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Existence and stability for a boundary value problem of Ambartsumian equation with Ξ-Hilfer generalized proportional fractional derivative

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ABSTRACT: In this paper, we prove the existence and uniqueness of solution for the mixed boundary value problem of Ambartsumian equation using Ξ-Hilfer generalized proportional fractional derivative(PFD). The main principles applied to investigate our results are based on the standard fixed point theorems. We do well in detail on some results concerning the Hyers-Ulam type stability. We verify our result with an illustrative example.

Key Words: Ambartsumian equation,Ξ-Hilfer generalized proportional fractional derivative, Mixed boundary value conditions, Existence, Uniqueness, Stability.

Contents

1. Introduction

Fractional differential equations (FDEs) are being used in various fields of science and engineering such as control system, electrochemistry, electromagnetics, viscoelasticity, physics, biophysics, porous media, blood flow phenomena, electrical circuits, biology, fitting of experimental data etc. Due to these features, models of fractional order become more practical and realistic than the models of integer-order. Recently, non-singular fractional operators have a significant role in the modelling of real-world problems. For more details, see [1,10,15].

The existence and uniqueness (EU) of solutions belong to the most important qualitative properties of FDEs. The EU of solutions to FDEs that include different types of fractional derivatives and initial/boundary conditions were tackled by several mathematicians, see [2,18,19,23]. Initially, the stability concept was developed by Ulam and Hyers-Ulam in [9,24]. In [7,25], the authors consider the stability of FDEs.

Motivated by the Hilfer and the Hilfer–Katugampola fractional derivative [13,14], the authors in [22] introduced a new Hilfer generalized PFD, which unifies the Riemann–Liouville (RL) and Caputo generalized PFD. More details about this derivative are given in [3,11,12,17,21].

Ambartsumian derived the standard Ambartsumian equation (AE). The absorption of light by interstellar matter has been defined in this equation. In the theory of surface brightness in the Milky Way, the Ambartsumian delay equation is used. The authors in [4,5,6,16,20] discussed about AE.

In this work, motivated by the research going on in this direction, we study a new class of boundary value problems of AE on Ξ-Hilfer generalized PFD with some mixed nonlocal boundary conditions of the form,

$$\begin{cases} \mathcal{D}_{0^+}^{p,q,\varrho,\Xi}\mathcal{A}(t) = \mathbb{Q}\left(t, \mathcal{A}\left(t\right), \mathcal{A}\left(\frac{t}{\eta}\right)\right), \quad t\in[0,T], \quad \eta>1, \\ \mathcal{A}(0)=0, \quad \sum_{i=1}^{m}\delta_i\mathcal{A}(\alpha_i)+\sum_{j=1}^{n}\omega_j\mathcal{I}_{0^+}^{\beta_j,\varrho,\Xi}\mathcal{A}(\theta_j)+\sum_{k=1}^{r}\lambda_k\mathcal{D}_{0^+}^{\mu_k,p,\varrho;\Xi}\mathcal{A}(\zeta_k)=\varpi, \end{cases} \tag{1.1}$$

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where, $\mathbb{Q}\left(t,\mathcal{A}\left(t\right),\mathcal{A}\left(\frac{t}{\eta}\right)\right)=\frac{1}{\eta}\mathcal{A}\left(\frac{t}{\eta}\right)-\mathcal{A}(t),\mathcal{D}_{a^+}^{u,q,\varrho,\Xi}$ is the Ξ-Hilfer generalized PFD of order $u=\{p,\mu_k\}$ with $1<\mu_k<p\leq 2$, $0\leq\varrho\leq 1,\mathcal{I}_{0^+}^{\beta_j,\varrho,\Xi}$ is Ξ-RL fractional integral with $\beta_j>0$, for $j=1,2,3...,n,\varpi,\delta_i,\omega_j,\lambda_k\in\mathbb{R}$ are given constants, the points $\alpha_i,\theta_j,\zeta_k\in[0,T]=J,i=1,2,3...,m,\quad j=1,2,3,...n,\quad k=1,2,3,...,r$ and $\mathbb{Q}:J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous function, and $J=[0,T],T>0$.

2. Auxillary Results

In this section, we investigate some notation, spaces, definitions and fundamental lemmas which are useful to the entire paper.

Let $C=C(J,\mathbb{R})$ denote the Banach space for all continuous functions from J into $\mathbb{R}$ with the norm defined by

$$\|\mathbb{Q}\|=\sup_{t\in J}\{|\mathbb{Q}(t)|\}.$$

On the other hand, we have n-times absolutely continuous functions given by

$$AC^n(J,\mathbb{R})=\left\{\mathbb{Q}:J\to\mathbb{R};\mathbb{Q}^{(n-1)}\in AC(J,\mathbb{R})\right\}.$$

Definition 2.1 *[17] If $\varrho\in(0,1]$ and $p\in\mathbb{R}^+$. Then the left-sided generalized proportional fractional integral of order p of the function $\mathbb{Q}$ with respect to the another function Ξ is defined by*

$$\left(\mathcal{I}_{0^+}^{p,\varrho,\Xi}\mathbb{Q}\right)(t)=\frac{1}{\varrho^p\Gamma(p)}\int_0^t e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(s))}(\Xi(t)-\Xi(s))^{p-1}\Xi^{'}(s)\mathbb{Q}(s)ds,\quad t>0. \tag{2.1}$$

