Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–11.
ISSN-0037-8712
doi:10.5269/bspm.64830

Metric Domains and Common Fixed Points with Application

Hakima Bouhadjera* and Iqbal H. Jebril

ABSTRACT: In this paper, in the first step, we prove the existence and uniqueness of common fixed points for four occasionally weakly biased maps of type (A) in a dislocated metric (d-metric) space. In the second step, we derive some results of one map, two and three maps. In the third step, we create two examples to clarify the validity and credibility of our results. In the fourth and last step, we furnish an application for solving an integral equation.

Key Words: Dislocated metric space, occasionally weakly biased maps of type (A), unique common fixed points, integral equation.

Contents

1. Introduction and preliminaries

In 1985, Matthews introduced the concept of dislocated metric spaces under the notion of **metric domains** in order to promote the notion of completeness in domain theory. Also, he pointed out that there is a one to one correspondence between the class of metric domains and the class of metric spaces. In order to justify his reformulation of metric spaces, he showed how the Banach Contraction Map Theorem from elementary metric space theory can be used in metric domain theory as a tool for program verification.

Definition 1.1 *([7]) A **Metric Domain** is a pair $< D, d >$ where D is a non-empty set, and d is a function from $D \times D$ to $\mathbb{R}^+$ such that*

1. $\forall\ x,\ y \in D\ d(x, y) = 0 \Rightarrow x = y$,
2. $\forall\ x,\ y \in D\ d(x, y) = d(y, x)$,
3. $\forall\ x,\ y,\ z \in D\ d(x, y) \leq d(x, z) + d(z, y)$.

After many years, precisely in 2012, Amini-Harandi introduced a new generalization of partial metric spaces which is called **metric-like spaces**. Then, he gave some fixed point results in such spaces which generalize and improve some well-known results in the literature even in the case of partial metric spaces.

Definition 1.2 *([1]) A map $\sigma : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$, where $\mathcal{X}$ is a nonempty set, is said to be metric-like on $\mathcal{X}$ if for any $x,\ y,\ z \in \mathcal{X}$, the following three conditions hold true:*

(σ1) $\sigma(x, y) = 0 \Rightarrow x = y$,

* Corresponding author
Submitted October 29, 2022. Published December 26, 2024
2010 *Mathematics Subject Classification:* 47H10, 54H25.

(σ2) $\sigma(x,y)=\sigma(y,x)$,

(σ3) $\sigma(x,y)\leq\sigma(x,z)+\sigma(z,y)$.

The pair $(\mathcal{X},\sigma)$ is then called a **metric-like space**. Then a metric-like on $\mathcal{X}$ satisfies all of the conditions of a metric except that $\sigma(x,x)$ may be positive for $x\in\mathcal{X}$.

Remark 1.1 *Closely looking at the two above definitions, we see that they are identical, only their names are different.*

Mention that several authors proved the existence and uniqueness of common fixed points in dislocated metric spaces using different conditions (see for example: [2,3,5,6,9,10]).

In a few months, we put in a new definition called **occasionally weakly biased maps of type** (A) and we asserted that our notion has an edge over weak and occasionally weak compatibility; that is, weakly compatible and occasionally weakly compatible maps are subclasses of occasionally weakly biased maps of type (A).

Definition 1.3 *([4]) Let $\mathfrak{M}$ and $\mathfrak{N}$ be self-maps of a non-empty set $\mathcal{X}$. The pair $(\mathfrak{M},\mathfrak{N})$ is said to be **occasionally weakly $\mathfrak{M}$-biased of type** (A) and **occasionally weakly $\mathfrak{N}$-biased of type** (A), respectively, if and only if, there exists a point p in $\mathcal{X}$ such that $\mathfrak{M}p=\mathfrak{N}p$ implies*

$$d(\mathfrak{M}\mathfrak{M}p,\mathfrak{N}p)\leq d(\mathfrak{N}\mathfrak{M}p,\mathfrak{M}p),$$
$$d(\mathfrak{N}\mathfrak{N}p,\mathfrak{M}p)\leq d(\mathfrak{M}\mathfrak{N}p,\mathfrak{N}p),$$

respectively.

Now, in their paper [8], Panthi and Jha established a common fixed point theorem for pairs of weakly compatible maps in a dislocated metric space which generalizes and improves similar fixed point results.

Theorem 1.1 *Let $(\mathcal{X},d)$ be a complete d-metric space. Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{S}$, $\mathfrak{T}:\mathcal{X}\to\mathcal{X}$ be continuous maps satisfying,*

1. *$\mathfrak{T}(\mathcal{X})\subset\mathfrak{A}(\mathcal{X})$, $\mathfrak{S}(\mathcal{X})\subset\mathfrak{B}(\mathcal{X})$,*
2. *the pairs $(\mathfrak{S},\mathfrak{A})$ and $(\mathfrak{T},\mathfrak{B})$ are weakly compatible and*
3. *$d(\mathfrak{S}x,\mathfrak{T}y)\leq\alpha[d(\mathfrak{A}x,\mathfrak{T}y)+d(\mathfrak{B}y,\mathfrak{S}x)]+\beta[d(\mathfrak{B}y,\mathfrak{T}y)+d(\mathfrak{A}x,\mathfrak{S}x)]+\gamma d(\mathfrak{A}x,\mathfrak{B}y)$*

for all x, $y\in\mathcal{X}$ where α, β, $\gamma\geq 0$ and $0\leq\alpha+\beta+\gamma<\frac{1}{4}$. Then $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{S}$, and $\mathfrak{T}$ have a unique common fixed point.

In this work, we will improve the above theorem and other similar results by deleting some conditions; in one hand by letting constants α, β and γ as they are and in other hand by extending them to real functions.

2. Main results

2.1. Existence and uniqueness of common fixed point for quadruple maps

Theorem 2.1 *Let $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{O}$ and $\mathfrak{P}$ be self-maps of a complete d-metric space $\mathcal{X}$ satisfying the following condition*

$$\begin{aligned} d(\mathfrak{M}x,\mathfrak{N}y) \quad \leq \quad & \alpha[d(\mathfrak{O}x,\mathfrak{N}y)+d(\mathfrak{P}y,\mathfrak{M}x)]+\beta[d(\mathfrak{P}y,\mathfrak{N}y)+d(\mathfrak{O}x,\mathfrak{M}x)] \\ & +\gamma d(\mathfrak{O}x,\mathfrak{P}y) \end{aligned}$$

for all x, $y\in\mathcal{X}$, where α, β, $\gamma\geq 0$ and $2\alpha+4\beta+\gamma<1$. If the pair $(\mathfrak{M},\mathfrak{O})$ as well as $(\mathfrak{N},\mathfrak{P})$ is occasionally weakly $\mathfrak{O}$-biased of type (A) and occasionally weakly $\mathfrak{P}$-biased of type (A), respectively, then $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{O}$ and $\mathfrak{P}$ have a unique common fixed point.

Proof: By hypotheses, there are two points u and v in $\mathcal{X}$ such that $\mathfrak{M}u = \mathfrak{O}u$ implies $d(\mathfrak{O}\mathfrak{O}u, \mathfrak{M}u) \leq d(\mathfrak{M}\mathfrak{O}u, \mathfrak{O}u)$ and $\mathfrak{N}v = \mathfrak{P}v$ implies $d(\mathfrak{P}\mathfrak{P}v, \mathfrak{N}v) \leq d(\mathfrak{N}\mathfrak{P}v, \mathfrak{P}v)$.

