy in F.

We put:

$$z_m = \frac{1}{m}\sum_{\phi(k)=1}^{m} x_{n_{\phi(k)}}.$$

We have $(T(z_m))$ is uo-Cauchy in F, and then there exists $(y_p) \downarrow 0$ in F such that:

$$|T(z_m) - T(z_n)| \wedge u \leq y_p \downarrow 0 \text{ for all } n, m \in \mathbb{N} \text{ and } u \in F^+.$$

Hence, from $(*)$ we have:

$$(**)\ ||T|(z_m) - |T|(z_n)| \wedge u = |T(z_m) - T(z_n)| \wedge u \leq y_p \downarrow 0 \text{ for all } n, m \in \mathbb{N} \text{ and } u \in F^+.$$

Since $(T(z_m))$ is uo-Cauchy, then $(|T|(z_m))$ is uo-Cauchy in F.
$\Longleftarrow$) From $(**)$ if $(|T|(z_m))$ is uo-Cauchy, then $(T(z_m))$ is uo-Cauchy in F. □

By the same way we investigate the following result:

Proposition 3.7 *Let E and F be two Banach lattices such that F is Dedekind-complet and $T : E \longrightarrow F$ be an order bounded onto preserving disjointness operator, then:*

T is Komlòs if and only if $|T|$ is Komlòs.

Theorem 3.7 *Let E, F be two Banach lattices and $S, T : E \longrightarrow F$ be two operators with $0 \leq S \leq T$. If T is a lattice homomorphism pre-Komlòs operator, then S is a pre-Komlòs operator.*

Proof: Let (x_n) be a norm bounded sequence of E. Since T is a pre-Komlòs operator, then there exists $(T(x_{n_k}))$ a subsequence of $(T(x_n))$ such that the Cesàro means of every subsequence of $(T(x_{n_k}))$ are uo-Cauchy in F. We put:

$$z_m = \frac{1}{m}\sum_{\phi(k)=1}^{m} x_{n_{\phi(k)}}.$$

We have $(T(z_m))$ is uo-Cauchy in F, then there exists $(y_p) \downarrow 0$ in E such that:

$$|T(z_m) - T(z_n)| \wedge u \leq y_p \downarrow 0 \text{ for all } n, m \in \mathbb{N} \text{ and } u \in F^+.$$

As T is a lattice homomorphism, then $|T(x)| = T(|x|)$ for all $x \in E$. So, for all $n, m \in \mathbb{N}$ and $u \in F^+$ we have

$$\begin{aligned} |S(z_m) - S(z_n)| \wedge u &\leq S|z_m - z_n| \wedge u \\ &\leq T|z_m - z_n| \wedge u \\ &= |T(z_m) - T(z_n)| \wedge u \\ &\leq y_p \downarrow 0 \end{aligned}$$

□

By the same way we investigate the following result:

Theorem 3.8 *Let E, F be two Banach lattices and $S, T : E \longrightarrow F$ be two operators with $0 \leq S \leq T$. If T is an onto lattice homomorphism Komlòs operator, then S is a Komlòs operator.*

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