Definition 2.2 *[17] If $\varrho\in(0,1]$ and $p\in\mathbb{R}^+$, and $\Xi\in C(J,\mathbb{R})$ where $\Xi^{'}(s)>0$. Then the left-sided generalized PFD of order p of the function $\mathbb{Q}$ with respect to the another function Ξ is defined by,*

$$\left(\mathcal{D}_{0^+}^{p,\varrho,\Xi}\mathbb{Q}\right)(t)=\frac{\mathcal{D}_t^{n,\varrho,\Xi}}{\varrho^{n-p}\Gamma(n-p)}\int_0^t e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(s))}(\Xi(t)-\Xi(s))^{n-p-1}\Xi^{'}(s)\mathbb{Q}(s)ds,\quad t>0,$$

where $\Gamma(.)$ is the gamma function and $n=[p]+1$, where $[p]$ denotes the integer part of the real number p.

Proposition 2.1 *[17] If $p\geq 0$ and $q>0$, then for any $\varrho>0$, we have,*

(i) $\left(\mathcal{I}_{0^+}^{p,\varrho,\Xi}e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(0))}(\Xi(t)-\Xi(0))^{q-1}\right)(t)=\frac{\Gamma(q)}{\varrho^p\Gamma(p+q)}e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(0))}(\Xi(t)-\Xi(0))^{p+q-1}$,

(ii) $\left(\mathcal{D}_{0^+}^{p,\varrho,\Xi}e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(0))}(\Xi(t)-\Xi(0))^{q-1}\right)(t)=\frac{\varrho^p\Gamma(q)}{\Gamma(q-p)}e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(0))}(\Xi(t)-\Xi(0))^{q-p-1}$.

Theorem 2.1 *[17] Suppose $\varrho\in(0,1],p>0$ and $q>0$. Then, if $\mathbb{Q}$ is continuous and defined for $t\geq 0$, we have*

$$\mathcal{I}_{0^+}^{p,\varrho,\Xi}\left(\mathcal{I}_{0^+}^{q,\varrho,\Xi}\mathbb{Q}\right)(t)=\mathcal{I}_{0^+}^{q,\varrho,\Xi}\left(\mathcal{I}_{0^+}^{p,\varrho,\Xi}\mathbb{Q}\right)(t)=\left(\mathcal{I}_{0^+}^{p+q,\varrho,\Xi}\mathbb{Q}\right)(t).$$

Theorem 2.2 *[17] Suppose $\varrho\in(0,1],0\leq n<[p]+1$ with $n\in\mathbb{N}$. If $\mathbb{Q}\in L_1(J,\mathbb{R})$, then*

$$\mathcal{D}_{0^+}^{n,\varrho,\Xi}\left(\mathcal{I}_{0^+}^{p,\varrho,\Xi}\mathbb{Q}\right)(t)=\left(\mathcal{I}_{0^+}^{p-n,\varrho,\Xi}\mathbb{Q}\right)(t). \tag{2.2}$$

In particular, for n=1, hence by using the Leibnitz rule, we have

$$\mathcal{D}_{0^+}^{1,\varrho,\Xi}\left(\mathcal{I}_{0^+}^{p,\varrho,\Xi}\mathbb{Q}\right)(t)=\frac{p-1}{\varrho^{p-1}\Gamma(p)}\int_0^t e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(s))}(\Xi(t)-\Xi(s))^{p-2}\Xi^{'}(s)\mathbb{Q}(s)ds. \tag{2.3}$$

Corollary 2.1 *[17] If $\varrho\in(0,1]$,$0\leq n<[p]<1$ with $n\in\mathbb{N}$ and $\mathbb{Q}\in L^1(J,\mathbb{R})$. Then we have*

$$\mathcal{D}_{0^+}^{q,\varrho,\Xi}\left(\mathcal{I}_{0^+}^{p,\varrho,\Xi}\mathbb{Q}\right)(t)=\left(\mathcal{I}_{0^+}^{p-q,\varrho,\Xi}\mathbb{Q}\right)(t). \tag{2.4}$$

Definition 2.3 *[17] Let J be an interval and $\mathbb{Q},\Xi \in C^n(J,\mathbb{R})$ be two functions such that Ξ is positive, strictly increasing and $\Xi^{'}(t) \neq 0, \forall t \in J$. The Ξ-Hilfer generalized PFD of order p and type q of $\mathbb{Q}$ with respect to the another function Ξ are defined by*

$$\left(\mathcal{D}_{0^+}^{p,q,\varrho,\Xi}\mathbb{Q}\right)(t) = \left(\mathcal{I}_{0^+}^{q(n-p),\varrho,\Xi}\left(\mathcal{D}^{n,\varrho,\Xi}\right)\mathcal{I}_{0^+}^{(1-q)(n-p),\varrho,\Xi}\mathbb{Q}\right)(t), \tag{2.5}$$

where $n-1 < p < n, 0 \leq q \leq 1$ with $n \in \mathbb{N}$ and $\varrho \in (0,1]$. Also $\mathcal{D}^{\varrho,\Xi}\mathbb{Q}(t) = (1-\varrho)\mathbb{Q}(t) + \varrho\frac{\mathbb{Q}^{'}(t)}{\Xi^{'}(t)}$ and $\mathcal{I}_{0^+}$ is the generalized proportional fractional integral defined in Eq.(2.1).
In particular, if $n=1$, then $0 < p < 1$ and $0 \leq q \leq 1$, so Eq.(2.5) becomes,

$$\left(\mathcal{D}_{0^+}^{p,q,\varrho,\Xi}\mathbb{Q}\right)(t) = \left(\mathcal{I}_{0^+}^{q(1-p),\varrho,\Xi}\left(\mathcal{D}^{1,\varrho,\Xi}\right)\mathcal{D}_{0^+}^{(1-q)(1-p),\varrho,\Xi}\mathbb{Q}\right)(t).$$