First, we are going to prove that $\mathfrak{M}u = \mathfrak{N}v$. We have

$$\begin{aligned}
d(\mathfrak{M}u, \mathfrak{N}v) &\leq \alpha[d(\mathfrak{O}u, \mathfrak{N}v) + d(\mathfrak{P}v, \mathfrak{M}u)] + \beta[d(\mathfrak{P}v, \mathfrak{N}v) + d(\mathfrak{O}u, \mathfrak{M}u)] \\
&\quad + \gamma d(\mathfrak{O}u, \mathfrak{P}v) \\
&= \alpha[d(\mathfrak{M}u, \mathfrak{N}v) + d(\mathfrak{N}v, \mathfrak{M}u)] + \beta[d(\mathfrak{N}v, \mathfrak{N}v) + d(\mathfrak{M}u, \mathfrak{M}u)] \\
&\quad + \gamma d(\mathfrak{M}u, \mathfrak{N}v) \\
&\leq \alpha[d(\mathfrak{M}u, \mathfrak{N}v) + d(\mathfrak{N}v, \mathfrak{M}u)] + \gamma d(\mathfrak{M}u, \mathfrak{N}v) \\
&\quad + \beta[d(\mathfrak{N}v, \mathfrak{M}u) + d(\mathfrak{M}u, \mathfrak{N}v) + d(\mathfrak{M}u, \mathfrak{N}v) + d(\mathfrak{N}v, \mathfrak{M}u)] \\
&= [2\alpha + 4\beta + \gamma] d(\mathfrak{M}u, \mathfrak{N}v) \\
&< d(\mathfrak{M}u, \mathfrak{N}v)
\end{aligned}$$

a contradiction, hence $\mathfrak{M}u = \mathfrak{N}v$.

Now, we assert that $\mathfrak{M}\mathfrak{M}u = \mathfrak{M}u$. If not, then we have

$$\begin{aligned}
d(\mathfrak{M}\mathfrak{M}u, \mathfrak{N}v) &\leq \alpha[d(\mathfrak{O}\mathfrak{M}u, \mathfrak{N}v) + d(\mathfrak{P}v, \mathfrak{M}\mathfrak{M}u)] + \beta[d(\mathfrak{P}v, \mathfrak{N}v) + d(\mathfrak{O}\mathfrak{M}u, \mathfrak{M}\mathfrak{M}u)] \\
&\quad + \gamma d(\mathfrak{O}\mathfrak{M}u, \mathfrak{P}v);
\end{aligned}$$

i.e.,

$$\begin{aligned}
d(\mathfrak{M}\mathfrak{M}u, \mathfrak{M}u) &\leq \alpha[d(\mathfrak{O}\mathfrak{O}u, \mathfrak{M}u) + d(\mathfrak{M}u, \mathfrak{M}\mathfrak{M}u)] + \beta[d(\mathfrak{M}u, \mathfrak{M}u) + d(\mathfrak{O}\mathfrak{O}u, \mathfrak{M}\mathfrak{M}u)] \\
&\quad + \gamma d(\mathfrak{O}\mathfrak{O}u, \mathfrak{M}u) \\
&\leq \alpha[d(\mathfrak{M}\mathfrak{O}u, \mathfrak{O}u) + d(\mathfrak{M}u, \mathfrak{M}\mathfrak{M}u)] + \gamma d(\mathfrak{M}\mathfrak{O}u, \mathfrak{O}u) \\
&\quad + \beta[d(\mathfrak{M}u, \mathfrak{M}\mathfrak{M}u) + d(\mathfrak{M}\mathfrak{M}u, \mathfrak{M}u) + d(\mathfrak{M}\mathfrak{O}u, \mathfrak{O}u) + d(\mathfrak{M}u, \mathfrak{M}\mathfrak{M}u)] \\
&= [2\alpha + 4\beta + \gamma] d(\mathfrak{M}\mathfrak{M}u, \mathfrak{M}u) \\
&< d(\mathfrak{M}\mathfrak{M}u, \mathfrak{M}u)
\end{aligned}$$

a contradiction, thus $\mathfrak{M}\mathfrak{M}u = \mathfrak{M}u$ and so $\mathfrak{O}\mathfrak{M}u = \mathfrak{M}u$.

Suppose that $\mathfrak{N}\mathfrak{N}v \neq \mathfrak{N}v$, then, we obtain

$$\begin{aligned}
d(\mathfrak{M}u, \mathfrak{N}\mathfrak{N}v) &\leq \alpha[d(\mathfrak{O}u, \mathfrak{N}\mathfrak{N}v) + d(\mathfrak{P}\mathfrak{N}v, \mathfrak{M}u)] + \beta[d(\mathfrak{P}\mathfrak{N}v, \mathfrak{N}\mathfrak{N}v) + d(\mathfrak{O}u, \mathfrak{M}u)] \\
&\quad + \gamma d(\mathfrak{O}u, \mathfrak{P}\mathfrak{N}v);
\end{aligned}$$

i.e.,

$$\begin{aligned}
d(\mathfrak{N}v, \mathfrak{N}\mathfrak{N}v) &\leq \alpha[d(\mathfrak{N}v, \mathfrak{N}\mathfrak{N}v) + d(\mathfrak{P}\mathfrak{N}v, \mathfrak{N}v)] + \beta[d(\mathfrak{P}\mathfrak{N}v, \mathfrak{N}\mathfrak{N}v) + d(\mathfrak{N}v, \mathfrak{N}v)] \\
&\quad + \gamma d(\mathfrak{N}v, \mathfrak{P}\mathfrak{N}v) \\
&= \alpha[d(\mathfrak{N}v, \mathfrak{N}\mathfrak{N}v) + d(\mathfrak{P}\mathfrak{P}v, \mathfrak{N}v)] + \beta[d(\mathfrak{P}\mathfrak{P}v, \mathfrak{N}\mathfrak{N}v) + d(\mathfrak{N}v, \mathfrak{N}v)] \\
&\quad + \gamma d(\mathfrak{N}v, \mathfrak{P}\mathfrak{P}v) \\
&\leq \alpha[d(\mathfrak{N}v, \mathfrak{N}\mathfrak{N}v) + d(\mathfrak{N}\mathfrak{P}v, \mathfrak{P}v)] + \gamma d(\mathfrak{P}v, \mathfrak{N}\mathfrak{P}v) \\
&\quad + \beta[d(\mathfrak{N}\mathfrak{P}v, \mathfrak{P}v) + d(\mathfrak{N}v, \mathfrak{N}\mathfrak{N}v) + d(\mathfrak{N}v, \mathfrak{N}\mathfrak{N}v) + d(\mathfrak{N}\mathfrak{N}v, \mathfrak{N}v)] \\
&= [2\alpha + 4\beta + \gamma] d(\mathfrak{N}v, \mathfrak{N}\mathfrak{N}v) \\
&< d(\mathfrak{N}v, \mathfrak{N}\mathfrak{N}v)
\end{aligned}$$

which implies that $\mathfrak{N}\mathfrak{N}v = \mathfrak{N}v$ and so $\mathfrak{P}\mathfrak{N}v = \mathfrak{N}v$; i.e., $\mathfrak{N}\mathfrak{M}u = \mathfrak{M}u$ and $\mathfrak{P}\mathfrak{M}u = \mathfrak{M}u$. Put $\mathfrak{M}u = \mathfrak{O}u = \mathfrak{N}v = \mathfrak{P}v = \omega$, therefore ω is a common fixed point of maps $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{O}$ and $\mathfrak{P}$.