Remark 2.1 *[17] From the Definition 2.3, we can view the operator $\mathcal{D}_{0^+}^{p,q,\varrho,\Xi}$ as an interpolate between the RL and Caputo generalized PFDs respectively, since*

$$\mathcal{D}_{0^+}^{p,q,\varrho,\Xi}\mathbb{Q} = \begin{cases}\mathcal{D}^{n,\varrho,\Xi}\mathcal{I}_{0^+}^{(n-p),\varrho,\Xi}\mathbb{Q}, & \text{if} \quad q=0,\\ \mathcal{I}_{0^+}^{q(n-p),\varrho,\Xi}\mathcal{D}^{n,\varrho,\Xi}\mathbb{Q}, & \text{if} \quad q=1.\end{cases}$$

Property 2.3 *[17] The Ξ-Hilfer generalized PFD $\mathcal{D}_{0^+}^{p,q,\varrho,\Xi}\mathbb{Q}$ is equivalent to*

$$\left(\mathcal{D}_{0^+}^{p,q,\varrho,\Xi}\mathbb{Q}\right)(t) = \left(\mathcal{I}_{0^+}^{q(n-p),\varrho,\Xi}\left(\mathcal{D}^{n,\varrho,\Xi}\right)\mathcal{I}_{0^+}^{(1-q)(n-p),\varrho,\Xi}\mathbb{Q}\right)(t) = \left(\mathcal{I}_{0^+}^{q(n-p),\varrho,\Xi}\mathcal{D}^{\vartheta,\varrho,\Xi}\mathbb{Q}\right)(t),$$

where, $\vartheta = p + q(n-p)$.

Property 2.4 *[17] Assume that p,q,ϑ satisfying the relations as*

$$\vartheta = p + q(n-p), n-1 < p, \vartheta \leq n, 0 \leq q \leq 1,$$

and

$$\vartheta \geq p, \vartheta > q, n - \vartheta < n - q(n-p).$$

Proposition 2.2 *[17] If $p,q \in \mathbb{R}$ with $p > 0, q > 0$ then for any $\varrho > 0$ and $n = [p]+1$, we obtained,*

$$\left({}^{C}\mathcal{D}_{0^+}^{p,\varrho,\Xi}e^{\frac{\varrho-1}{\varrho}(\Xi(s)-\Xi(0))}(\Xi(s)-\Xi(0))^{q-1}\right)(t) = \frac{\varrho^p\Gamma(q)}{\Gamma(q-p)}e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(0))}(\Xi(t)-\Xi(0))^{q-p-1}.$$

For k=0,1,2,........,n-1, we have

$$\left({}^{C}\mathcal{D}_{0^+}^{p,\varrho,\Xi}e^{\frac{\varrho-1}{\varrho}(\Xi(s)-\Xi(0))}(\Xi(s)-\Xi(0))^{k}\right)(t) = 0.$$

In particular, $\left({}^{C}\mathcal{D}_{0^+}^{p,\varrho,\Xi}e^{\frac{\varrho-1}{\varrho}(\Xi(s)-\Xi(0))}\right)(t) = 0$.

Lemma 2.1 *[11] Let $n-1 < p < n$ with $n \in \mathbb{N}, 0 \leq q \leq 1, \varrho \in (0,1]$ with $\vartheta = p + q(n-p)$ be such that $n-1 < \vartheta < n$. If $\mathbb{Q} \in C[J,\mathbb{R}]$, and $\mathcal{I}_{0^+}^{n-\vartheta,\varrho,\Xi}\mathbb{Q} \in C^n[J,\mathbb{R}]$, then*

$$\left(\mathcal{I}_{0^+}^{p,\varrho,\Xi}\mathcal{D}_{0^+}^{p,q,\varrho,\Xi}\mathbb{Q}\right)(t) = \mathbb{Q}(t) - \sum_{k=0}^{n}\frac{e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(0))}(\Xi(t)-\Xi(0))^{\vartheta-k}}{\varrho^{\vartheta-k}\Gamma(\vartheta-k+1)}\left(\mathcal{I}_{0^+}^{k-\vartheta,\varrho,\Xi}\mathbb{Q}\right)(0). \tag{2.6}$$

Lemma 2.2 *[8](Banach contraction principle) Let D be a non-empty closed subset of a Banach space E. Then any contraction mapping T from D into itself has a unique fixed point.*

Theorem 2.5 *[8](Krasnoselskii's fixed point theorem) Let B be a non empty bounded, closed, convex subset of a Banach space X. Let $N, M : B \to X$ be two continuos operators satisfying:*

- *$Nx + My \in B$ whenever $x, y \in B$,*
- *N is compact and continuous,*
- *M is contraction mapping.*

Then, there exist $u \in B$ such that $u = Nu + Mu$.

Now we have to show that the equivalence relation between the mixed boundary value problem (1.1) and the Volterra integral equation.

Lemma 2.3 *Let $\mathbb{Q} \in C(J, \mathbb{R}), 1 \leq \mu_k < p \leq 2, 0 \leq q \leq 1, \varrho \in (0,1], \vartheta = p + q(2-p), i = 1, 2, ..., m, \quad j = 1, 2, ..., n$ $k = 1, 2, 3..., r$ and $m, n, r \neq 0$. Suppose that $\mathbb{Q} \in C$. Then $\mathcal{A} \in C^2$ is a solution of the problem (1.1) if and only if $\mathcal{A}$ satisfies the integral equation*

$$\begin{aligned}\mathcal{A}(t) = &\mathcal{I}_{0^+}^{p,\varrho,\Xi}\mathbb{Q}\left(t, \mathcal{A}\left(t\right), \mathcal{A}\left(\frac{t}{\eta}\right)\right) + \frac{\wedge}{\varrho^{\vartheta-1}\Gamma(\vartheta)} e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(0))} \left(\Xi(t) - \Xi(0)\right)^{\vartheta-1} \\ &\times \left(k - \sum_{i=1}^{m} \delta_i \mathcal{I}_{0^+}^{p,\varrho,\Xi}\mathbb{Q}\left(\alpha_i, \mathcal{A}\left(\alpha_i\right), \mathcal{A}\left(\frac{\alpha_i}{\eta}\right)\right) + \sum_{j=1}^{n} \omega_j \mathcal{I}_{0^+}^{p+\beta_j;\varrho,\Xi}\mathbb{Q}\left(\theta_j, \mathcal{A}\left(\theta_j\right), \mathcal{A}\left(\frac{\theta_j}{\eta}\right)\right)\right. \\ &\left. + \sum_{k=1}^{r} \lambda_k \mathcal{I}_{0^+}^{p-\mu_k,\varrho;\Xi}\mathbb{Q}\left(\zeta_k, \mathcal{A}\left(\zeta_k\right), \mathcal{A}\left(\frac{\zeta_k}{\eta}\right)\right)\right), \quad t \in J,\end{aligned} \tag{2.7}$$

where,

$$\begin{aligned}\frac{1}{\wedge} = &\sum_{i=1}^{m} \delta_i \frac{e^{\frac{\varrho-1}{\varrho}(\Xi(\alpha_i)-\Xi(0))} \left(\Xi(\alpha_i) - \Xi(0)\right)^{\vartheta-1}}{\varrho^{\vartheta-1}\Gamma(\vartheta)} + \sum_{j=1}^{n} \omega_j \frac{e^{\frac{\varrho-1}{\varrho}(\Xi(\theta_j)-\Xi(0))} \left(\Xi(\theta_j) - \Xi(0)\right)^{\vartheta+\beta_j-1}}{\varrho^{\vartheta+\beta_j-1}\Gamma(\vartheta+\beta_j)} \\ &+ \sum_{k=1}^{r} \lambda_k \frac{e^{\frac{\varrho-1}{\varrho}(\Xi(\zeta_k)-\Xi(0))} \left(\Xi(\zeta_k) - \Xi(0)\right)^{\vartheta-\mu_k-1}}{\varrho^{\vartheta-\mu_k-1}\Gamma(\vartheta-\mu_k)}\end{aligned} \tag{2.8}$$

Proof: Let $\mathcal{A} \in C$ be a solution of the problem (1.1). By using Lemma 2.1, we have

$$\begin{aligned}\mathcal{A}(t) = &\mathcal{I}_{0^+}^{p,\varrho,\Xi}\mathbb{Q}\left(t, \mathcal{A}\left(t\right), \mathcal{A}\left(\frac{t}{\eta}\right)\right) + \frac{e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(0))} \left(\Xi(t) - \Xi(0)\right)^{\vartheta-1}}{\varrho^{\vartheta-1}\Gamma(\vartheta)} c_1 \\ &+ \frac{e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(0))} \left(\Xi(t) - \Xi(0)\right)^{\vartheta-2}}{\varrho^{\vartheta-1}\Gamma(\vartheta)} c_2,\end{aligned} \tag{2.9}$$

where, $c_1, c_2 \in \mathbb{R}$ are arbitrary constants.
For $t = 0$, we get $c_2 = 0$ and thus

$$\mathcal{A}(t) = \mathcal{I}_{0^+}^{p,\varrho,\Xi}\mathbb{Q}\left(t, \mathcal{A}\left(t\right), \mathcal{A}\left(\frac{t}{\eta}\right)\right) + \frac{e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(0))} \left(\Xi(t) - \Xi(0)\right)^{\vartheta-1}}{\varrho^{\vartheta-1}\Gamma(\vartheta)} c_1. \tag{2.10}$$