Finally, let ω and ϖ be two distinct common fixed points of maps $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{O}$ and $\mathfrak{P}$. Then, $\omega = \mathfrak{M}\omega = \mathfrak{N}\omega = \mathfrak{O}\omega = \mathfrak{P}\omega$ and $\varpi = \mathfrak{M}\varpi = \mathfrak{N}\varpi = \mathfrak{O}\varpi = \mathfrak{P}\varpi$. We have

$$\begin{aligned} d(\mathfrak{M}\varpi, \mathfrak{N}\omega) \leq\ & \alpha[d(\mathfrak{O}\varpi, \mathfrak{N}\omega) + d(\mathfrak{P}\omega, \mathfrak{M}\varpi)] + \beta[d(\mathfrak{P}\omega, \mathfrak{N}\omega) + d(\mathfrak{O}\varpi, \mathfrak{M}\varpi)] \\ & + \gamma d(\mathfrak{O}\varpi, \mathfrak{P}\omega); \end{aligned}$$

i.e.,

$$\begin{aligned} d(\varpi, \omega) &\leq \alpha[d(\varpi, \omega) + d(\omega, \varpi)] + \beta[d(\omega, \omega) + d(\varpi, \varpi)] \\ &\quad + \gamma d(\varpi, \omega) \\ &\leq \alpha[d(\varpi, \omega) + d(\omega, \varpi)] + \gamma d(\varpi, \omega) \\ &\quad + \beta[d(\omega, \varpi) + d(\varpi, \omega) + d(\varpi, \omega) + d(\omega, \varpi)] \\ &= [2\alpha + 4\beta + \gamma] d(\varpi, \omega) \\ &< d(\varpi, \omega) \end{aligned}$$

which is a contradiction, this implies that $\varpi = \omega$. □

Theorem 2.2 *Let $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{O}$ and $\mathfrak{P}$ be self-maps of a complete d-metric space $\mathcal{X}$ satisfying*

$$\begin{aligned} d(\mathfrak{M}x, \mathfrak{N}y) \leq\ & \alpha(d(\mathfrak{O}x, \mathfrak{P}y))[d(\mathfrak{O}x, \mathfrak{N}y) + d(\mathfrak{P}y, \mathfrak{M}x)] \\ & + \beta(d(\mathfrak{O}x, \mathfrak{P}y))[d(\mathfrak{P}y, \mathfrak{N}y) + d(\mathfrak{O}x, \mathfrak{M}x)] \\ & + \gamma(d(\mathfrak{O}x, \mathfrak{P}y)) d(\mathfrak{O}x, \mathfrak{P}y) \end{aligned} \tag{2.1}$$

for all x, $y \in \mathcal{X}$, where α, β, $\gamma : [0, +\infty) \to [0, 1)$ are non-decreasing functions which satisfying the following condition

$$4\alpha(t) + 2\beta(t) + \gamma(t) < 1 \ \forall t > 0.$$

If the pair $(\mathfrak{M}, \mathfrak{O})$ as well as $(\mathfrak{N}, \mathfrak{P})$ is occasionally weakly $\mathfrak{O}$-biased of type (A) and occasionally weakly $\mathfrak{P}$-biased of type (A), respectively, then $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{O}$ and $\mathfrak{P}$ have a unique common fixed point.

Proof: As in the proof of Theorem 2.1, there exist two elements u and v in $\mathcal{X}$ such that $\mathfrak{M}u = \mathfrak{O}u$ implies $d(\mathfrak{O}\mathfrak{O}u, \mathfrak{M}u) \leq d(\mathfrak{M}\mathfrak{O}u, \mathfrak{O}u)$ and $\mathfrak{N}v = \mathfrak{P}v$ implies $d(\mathfrak{P}\mathfrak{P}v, \mathfrak{N}v) \leq d(\mathfrak{N}\mathfrak{P}v, \mathfrak{P}v)$.

First, we are going to prove that $\mathfrak{M}u = \mathfrak{N}v$. From inequality (2.1) we have

$$\begin{aligned} d(\mathfrak{M}u, \mathfrak{N}v) &\leq \alpha(d(\mathfrak{O}u, \mathfrak{P}v))[d(\mathfrak{O}u, \mathfrak{N}v) + d(\mathfrak{P}v, \mathfrak{M}u)] \\ &\quad + \beta(d(\mathfrak{O}u, \mathfrak{P}v))[d(\mathfrak{P}v, \mathfrak{N}v) + d(\mathfrak{O}u, \mathfrak{M}u)] \\ &\quad + \gamma(d(\mathfrak{O}u, \mathfrak{P}v)) d(\mathfrak{O}u, \mathfrak{P}v) \\ &= \alpha(d(\mathfrak{M}u, \mathfrak{N}v))[d(\mathfrak{M}u, \mathfrak{N}v) + d(\mathfrak{N}v, \mathfrak{M}u)] \\ &\quad + \beta(d(\mathfrak{M}u, \mathfrak{N}v))[d(\mathfrak{N}v, \mathfrak{N}v) + d(\mathfrak{M}u, \mathfrak{M}u)] \\ &\quad + \gamma(d(\mathfrak{M}u, \mathfrak{N}v)) d(\mathfrak{M}u, \mathfrak{N}v) \\ &\leq \alpha(d(\mathfrak{M}u, \mathfrak{N}v))[d(\mathfrak{M}u, \mathfrak{N}v) + d(\mathfrak{N}v, \mathfrak{M}u)] \\ &\quad + \beta(d(\mathfrak{M}u, \mathfrak{N}v))[d(\mathfrak{N}v, \mathfrak{M}u) + d(\mathfrak{M}u, \mathfrak{N}v) + d(\mathfrak{M}u, \mathfrak{N}v) + d(\mathfrak{N}v, \mathfrak{M}u)] \\ &\quad + \gamma(d(\mathfrak{M}u, \mathfrak{N}v)) d(\mathfrak{M}u, \mathfrak{N}v) \\ &= [2\alpha(d(\mathfrak{M}u, \mathfrak{N}v)) + 4\beta(d(\mathfrak{M}u, \mathfrak{N}v)) + \gamma(d(\mathfrak{M}u, \mathfrak{N}v))] d(\mathfrak{M}u, \mathfrak{N}v) \\ &< d(\mathfrak{M}u, \mathfrak{N}v) \end{aligned}$$

a contradiction, hence $\mathfrak{M}u = \mathfrak{N}v$.

Now, we assert that $\mathfrak{M}\mathfrak{M}u = \mathfrak{M}u$. If not, then the use of condition (2.1) gives

$$\begin{aligned} d(\mathfrak{M}\mathfrak{M}u, \mathfrak{N}v) \leq\ & \alpha(d(\mathfrak{O}\mathfrak{M}u, \mathfrak{P}v))[d(\mathfrak{O}\mathfrak{M}u, \mathfrak{N}v) + d(\mathfrak{P}v, \mathfrak{M}\mathfrak{M}u)] \\ & + \beta(d(\mathfrak{O}\mathfrak{M}u, \mathfrak{P}v))[d(\mathfrak{P}v, \mathfrak{N}v) + d(\mathfrak{O}\mathfrak{M}u, \mathfrak{M}\mathfrak{M}u)] \\ & + \gamma(d(\mathfrak{O}\mathfrak{M}u, \mathfrak{P}v)) d(\mathfrak{O}\mathfrak{M}u, \mathfrak{P}v); \end{aligned}$$

i.e.,

$$\begin{aligned}
d(\mathfrak{M}\mathfrak{M}u,\mathfrak{M}u) &\leq \alpha(d(\mathfrak{O}\mathfrak{M}u,\mathfrak{M}u))[d(\mathfrak{O}\mathfrak{M}u,\mathfrak{M}u)+d(\mathfrak{M}u,\mathfrak{M}\mathfrak{M}u)]\\
&\quad+\beta(d(\mathfrak{O}\mathfrak{M}u,\mathfrak{M}u))[d(\mathfrak{M}u,\mathfrak{M}u)+d(\mathfrak{O}\mathfrak{M}u,\mathfrak{M}\mathfrak{M}u)]\\
&\quad+\gamma(d(\mathfrak{O}\mathfrak{M}u,\mathfrak{M}u))d(\mathfrak{O}\mathfrak{M}u,\mathfrak{M}u)\\
&= \alpha(d(\mathfrak{O}\mathfrak{O}u,\mathfrak{M}u))[d(\mathfrak{O}\mathfrak{O}u,\mathfrak{M}u)+d(\mathfrak{M}u,\mathfrak{M}\mathfrak{M}u)]\\
&\quad+\beta(d(\mathfrak{O}\mathfrak{O}u,\mathfrak{M}u))[d(\mathfrak{M}u,\mathfrak{M}u)+d(\mathfrak{O}\mathfrak{O}u,\mathfrak{M}\mathfrak{M}u)]\\
&\quad+\gamma(d(\mathfrak{O}\mathfrak{O}u,\mathfrak{M}u))d(\mathfrak{O}\mathfrak{O}u,\mathfrak{M}u)\\
&\leq \alpha(d(\mathfrak{M}\mathfrak{O}u,\mathfrak{O}u))[d(\mathfrak{M}\mathfrak{O}u,\mathfrak{O}u)+d(\mathfrak{M}u,\mathfrak{M}\mathfrak{M}u)]\\
&\quad+\beta(d(\mathfrak{M}\mathfrak{O}u,\mathfrak{O}u))[d(\mathfrak{M}u,\mathfrak{M}\mathfrak{M}u)+d(\mathfrak{M}\mathfrak{M}u,\mathfrak{M}u)\\
&\quad+d(\mathfrak{M}\mathfrak{O}u,\mathfrak{O}u)+d(\mathfrak{M}u,\mathfrak{M}\mathfrak{M}u)]\\
&\quad+\gamma(d(\mathfrak{M}\mathfrak{O}u,\mathfrak{O}u))d(\mathfrak{M}\mathfrak{O}u,\mathfrak{O}u)\\
&= \alpha(d(\mathfrak{M}\mathfrak{M}u,\mathfrak{M}u))[d(\mathfrak{M}\mathfrak{M}u,\mathfrak{M}u)+d(\mathfrak{M}u,\mathfrak{M}\mathfrak{M}u)]\\
&\quad+\beta(d(\mathfrak{M}\mathfrak{M}u,\mathfrak{M}u))[d(\mathfrak{M}u,\mathfrak{M}\mathfrak{M}u)+d(\mathfrak{M}\mathfrak{M}u,\mathfrak{M}u)\\
&\quad+d(\mathfrak{M}\mathfrak{M}u,\mathfrak{M}u)+d(\mathfrak{M}u,\mathfrak{M}\mathfrak{M}u)]\\
&\quad+\gamma(d(\mathfrak{M}\mathfrak{M}u,\mathfrak{M}u))d(\mathfrak{M}\mathfrak{M}u,\mathfrak{M}u)\\
&= [2\alpha(d(\mathfrak{M}\mathfrak{M}u,\mathfrak{M}u))+4\beta(d(\mathfrak{M}\mathfrak{M}u,\mathfrak{M}u))\\
&\quad+\gamma(d(\mathfrak{M}\mathfrak{M}u,\mathfrak{M}u))]d(\mathfrak{M}\mathfrak{M}u,\mathfrak{M}u)\\
&< d(\mathfrak{M}\mathfrak{M}u,\mathfrak{M}u)
\end{aligned}$$

a contradiction, thus, $\mathfrak{M}\mathfrak{M}u=\mathfrak{M}u$ and so $\mathfrak{O}\mathfrak{M}u=\mathfrak{M}u$.

Suppose that $\mathfrak{N}\mathfrak{N}v\neq\mathfrak{N}v$. Using inequality (2.1) we obtain

$$\begin{aligned}
d(\mathfrak{M}u,\mathfrak{N}\mathfrak{N}v) &\leq \alpha(d(\mathfrak{O}u,\mathfrak{P}\mathfrak{N}v))[d(\mathfrak{O}u,\mathfrak{N}\mathfrak{N}v)+d(\mathfrak{P}\mathfrak{N}v,\mathfrak{M}u)]\\
&\quad+\beta(d(\mathfrak{O}u,\mathfrak{P}\mathfrak{N}v))[d(\mathfrak{P}\mathfrak{N}v,\mathfrak{N}\mathfrak{N}v)+d(\mathfrak{O}u,\mathfrak{M}u)]\\
&\quad+\gamma(d(\mathfrak{O}u,\mathfrak{P}\mathfrak{N}v))d(\mathfrak{O}u,\mathfrak{P}\mathfrak{N}v);
\end{aligned}$$

i.e.,

$$\begin{aligned}
d(\mathfrak{N}v,\mathfrak{N}\mathfrak{N}v) &\leq \alpha(d(\mathfrak{N}v,\mathfrak{P}\mathfrak{N}v))[d(\mathfrak{N}v,\mathfrak{N}\mathfrak{N}v)+d(\mathfrak{P}\mathfrak{N}v,\mathfrak{N}v)]\\
&\quad+\beta(d(\mathfrak{N}v,\mathfrak{P}\mathfrak{N}v))[d(\mathfrak{P}\mathfrak{N}v,\mathfrak{N}\mathfrak{N}v)+d(\mathfrak{N}v,\mathfrak{N}v)]\\
&\quad+\gamma(d(\mathfrak{N}v,\mathfrak{P}\mathfrak{N}v))d(\mathfrak{N}v,\mathfrak{P}\mathfrak{N}v)\\
&= \alpha(d(\mathfrak{N}v,\mathfrak{P}\mathfrak{P}v))[d(\mathfrak{N}v,\mathfrak{N}\mathfrak{N}v)+d(\mathfrak{P}\mathfrak{P}v,\mathfrak{N}v)]\\
&\quad+\beta(d(\mathfrak{N}v,\mathfrak{P}\mathfrak{P}v))[d(\mathfrak{P}\mathfrak{P}v,\mathfrak{N}\mathfrak{N}v)+d(\mathfrak{N}v,\mathfrak{N}v)]\\
&\quad+\gamma(d(\mathfrak{N}v,\mathfrak{P}\mathfrak{P}v))d(\mathfrak{N}v,\mathfrak{P}\mathfrak{P}v)\\
&\leq \alpha(d(\mathfrak{P}v,\mathfrak{N}\mathfrak{P}v))[d(\mathfrak{N}v,\mathfrak{N}\mathfrak{N}v)+d(\mathfrak{N}\mathfrak{P}v,\mathfrak{P}v)]\\
&\quad+\beta(d(\mathfrak{P}v,\mathfrak{N}\mathfrak{P}v))[d(\mathfrak{N}\mathfrak{P}v,\mathfrak{P}v)+d(\mathfrak{N}v,\mathfrak{N}\mathfrak{N}v)\\
&\quad+d(\mathfrak{N}v,\mathfrak{N}\mathfrak{N}v)+d(\mathfrak{N}\mathfrak{N}v,\mathfrak{N}v)]\\
&\quad+\gamma(d(\mathfrak{P}v,\mathfrak{N}\mathfrak{P}v))d(\mathfrak{P}v,\mathfrak{N}\mathfrak{P}v)\\
&= [2\alpha(d(\mathfrak{N}v,\mathfrak{N}\mathfrak{N}v))+4\beta(d(\mathfrak{N}v,\mathfrak{N}\mathfrak{N}v))\\
&\quad+\gamma(d(\mathfrak{N}v,\mathfrak{N}\mathfrak{N}v))]d(\mathfrak{N}v,\mathfrak{N}\mathfrak{N}v)\\
&< d(\mathfrak{N}v,\mathfrak{N}\mathfrak{N}v)
\end{aligned}$$