Taking the operators $\mathcal{D}_{0^+}^{\mu_k,p,\varrho;\Xi}$ and $\mathcal{I}_{0^+}^{\beta_j,\varrho,\Xi}$ into (3.1), we obtain

$$\begin{aligned}\mathcal{D}_{0^+}^{\mu_k,p,\varrho;\Xi}\mathcal{A}(t) = &\mathcal{I}_{0^+}^{p-\mu_k,\varrho,\Xi}\mathbb{Q}\left(t, \mathcal{A}\left(t\right), \mathcal{A}\left(\frac{t}{\eta}\right)\right) + \frac{e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(0))} \left(\Xi(t) - \Xi(0)\right)^{\vartheta-\mu_k-1}}{\varrho^{\vartheta-\mu_k}\Gamma(\vartheta-\mu_k)} c_1, \\ \mathcal{I}_{0^+}^{\beta_j,\varrho,\Xi}\mathcal{A}(t) = &\mathcal{I}_{0^+}^{p+\beta_j,\varrho,\Xi}\mathbb{Q}\left(t, \mathcal{A}\left(t\right), \mathcal{A}\left(\frac{t}{\eta}\right)\right) + \frac{e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(0))} \left(\Xi(t) - \Xi(0)\right)^{\vartheta+\beta_j-1}}{\varrho^{\vartheta+\beta_j}\Gamma(\vartheta+\beta_j)} c_1.\end{aligned}$$

Applying the second boundary condition in (1.1), we have

$$c_1\left[\sum_{i=1}^{m}\delta_i\frac{e^{\frac{\varrho-1}{\varrho}(\Xi(\alpha_i)-\Xi(0))}\left(\Xi(\alpha_i)-\Xi(0)\right)^{\vartheta-1}}{\varrho^{\vartheta}\Gamma(\vartheta)}+\sum_{j=1}^{n}\omega_j\frac{e^{\frac{\varrho-1}{\varrho}(\Xi(\theta_j)-\Xi(0))}\left(\Xi(\theta_j)-\Xi(0)\right)^{\vartheta+\beta_j-1}}{\varrho^{\vartheta+\beta_j}\Gamma(\vartheta+\beta_j)}\right.$$
$$\left.+\sum_{k=1}^{r}\lambda_k\frac{e^{\frac{\varrho-1}{\varrho}(\Xi(\zeta_k)-\Xi(0))}\left(\Xi(\zeta_k)-\Xi(0)\right)^{\vartheta-\mu_k-1}}{\varrho^{\vartheta-\mu_k}\Gamma(\vartheta-\mu_k)}\right]+\left(\sum_{i=1}^{m}\delta_i\mathcal{I}_{0^+}^{p,\varrho,\Xi}\mathbb{Q}\left(\alpha_i,\mathcal{A}\left(\alpha_i\right),\mathcal{A}\left(\frac{\alpha_i}{\eta}\right)\right)\right.$$
$$\left.+\sum_{j=1}^{n}\omega_j\mathcal{I}_{0^+}^{p+\beta_j;\varrho,\Xi}\mathbb{Q}\left(\theta_j,\mathcal{A}\left(\theta_j\right),\mathcal{A}\left(\frac{\theta_j}{\eta}\right)\right)+\sum_{k=1}^{r}\lambda_k\mathcal{I}_{0^+}^{p-\mu_k,\varrho;\Xi}\mathbb{Q}\left(\zeta_k,\mathcal{A}\left(\zeta_k\right),\mathcal{A}\left(\frac{\zeta_k}{\eta}\right)\right)\right)=\varpi,$$

which gives

$$c_1=\wedge\left[\varpi-\left(\sum_{i=1}^{m}\delta_i\mathcal{I}_{0^+}^{p,\varrho,\Xi}\mathbb{Q}\left(\alpha_i,\mathcal{A}\left(\alpha_i\right),\mathcal{A}\left(\frac{\alpha_i}{\eta}\right)\right)\right.\right.$$
$$\left.\left.+\sum_{j=1}^{n}\omega_j\mathcal{I}_{0^+}^{p+\beta_j;\varrho,\Xi}\mathbb{Q}\left(\theta_j,\mathcal{A}\left(\theta_j\right),\mathcal{A}\left(\frac{\theta_j}{\eta}\right)\right)+\sum_{k=1}^{r}\lambda_k\mathcal{I}_{0^+}^{p-\mu_k,\varrho;\Xi}\mathbb{Q}\left(\zeta_k,\mathcal{A}\left(\zeta_k\right),\mathcal{A}\left(\frac{\zeta_k}{\eta}\right)\right)\right)\right],$$

where $\frac{1}{\wedge}$ defined by Eq.(2.8). By the substitution of the value of c_1 in Eq.(2.10), we get Eq.(2.7).

Conversely, it is easily to shown, by a direct calculation, that the solution $\mathcal{A}$ given by Eq.(2.7) satisfies the problem (1.1). Hence the Lemma is proved. □

3. Existence and Uniqueness

Now, we present the existence result to the considered problem (1.1) by using Theorem 2.5. For the sake of convenience, we use the following notations:

$$x(\chi,p)=\frac{e^{\frac{\varrho-1}{\varrho}(\Xi(\chi)-\Xi(0))}\left(\Xi(\chi)-\Xi(0)\right)^{p}}{\varrho^{p+1}\Gamma(p+1)},\tag{3.1}$$

$$\Omega_1=\begin{cases}x(T,p)+x(T,\vartheta-1)\left|\wedge\right|\left(\sum_{i=1}^{m}\left|\delta_i\right|x(\alpha_i,p)\right.\\ \left.+\sum_{j=1}^{n}\left|\omega_j\right|x(\theta_j,p+\beta_j)+\sum_{k=1}^{r}\left|\lambda_k\right|x(\zeta_k,p-\mu_k)\right).\end{cases}\tag{3.2}$$