which implies that $\mathfrak{N}\mathfrak{N}v=\mathfrak{N}v$ and so $\mathfrak{P}\mathfrak{N}v=\mathfrak{N}v$; i.e., $\mathfrak{N}\mathfrak{M}u=\mathfrak{M}u$ and $\mathfrak{P}\mathfrak{M}u=\mathfrak{M}u$. Put $\mathfrak{M}u=\mathfrak{O}u=\mathfrak{N}v=\mathfrak{P}v=\omega$, therefore ω is a common fixed point of maps $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{O}$ and $\mathfrak{P}$.

Finally, let ω and ϖ be two distinct common fixed points of maps $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{O}$ and $\mathfrak{P}$. Then, $\omega = \mathfrak{M}\omega = \mathfrak{N}\omega = \mathfrak{O}\omega = \mathfrak{P}\omega$ and $\varpi = \mathfrak{M}\varpi = \mathfrak{N}\varpi = \mathfrak{O}\varpi = \mathfrak{P}\varpi$. From (2.1) we have

$$\begin{aligned} d(\mathfrak{M}\varpi, \mathfrak{N}\omega) &\leq \alpha(d(\mathfrak{O}\varpi, \mathfrak{P}\omega))[d(\mathfrak{O}\varpi, \mathfrak{N}\omega) + d(\mathfrak{P}\omega, \mathfrak{M}\varpi)] \\ &\quad +\beta(d(\mathfrak{O}\varpi, \mathfrak{P}\omega))[d(\mathfrak{P}\omega, \mathfrak{N}\omega) + d(\mathfrak{O}\varpi, \mathfrak{M}\varpi)] \\ &\quad +\gamma(d(\mathfrak{O}\varpi, \mathfrak{P}\omega))d(\mathfrak{O}\varpi, \mathfrak{P}\omega); \end{aligned}$$

i.e.,

$$\begin{aligned} d(\varpi, \omega) &\leq \alpha(d(\varpi, \omega))[d(\varpi, \omega) + d(\omega, \varpi)] \\ &\quad +\beta(d(\varpi, \omega))[d(\omega, \omega) + d(\varpi, \varpi)] \\ &\quad +\gamma(d(\varpi, \omega))d(\varpi, \omega) \\ &\leq \alpha(d(\varpi, \omega))[d(\varpi, \omega) + d(\omega, \varpi)] \\ &\quad +\beta(d(\varpi, \omega))[d(\omega, \varpi) + d(\varpi, \omega) + d(\varpi, \omega) + d(\omega, \varpi)] \\ &\quad +\gamma(d(\varpi, \omega))d(\varpi, \omega) \\ &= [2\alpha(d(\varpi, \omega)) + 4\beta(d(\varpi, \omega)) + \gamma(d(\varpi, \omega))]d(\varpi, \omega) \\ &< d(\varpi, \omega) \end{aligned}$$

which is a contradiction, this implies that $\varpi = \omega$. □

2.2. Some results

Corollary 2.1 *Let $\mathfrak{M}$ be a self-map of a complete d-metric space $\mathcal{X}$ satisfying the following condition*

$$\begin{aligned} d(\mathfrak{M}x, \mathfrak{M}y) &\leq \alpha[d(x, \mathfrak{M}y) + d(y, \mathfrak{M}x)] + \beta[d(y, \mathfrak{M}y) + d(x, \mathfrak{M}x)] \\ &\quad +\gamma d(x, y) \end{aligned}$$

for all x, $y \in \mathcal{X}$, where α, β, $\gamma \geq 0$ and $2\alpha + 4\beta + \gamma < 1$, then $\mathfrak{M}$ has a unique fixed point.

Corollary 2.2 *Let $\mathfrak{M}$ and $\mathfrak{O}$ be two self-maps of a complete d-metric space $\mathcal{X}$ satisfying the following condition*

$$\begin{aligned} d(\mathfrak{M}x, \mathfrak{M}y) &\leq \alpha[d(\mathfrak{O}x, \mathfrak{M}y) + d(\mathfrak{O}y, \mathfrak{M}x)] + \beta[d(\mathfrak{O}y, \mathfrak{M}y) + d(\mathfrak{O}x, \mathfrak{M}x)] \\ &\quad +\gamma d(\mathfrak{O}x, \mathfrak{O}y) \end{aligned}$$

for all x, $y \in \mathcal{X}$, where α, β, $\gamma \geq 0$ and $2\alpha + 4\beta + \gamma < 1$. If $\mathfrak{M}$ and $\mathfrak{O}$ are occasionally weakly $\mathfrak{O}$-biased of type (A), then $\mathfrak{M}$ and $\mathfrak{O}$ have a unique common fixed point.

Corollary 2.3 *Let $\mathfrak{M}$, $\mathfrak{N}$ and $\mathfrak{O}$ be three self-maps of a complete d-metric space $\mathcal{X}$ satisfying the following condition*

$$\begin{aligned} d(\mathfrak{M}x, \mathfrak{N}y) &\leq \alpha[d(\mathfrak{O}x, \mathfrak{N}y) + d(\mathfrak{O}y, \mathfrak{M}x)] + \beta[d(\mathfrak{O}y, \mathfrak{N}y) + d(\mathfrak{O}x, \mathfrak{M}x)] \\ &\quad +\gamma d(\mathfrak{O}x, \mathfrak{O}y) \end{aligned}$$

for all x, $y \in \mathcal{X}$, where α, β, $\gamma \geq 0$ and $2\alpha + 4\beta + \gamma < 1$. If $\mathfrak{M}$ and $\mathfrak{O}$ as well as $\mathfrak{N}$ and $\mathfrak{O}$ are occasionally weakly $\mathfrak{O}$-biased of type (A), then $\mathfrak{M}$, $\mathfrak{N}$ and $\mathfrak{O}$ have a unique common fixed point.

Corollary 2.4 *Let $\mathfrak{M}$ be a self-map of a complete d-metric space $\mathcal{X}$ satisfying*

$$\begin{aligned} d(\mathfrak{M}x, \mathfrak{M}y) &\leq \alpha(d(x, y))[d(x, \mathfrak{M}y) + d(y, \mathfrak{M}x)] \\ &\quad +\beta(d(x, y))[d(y, \mathfrak{M}y) + d(x, \mathfrak{M}x)] \\ &\quad +\gamma(d(x, y))d(x, y) \end{aligned}$$

for all x, $y \in \mathcal{X}$, where α, β, $\gamma : [0, +\infty) \to [0, 1)$ are non-decreasing continuous functions which satisfying the following condition

$$4\alpha(t) + 2\beta(t) + \gamma(t) < 1 \ \forall t > 0,$$

then $\mathfrak{M}$ has a unique fixed point.

Corollary 2.5 *Let $\mathfrak{M}$ and $\mathfrak{O}$ be self-maps of a complete d-metric space $\mathcal{X}$ satisfying*

$$\begin{aligned} d(\mathfrak{M}x, \mathfrak{M}y) \leq\ & \alpha(d(\mathfrak{O}x, \mathfrak{O}y))[d(\mathfrak{O}x, \mathfrak{M}y) + d(\mathfrak{O}y, \mathfrak{M}x)] \\ & +\beta(d(\mathfrak{O}x, \mathfrak{O}y))[d(\mathfrak{O}y, \mathfrak{M}y) + d(\mathfrak{O}x, \mathfrak{M}x)] \\ & +\gamma(d(\mathfrak{O}x, \mathfrak{O}y))d(\mathfrak{O}x, \mathfrak{O}y) \end{aligned}$$

for all x, $y \in \mathcal{X}$, where α, β, $\gamma : [0, +\infty) \to [0, 1)$ are non-decreasing functions which satisfying the following condition

$$4\alpha(t) + 2\beta(t) + \gamma(t) < 1 \ \forall t > 0.$$

If $\mathfrak{M}$ and $\mathfrak{O}$ are occasionally weakly $\mathfrak{O}$-biased of type (A), then $\mathfrak{M}$ and $\mathfrak{O}$ have a unique common fixed point.

Corollary 2.6 *Let $\mathfrak{M}$, $\mathfrak{N}$ and $\mathfrak{O}$ be self-maps of a complete d-metric space $\mathcal{X}$ satisfying*

$$\begin{aligned} d(\mathfrak{M}x, \mathfrak{N}y) \leq\ & \alpha(d(\mathfrak{O}x, \mathfrak{O}y))[d(\mathfrak{O}x, \mathfrak{N}y) + d(\mathfrak{O}y, \mathfrak{M}x)] \\ & +\beta(d(\mathfrak{O}x, \mathfrak{O}y))[d(\mathfrak{O}y, \mathfrak{N}y) + d(\mathfrak{O}x, \mathfrak{M}x)] \\ & +\gamma(d(\mathfrak{O}x, \mathfrak{O}y))d(\mathfrak{O}x, \mathfrak{O}y) \end{aligned}$$

for all x, $y \in \mathcal{X}$, where α, β, $\gamma : [0, +\infty) \to [0, 1)$ are non-decreasing functions which satisfying the following condition

$$4\alpha(t) + 2\beta(t) + \gamma(t) < 1 \ \forall t > 0.$$

If the pair $(\mathfrak{M}, \mathfrak{O})$ as well as $(\mathfrak{N}, \mathfrak{O})$ is occasionally weakly $\mathfrak{O}$-biased of type (A), then $\mathfrak{M}$, $\mathfrak{N}$ and $\mathfrak{O}$ have a unique common fixed point.

2.3. Illustrative examples

Example 2.1 *Let $\mathcal{X} = \mathbb{R}$ with the d-metric $d(x, y) = \max\{|x|, |y|\}$. Define*

$$\mathfrak{M}x = \begin{cases} 0 \ \text{if } x \in (-\infty, 0] \\ -\dfrac{1}{4} \ \text{if } x \in (0, +\infty), \end{cases} \quad \mathfrak{N}x = \begin{cases} 0 \ \text{if } x \in (-\infty, 0] \\ -\dfrac{1}{3} \ \text{if } x \in (0, +\infty), \end{cases}$$

$$\mathfrak{O}x = \begin{cases} -30x \ \text{if } x \in (-\infty, 0] \\ 40 \ \text{if } x \in (0, +\infty), \end{cases} \quad \mathfrak{P}x = \begin{cases} -50x \ \text{if } x \in (-\infty, 0] \\ 100 \ \text{if } x \in (0, +\infty). \end{cases}$$

First, it is clear to see that $\mathfrak{M}$ and $\mathfrak{O}$ are occasionally weakly $\mathfrak{O}$-biased of type (A) and $\mathfrak{N}$ and $\mathfrak{P}$ are occasionally weakly $\mathfrak{P}$-biased of type (A). Take $\alpha = \beta = \dfrac{1}{25}$ and $\gamma = \dfrac{3}{4}$, we get

1. *for x, $y \in (-\infty, 0]$, we have $\mathfrak{M}x = 0$, $\mathfrak{N}y = 0$, $\mathfrak{O}x = -30x$, $\mathfrak{P}y = -50y$ and*

$$\begin{aligned} d(\mathfrak{M}x, \mathfrak{N}y) &= 0 \\ &\leq \frac{1}{25}[-30x - 50y] + \frac{1}{25}[-50y - 30x] + \frac{3}{4} \times \max\{-30x, -50y\} \\ &= \alpha[d(\mathfrak{O}x, \mathfrak{N}y) + d(\mathfrak{P}y, \mathfrak{M}x)] + \beta[d(\mathfrak{P}y, \mathfrak{N}y) + d(\mathfrak{O}x, \mathfrak{M}x)] \\ &\quad +\gamma d(\mathfrak{O}x, \mathfrak{P}y), \end{aligned}$$

2. *for x, $y \in (0, +\infty)$, we have $\mathfrak{M}x = -\dfrac{1}{4}$, $\mathfrak{N}y = -\dfrac{1}{3}$, $\mathfrak{O}x = 40$, $\mathfrak{P}y = 100$ and*

$$\begin{aligned} d(\mathfrak{M}x, \mathfrak{N}y) &= \frac{1}{3} \\ &\leq \frac{1}{25}[40 + 100] + \frac{1}{25}[100 + 40] + \frac{3}{4} \times 100 \\ &= \frac{431}{5} \\ &= \alpha[d(\mathfrak{O}x, \mathfrak{N}y) + d(\mathfrak{P}y, \mathfrak{M}x)] + \beta[d(\mathfrak{P}y, \mathfrak{N}y) + d(\mathfrak{O}x, \mathfrak{M}x)] \\ &\quad +\gamma d(\mathfrak{O}x, \mathfrak{P}y), \end{aligned}$$

3. for $x \in (-\infty, 0]$, $y \in (0, +\infty)$, *we have* $\mathfrak{M}x = 0$, $\mathfrak{N}y = -\frac{1}{3}$, $\mathfrak{O}x = -30x$, $\mathfrak{P}y = 100$ *and*

$$\begin{aligned} d(\mathfrak{M}x, \mathfrak{N}y) &= \frac{1}{3} \\ &\leq \frac{1}{25}\left[\max\left\{-30x, \frac{1}{3}\right\} + 100\right] + \frac{1}{25}[100 - 30x] \\ &\quad + \frac{3}{4} \times \max\{-30x, 100\} \\ &= \alpha[d(\mathfrak{O}x, \mathfrak{N}y) + d(\mathfrak{P}y, \mathfrak{M}x)] + \beta[d(\mathfrak{P}y, \mathfrak{N}y) + d(\mathfrak{O}x, \mathfrak{M}x)] \\ &\quad + \gamma d(\mathfrak{O}x, \mathfrak{P}y), \end{aligned}$$

4. for $x \in (0, +\infty)$, $y \in (-\infty, 0]$, *we have* $\mathfrak{M}x = -\frac{1}{4}$, $\mathfrak{N}y = 0$, $\mathfrak{O}x = 40$, $\mathfrak{P}y = -50y$ *and*

$$\begin{aligned} d(\mathfrak{M}x, \mathfrak{N}y) &= \frac{1}{4} \\ &\leq \frac{1}{25}\left[40 + \max\left\{-50y, \frac{1}{4}\right\}\right] + \frac{1}{25}[-50y + 40] \\ &\quad + \frac{3}{4} \times \max\{40, -50y\} \\ &= \alpha[d(\mathfrak{O}x, \mathfrak{N}y) + d(\mathfrak{P}y, \mathfrak{M}x)] + \beta[d(\mathfrak{P}y, \mathfrak{N}y) + d(\mathfrak{O}x, \mathfrak{M}x)] \\ &\quad + \gamma d(\mathfrak{O}x, \mathfrak{P}y), \end{aligned}$$

so, all hypotheses of Theorem 2.1 are satisfied and 0 *is the unique common fixed point of maps* $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{O}$ *and* $\mathfrak{P}$.