In view of Lemma 2.3, an operator $\mathcal{T}:C\to C$ is defined by

$$\mathcal{T}\mathcal{A}(t)=\begin{cases}\mathcal{I}_{0^+}^{p,\varrho,\Xi}\mathbb{Q}\left(t,\mathcal{A}\left(t\right),\mathcal{A}\left(\frac{t}{\eta}\right)\right)+\wedge x(t,\vartheta-1)\\ \times\left[\varpi-\left(\sum_{i=1}^{m}\delta_i\mathcal{I}_{0^+}^{p+j,\varrho,\Xi}\mathbb{Q}\left(\alpha_i,\mathcal{A}\left(\alpha_i\right),\mathcal{A}\left(\frac{\alpha_i}{\eta}\right)\right)\right.\right.\\ +\sum_{j=1}^{n}\omega_j\mathcal{I}_{0^+}^{p+\beta_j;\varrho,\Xi}\mathbb{Q}\left(\theta_j,\mathcal{A}\left(\theta_j\right),\mathcal{A}\left(\frac{\theta_j}{\eta}\right)\right)\\ \left.\left.+\sum_{k=1}^{r}\lambda_k\mathcal{I}_{0^+}^{p-\mu_k,\varrho;p,\Xi}\mathbb{Q}\left(\zeta_k,\mathcal{A}\left(\zeta_k\right),\mathcal{A}\left(\frac{\zeta_k}{\eta}\right)\right)\right)\right],t\in J.\end{cases}\tag{3.3}$$

Theorem 3.1 *Assume that $\mathbb{Q}:J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous function such that:*
$(H_1):$ *There exist a constant $L_1>0$ such that*

$$\left|\mathbb{Q}\left(t,u_1,v_1\right)-\mathbb{Q}\left(t,u_1,v_1\right)\right|\leq L_1\left(\left|u_1-u_2\right|+\left|v_1-v_2\right|\right),\tag{3.4}$$

for any $u_i,v_i\in\mathbb{R},\quad i=1,2$ and $t\in J$.
If

$$\Omega_1L_1<1.\tag{3.5}$$

Then, the problem (1.1) has a unique solution on J.

Proof: Firstly, we transform the problem (1.1) into a fixed point problem $\mathcal{A} = \mathcal{T}\mathcal{A}$, where the operator $\mathcal{T}$ is defined in Eq.(3.3). Applying the Banach contraction mapping principle, we shall show that the operator $\mathcal{T}$ has a unique fixed point, which is the unique solution of the problem (1.1).

Let $\sup_{t\in J} |\mathbb{Q}(t,0,0)| := M_1 < \infty$. Next we set $B_{r_1} := \{\mathcal{A} \in C : \|\mathcal{A}\| \leq r_1\}$ with

$$r_1 \geq \frac{\Omega_1 M_1 + (|\varpi| \, |\wedge| \, x(T,\vartheta-1))}{1-\Omega_1 L_1}, \tag{3.6}$$

where $\frac{1}{\wedge}, x(T,\vartheta-1), \Omega_1$ are given by (2.8),(3.1) and (3.2) respectively.
Observe that B_{r_1} is a bounded, closed and convex subset of C. The proof is divided into two steps:
Step:1 We show that $\mathcal{T}B_{r_1} \subset B_{r_1}$.
For any $\mathcal{A} \in B_{r_1}$, we have

$$\begin{aligned}
|\mathcal{T}\mathcal{A}(t)| \leq & \mathcal{I}_{0^+}^{p,\varrho,\Xi} \left|\mathbb{Q}\left(T, \mathcal{A}(T), \mathcal{A}\left(\frac{T}{\eta}\right)\right)\right| + |\wedge| \, x(T,\vartheta-1)\left(|\varpi|\right. \\
& + \sum_{i=1}^{m} |\delta_i| \, \mathcal{I}_{0^+}^{p,\varrho,\Xi} \left|\mathbb{Q}\left(\alpha_i, \mathcal{A}(\alpha_i), \mathcal{A}\left(\frac{\alpha_i}{\eta}\right)\right)\right| \\
& + \sum_{j=1}^{n} |\omega_j| \, \mathcal{I}_{0^+}^{p+\beta_j;\varrho,\Xi} \left|\mathbb{Q}\left(\theta_j, \mathcal{A}(\theta_j), \mathcal{A}\left(\frac{\theta_j}{\eta}\right)\right)\right| \\
& \left. + \sum_{k=1}^{r} |\lambda_k| \, \mathcal{I}_{0^+}^{p-\mu_k,\varrho;p,\Xi} \left|\mathbb{Q}\left(\zeta_k, \mathcal{A}(\zeta_k), \mathcal{A}\left(\frac{\zeta_k}{\eta}\right)\right)\right|\right).
\end{aligned}$$

It follows from the condition (H_1) that

$$\begin{aligned}
\left|\mathbb{Q}\left(t, \mathcal{A}(t), \mathcal{A}\left(\frac{t}{\eta}\right)\right)\right| \leq & \left|\mathbb{Q}\left(t, \mathcal{A}(t), \mathcal{A}\left(\frac{t}{\eta}\right)\right) - \mathbb{Q}(t,0,0)\right| + |\mathbb{Q}(t,0,0)| \\
\leq & L_1 \left\{ l_{\mathbb{Q}} \, |\mathcal{A}(t)| + c_{\mathbb{Q}} \left|\mathcal{A}\left(\frac{t}{\eta}\right)\right| \right\} + M_1
\end{aligned}$$