Remark 2.1 *Note that* $\mathfrak{M}\mathcal{X} = \left\{0, -\frac{1}{4}\right\} \not\subseteq \mathfrak{P}\mathcal{X} = [0, +\infty)$ *and* $\mathfrak{N}\mathcal{X} = \left\{0, -\frac{1}{3}\right\} \not\subseteq \mathfrak{O}\mathcal{X} = [0, +\infty)$.

Example 2.2 *Let* $\mathcal{X} = \left[0, \frac{\pi}{2}\right]$ *with the d-metric* $d(x, y) = x + y$. *Define*

$$\mathfrak{M}x = \begin{cases} 0 \text{ if } x \in \left[0, \frac{\pi}{4}\right] \\ \frac{13\pi}{50} \text{ if } x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right] \end{cases}, \quad \mathfrak{N}x = \begin{cases} 0 \text{ if } x \in \left[0, \frac{\pi}{4}\right] \\ \frac{7\pi}{25} \text{ if } x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right] \end{cases},$$

$$\mathfrak{O}x = \begin{cases} x \text{ if } x \in \left[0, \frac{\pi}{4}\right] \\ \frac{\pi}{2} \text{ if } x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right] \end{cases}, \quad \mathfrak{P}x = \begin{cases} x \text{ if } x \in \left[0, \frac{\pi}{4}\right] \\ \frac{3\pi}{8} \text{ if } x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right] \end{cases}.$$

First, it is clear to see that $\mathfrak{M}$ *and* $\mathfrak{O}$ *are occasionally weakly* $\mathfrak{O}$*-biased of type* (A) *and* $\mathfrak{N}$ *and* $\mathfrak{P}$ *are occasionally weakly* $\mathfrak{P}$*-biased of type* (A). *Take* $\alpha(t) = \frac{\sin t}{25} = \beta(t)$ *and* $\gamma(t) = \frac{3}{4}$, *we get*

1. for $x, y \in \left[0, \frac{\pi}{4}\right]$, *we have* $\mathfrak{M}x = 0$, $\mathfrak{N}y = 0$, $\mathfrak{O}x = x$, $\mathfrak{P}y = y$ *and*

$$\begin{aligned} d(\mathfrak{M}x, \mathfrak{N}y) &= 0 \\ &\leq \frac{\sin(x+y)}{25}[x+y] + \frac{\sin(x+y)}{25}[y+x] \\ &\quad + \frac{3}{4} \times (x+y) \\ &= (x+y)\left[\frac{2}{25}\sin(x+y) + \frac{3}{4}\right] \\ &= \alpha(d(\mathfrak{O}x, \mathfrak{P}y))[d(\mathfrak{O}x, \mathfrak{N}y) + d(\mathfrak{P}y, \mathfrak{M}x)] \\ &\quad + \beta(d(\mathfrak{O}x, \mathfrak{P}y))[d(\mathfrak{P}y, \mathfrak{N}y) + d(\mathfrak{O}x, \mathfrak{M}x)] \\ &\quad + \gamma(d(\mathfrak{O}x, \mathfrak{P}y))d(\mathfrak{O}x, \mathfrak{P}y), \end{aligned}$$

2. *for* $x,\ y \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right]$, *we have* $\mathfrak{M}x = \frac{13\pi}{50}$, $\mathfrak{N}y = \frac{7\pi}{25}$, $\mathfrak{O}x = \frac{\pi}{2}$, $\mathfrak{P}y = \frac{3\pi}{8}$ *and*

$$\begin{aligned}
d(\mathfrak{M}x, \mathfrak{N}y) &= \frac{27\pi}{50} \\
&\leq \frac{\sin\left(\frac{7\pi}{8}\right)}{25}\left[\frac{283\pi}{200}\right] + \frac{\sin\left(\frac{7\pi}{8}\right)}{25}\left[\frac{283\pi}{200}\right] \\
&\quad + \frac{3}{4} \times \left(\frac{7\pi}{8}\right) \\
&= \frac{\sin\left(\frac{7\pi}{8}\right)}{25}\left[\frac{283\pi}{100}\right] + \frac{21\pi}{32} \\
&= \alpha(d(\mathfrak{O}x, \mathfrak{P}y))[d(\mathfrak{O}x, \mathfrak{N}y) + d(\mathfrak{P}y, \mathfrak{M}x)] \\
&\quad + \beta(d(\mathfrak{O}x, \mathfrak{P}y))[d(\mathfrak{P}y, \mathfrak{N}y) + d(\mathfrak{O}x, \mathfrak{M}x)] \\
&\quad + \gamma(d(\mathfrak{O}x, \mathfrak{P}y))d(\mathfrak{O}x, \mathfrak{P}y),
\end{aligned}$$

3. *for* $x \in \left[0, \frac{\pi}{4}\right]$, $y \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right]$ *we have* $\mathfrak{M}x = 0$, $\mathfrak{N}y = \frac{7\pi}{25}$, $\mathfrak{O}x = x$, $\mathfrak{P}y = \frac{3\pi}{8}$ *and*

$$\begin{aligned}
d(\mathfrak{M}x, \mathfrak{N}y) &= \frac{7\pi}{25} \\
&\leq \frac{\sin\left(x + \frac{3\pi}{8}\right)}{25}\left[x + \frac{131\pi}{200}\right] + \frac{\sin\left(x + \frac{3\pi}{8}\right)}{25}\left[x + \frac{131\pi}{200}\right] \\
&\quad + \frac{3}{4} \times \left(x + \frac{3\pi}{8}\right) \\
&= \frac{2}{25}\sin\left(x + \frac{3\pi}{8}\right)\left[x + \frac{131\pi}{200}\right] + \frac{3}{4} \times \left(x + \frac{3\pi}{8}\right) \\
&= \alpha(d(\mathfrak{O}x, \mathfrak{P}y))[d(\mathfrak{O}x, \mathfrak{N}y) + d(\mathfrak{P}y, \mathfrak{M}x)] \\
&\quad + \beta(d(\mathfrak{O}x, \mathfrak{P}y))[d(\mathfrak{P}y, \mathfrak{N}y) + d(\mathfrak{O}x, \mathfrak{M}x)] \\
&\quad + \gamma(d(\mathfrak{O}x, \mathfrak{P}y))d(\mathfrak{O}x, \mathfrak{P}y),
\end{aligned}$$

4. *for* $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right]$, $y \in \left[0, \frac{\pi}{4}\right]$, *we have* $\mathfrak{M}x = \frac{13\pi}{50}$, $\mathfrak{N}y = 0$, $\mathfrak{O}x = \frac{\pi}{2}$, $\mathfrak{P}y = y$ *and*

$$\begin{aligned}
d(\mathfrak{M}x, \mathfrak{N}y) &= \frac{13\pi}{50} \\
&\leq \frac{\sin\left(\frac{\pi}{2} + y\right)}{25}\left[\frac{19\pi}{25} + y\right] + \frac{\sin\left(\frac{\pi}{2} + y\right)}{25}\left[\frac{19\pi}{25} + y\right] \\
&\quad + \frac{3}{4} \times \left(\frac{\pi}{2} + y\right) \\
&= \frac{2}{25}\sin\left(\frac{\pi}{2} + y\right)\left[\frac{19\pi}{25} + y\right] + \frac{3}{4} \times \left(\frac{\pi}{2} + y\right) \\
&= \alpha(d(\mathfrak{O}x, \mathfrak{P}y))[d(\mathfrak{O}x, \mathfrak{N}y) + d(\mathfrak{P}y, \mathfrak{M}x)] \\
&\quad + \beta(d(\mathfrak{O}x, \mathfrak{P}y))[d(\mathfrak{P}y, \mathfrak{N}y) + d(\mathfrak{O}x, \mathfrak{M}x)] \\
&\quad + \gamma(d(\mathfrak{O}x, \mathfrak{P}y))d(\mathfrak{O}x, \mathfrak{P}y),
\end{aligned}$$

so, all hypotheses of Theorem 2.2 are satisfied and 0 *is the unique common fixed point of maps* $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{O}$ *and* $\mathfrak{P}$.

Remark 2.2 *Note that* $\mathfrak{M}\mathcal{X} = \left\{0, \frac{13\pi}{50}\right\} \nsubseteq \mathfrak{P}\mathcal{X} = \left[0, \frac{\pi}{4}\right] \cup \left\{\frac{3\pi}{8}\right\}$ *and* $\mathfrak{N}\mathcal{X} = \left\{0, \frac{7\pi}{25}\right\} \nsubseteq \mathfrak{O}\mathcal{X} = \left[0, \frac{\pi}{4}\right] \cup \left\{\frac{\pi}{2}\right\}$.

2.4. Application

Now, we furnish an application for solving the following integral equation:

$$\rho(l) = \int_0^1 \mathfrak{K}(l, q, \rho(q))dq, \tag{2.2}$$

where $\mathfrak{K} : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ is a nonnegative continuous function.

Let $\mathcal{X} = C[0,1]$ be the set of real nonnegative continuous functions defined on $[0,1]$. Take the d-metric $d : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$ defined by

$$d(\mu, \nu) = \max_{0 \le l \le 1} (|\mu(l)| + |\nu(l)|)$$

for all μ, $\nu \in \mathcal{X}$. Take the operator $\mathfrak{F}(\rho(l)) = \int_0^1 \mathfrak{K}(l, q, \rho(q))dq$ for all $\rho \in \mathcal{X}$ and for all $l \in [0,1]$. Mention that integral equation (2.2) has a unique solution if and only if operator $\mathfrak{F}$ has a unique fixed point. We present the following result.

Theorem 2.3 *Suppose that there is* $\gamma \in [0,1)$ *such that for all* l, $q \in [0,1]$ *and* ρ, $\varrho \in \mathcal{X}$, *we have*

$$|\mathfrak{K}(l, q, \rho(q))| + |\mathfrak{K}(l, q, \varrho(q))| \le \gamma(|\rho(q)| + |\varrho(q)|),$$

then $\mathfrak{F}$ *has a unique fixed point; i.e., integral equation (2.2) has a unique solution* $\rho \in \mathcal{X}$.

Proof: For all $l \in [0,1]$, we have

$$\begin{aligned}
|\mathfrak{F}(\rho(l))| + |\mathfrak{F}(\varrho(l))| &= \left|\int_0^1 \mathfrak{K}(l, q, \rho(q))dq\right| + \left|\int_0^1 \mathfrak{K}(l, q, \varrho(q))dq\right| \\
&\le \int_0^1 |\mathfrak{K}(l, q, \rho(q))|\, dq + \int_0^1 |\mathfrak{K}(l, q, \varrho(q))|\, dq \\
&= \int_0^1 (|\mathfrak{K}(l, q, \rho(q))| + |\mathfrak{K}(l, q, \varrho(q))|)dq \\
&\le \gamma \int_0^1 (|\rho(q)| + |\varrho(q)|)dq \\
&\le \gamma d(\rho(l), \varrho(l)) \\
&\le \alpha[d(\rho(l), \mathfrak{F}(\varrho(l))) + d(\varrho(l), \mathfrak{F}(\rho(l)))] \\
&\quad + \beta[d(\varrho(l), \mathfrak{F}(\varrho(l))) + d(\rho(l), \mathfrak{F}(\rho(l)))] + \gamma d(\rho(l), \varrho(l)),
\end{aligned}$$

therefore, all the conditions of Corollary 2.1 are satisfied. Hence, $\mathfrak{F}$ has a unique fixed point; i.e., integral equation (2.2) has a unique solution. □

3. Conclusion

We conclude this work by mentioning that our presented results extend and improve some existing results in fixed point literature among them, for a few, the main result of [8]. Again, we know that metric spaces yield partial metric spaces and partial metric spaces yield d-metric spaces, consequently, our theorems improve and/or extend many similar results on the three mentioned spaces.

References

1. Amini-Harandi, A., *Metric-like spaces, partial metric spaces and fixed points,* Fixed Point Theory Appl. 204, 1-10, (2012).
2. Aydi, H., Felhi, A., Karapinar, E. and Sahmim, S., *A Nadler-type fixed point theorem in dislocated spaces and applications,* Miskolc Math. Notes 19(1), 111-124, (2018).
3. Beloul, S., *A common fixed point theorem for generalized almost contractions in metric-like spaces,* Appl. Math. E-Notes 18, 127-139, (2018).
4. Bouhadjera, H., *Fixed points for occasionally weakly biased mappings of type* (A)*,* Math. Morav. 36(1), 113-122, (2022).
5. Jungck, G. F. and Rhoades, B. E., *General fixed point results in dislocated metric spaces,* Fixed Point Theory 18(2), 615-624, (2017).
6. Karapinar, E. and Salimi, P., *Dislocated metric space to metric spaces with some fixed point theorems,* Fixed Point Theory Appl. 222, 1-19, (2013).
7. Matthews, S. G., *Metric domains for completness,* Technical report 76, department of computer science, University of Warwick, UK, April 86. Ph.D. Thesis, 1985.
8. Panthi, D. and Jha, K., *A common fixed point of weakly compatible mappings in dislocated metric space,* Kathmandu University Journal of Science, Engineering and Technology 8(2), 25-30, (2012).
9. Pasicki, L., *Dislocated metric and fixed point theorems,* Fixed Point Theory Appl. 82, 1-14, (2015).
10. Ofoedu, E. U. and Odiliobi, C. S., *New fixed point theorems in dislocated metric spaces,* J. Nigerian Math. Soc. 38(3), 547-567, (2019).

Hakima Bouhadjera,
Laboratory of Applied Mathematics,
Badji Mokhtar-Annaba University,
P.O. Box 12, 23000 Annaba,
Algeria.
E-mail address: b_hakima2000@yahoo.fr

and

Iqbal H. Jebril,
Faculty of Science and Information Technology,
Al-Zaytoonah University of Jordan,
P.O. Box 130 Amman 11733, Jordan.
E-mail address: i.jebril@zuj.edu.jo