Then we have

$$\begin{aligned}
|\mathcal{T}\mathcal{A}(t)| \leq & \left(L_1 \left\{ l_{\mathbb{Q}} \, |\mathcal{A}(t)| + c_{\mathbb{Q}} \left|\mathcal{A}\left(\frac{t}{\eta}\right)\right| \right\} + M_1\right) \frac{e^{\frac{\varrho-1}{\varrho}(\Xi(T)-\Xi(0))} \left(\Xi(T)-\Xi(0)\right)^{\phi}}{\varrho^{\phi+1}\Gamma(\phi+1)} \\
& + |\wedge| \, x(T,\vartheta-1) \left[|\varpi| + \left(L_1 \left\{ l_{\mathbb{Q}} \, |\mathcal{A}(t)| + c_{\mathbb{Q}} \left|\mathcal{A}\left(\frac{t}{\eta}\right)\right| \right\} + M_1\right)\right. \\
& \left(\sum_{i=1}^{m} \delta_i \frac{e^{\frac{\varrho-1}{\varrho}(\Xi(\alpha_i)-\Xi(0))} \left(\Xi(\alpha_i)-\Xi(0)\right)^{\vartheta}}{\varrho^{\vartheta+1}\Gamma(\vartheta+1)}\right. \\
& + \sum_{j=1}^{n} \omega_j \frac{e^{\frac{\varrho-1}{\varrho}(\Xi(\theta_j)-\Xi(0))} \left(\Xi(\theta_j)-\Xi(0)\right)^{\vartheta+\beta_j}}{\varrho^{\vartheta+\beta_j+1}\Gamma(\vartheta+\beta_j+1)} \\
& \left.\left. + \sum_{k=1}^{r} \lambda_k \frac{e^{\frac{\varrho-1}{\varrho}(\Xi(\zeta_k)-\Xi(0))} \left(\Xi(\zeta_k)-\Xi(0)\right)^{\vartheta-\mu_k}}{\varrho^{\vartheta-\mu_k+1}\Gamma(\vartheta-\mu_k+1)}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
|\mathcal{T}\mathcal{A}(t)| \leq & L_1 \left\{ l_{\mathbb{Q}} |\mathcal{A}(t)| + c_{\mathbb{Q}} \left| \mathcal{A}\left(\frac{t}{\eta}\right) \right| \right\} \left\{ x(T,p) + |\wedge| \, x(T,\vartheta-1) + \left(\sum_{i=1}^{m} |\delta_i| \, x(\alpha_i,p) \right.\right. \\
& \left.\left. + \sum_{j=1}^{n} |\omega_j| \, x(\theta_j, p+\beta_j) + \sum_{k=1}^{r} |\lambda_k| \, x(\zeta_k, p-\mu_k) \right) \right\} + \{ x(T,p) + |\wedge| \, x(T,\vartheta-1) \\
& + \left. \left(\sum_{i=1}^{m} |\delta_i| \, x(\alpha_i,p) + \sum_{j=1}^{n} |\omega_j| \, x(\theta_j, p+\beta_j) + \sum_{k=1}^{r} |\lambda_k| \, x(\zeta_k, p-\mu_k) \right) \right\} M_1 \\
& + |\wedge| \, |\varpi| \, x(T,\vartheta-1) \\
\leq & \Omega_1 L_1 \{ l_{\mathbb{Q}} + c_{\mathbb{Q}} \} r_1 + \Omega_1 M_1 + |\wedge| \, |\varpi| \, x(T,\vartheta-1) \leq r_1, \quad l_{\mathbb{Q}}, c_{\mathbb{Q}} > 0,
\end{aligned}
$$

which implies that $\mathcal{T} B_{r_1} \subset B_{r_1}$.
Step:2 We show that $\mathcal{T} : C \to C$ is a contraction.
For any $\mathcal{A}, \overline{\mathcal{A}} \in C$ and for each $t \in J$, we have

$$
\begin{aligned}
\left| \mathcal{T}\mathcal{A}(t) - \mathcal{T}\overline{\mathcal{A}}(t) \right| \leq & \mathcal{I}_{0^+}^{p,\varrho,\Xi} \left| \mathbb{Q}\left(T, \mathcal{A}(T), \mathcal{A}\left(\frac{T}{\eta}\right)\right) - \mathbb{Q}\left(t, \overline{\mathcal{A}}(t), \overline{\mathcal{A}}\left(\frac{t}{\eta}\right)\right) \right| + |\wedge| \, x(T,\vartheta-1) \\
& \left(\sum_{i=1}^{m} |\delta_i| \, \mathcal{I}_{0^+}^{p,\varrho,\Xi} \left| \mathbb{Q}\left(\alpha_i, \mathcal{A}(\alpha_i), \mathcal{A}\left(\frac{\alpha_i}{\eta}\right)\right) - \mathbb{Q}\left(\alpha_i, \overline{\mathcal{A}}(\alpha_i), \overline{\mathcal{A}}\left(\frac{\alpha_i}{\eta}\right)\right) \right| \right. \\
& + \sum_{j=1}^{n} |\omega_j| \, \mathcal{I}_{0^+}^{p+\beta_j;\varrho,\Xi} \left| \mathbb{Q}\left(\theta_j, \mathcal{A}(\theta_j), \mathcal{A}\left(\frac{\theta_j}{\eta}\right)\right) - \mathbb{Q}\left(\theta_j, \overline{\mathcal{A}}(\theta_j), \overline{\mathcal{A}}\left(\frac{\theta_j}{\eta}\right)\right) \right| \\
& \left. + \sum_{k=1}^{r} |\lambda_k| \, \mathcal{I}_{0^+}^{p-\mu_k,\varrho;p,\Xi} \left| \mathbb{Q}\left(\zeta_k, \mathcal{A}(\zeta_k), \mathcal{A}\left(\frac{\zeta_k}{\eta}\right)\right) - \mathbb{Q}\left(\zeta_k, \overline{\mathcal{A}}(\zeta_k), \overline{\mathcal{A}}\left(\frac{\zeta_k}{\eta}\right)\right) \right| \right) \\
\leq & \left\{ x(T,p) + |\wedge| \, x(T,\vartheta-1) \left(\sum_{i=1}^{m} |\delta_i| \, x(\alpha_i,p) + \sum_{j=1}^{n} |\omega_j| \, x(\theta_j, p+\beta_j) \right.\right. \\
& \left.\left. + \sum_{k=1}^{r} |\lambda_k| \, x(\zeta_k, p-\mu_k) \right) \right\} L_1 \left\{ l_{\mathbb{Q}} \left\| \mathcal{A}(t) - \overline{\mathcal{A}}(t) \right\| + c_{\mathbb{Q}} \left\| \mathcal{A}\left(\frac{t}{\eta}\right) - \overline{\mathcal{A}}\left(\frac{t}{\eta}\right) \right\| \right\} \\
= & \Omega_1 L_1 \left\{ l_{\mathbb{Q}} \left\| \mathcal{A}(t) - \overline{\mathcal{A}}(t) \right\| + c_{\mathbb{Q}} \left\| \mathcal{A}\left(\frac{t}{\eta}\right) - \overline{\mathcal{A}}\left(\frac{t}{\eta}\right) \right\| \right\}, \\
\left| \mathcal{T}\mathcal{A}(t) - \mathcal{T}\overline{\mathcal{A}}(t) \right| \leq & \Omega_1 L_1 \left\{ l_{\mathbb{Q}} \left\| \mathcal{A}(t) - \overline{\mathcal{A}}(t) \right\| + c_{\mathbb{Q}} \left\| \mathcal{A}\left(\frac{t}{\eta}\right) - \overline{\mathcal{A}}\left(\frac{t}{\eta}\right) \right\| \right\}.
\end{aligned}
$$

As $\Omega_1 L_1 < 1$, hence the operator $\mathcal{T}$ is contraction. Hence, by the Banach contraction mapping principle, the operator $\mathcal{T}$ has a fixed point and the problem (1.1) has a unique solution on J. The proof is completed. □

Theorem 3.2 *Assume that $\mathbb{Q} : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and satisfying (H_1). In addition, we assume that*
(H_2) :

$$
|\mathbb{Q}(t,u,v)| \leq \sigma(t), \forall (t,u,v) \in J \times \mathbb{R} \times \mathbb{R} \quad \textit{and} \quad \sigma \in C(J, \mathbb{R}^+). \tag{3.7}
$$

If

$$
L_1 \left[\Omega_1 - x(T,p) \right] < 1, \tag{3.8}
$$

where, $\Omega_1, x(T,p)$ are defined in (3.2) and (3.1) respectively, then the problem (1.1) has atleast one solution on J.

Proof: Fix $\sup_{t\in J}|\sigma(t)| = \|\sigma\|$ and $B_{r_2} := \{\mathcal{A} \in C : \|\mathcal{A}\| \leq r_2\}$, where

$$r_2 \geq \|\sigma\| \, \Omega_1 + |\wedge| \, |\varpi| \, x(T, \vartheta - 1).$$

We define the operators $\mathcal{G}$ and $\mathcal{H}$ on B_{r_2} by

$$\begin{aligned}
\mathcal{G}\mathcal{A}(t) =& \mathcal{I}_{0^+}^{p,\varrho,\Xi}\mathbb{Q}\left(t, \mathcal{A}(t), \mathcal{A}\left(\frac{t}{\eta}\right)\right),\\
\mathcal{H}\mathcal{A}(t) =& \wedge\, x(t,\vartheta-1)\left(\varpi - \sum_{i=1}^{m}\delta_i\mathcal{I}_{0^+}^{p,\varrho,\Xi}\mathbb{Q}\left(\alpha_i, \mathcal{A}(\alpha_i), \mathcal{A}\left(\frac{\alpha_i}{\eta}\right)\right) - \sum_{j=1}^{n}\omega_j\mathcal{I}_{0^+}^{p+\beta_j;\varrho,\Xi}\right.\\
&\left.\mathbb{Q}\left(\theta_j, \mathcal{A}(\theta_j), \mathcal{A}\left(\frac{\theta_j}{\eta}\right)\right) - \sum_{k=1}^{r}\lambda_k\mathcal{I}_{0^+}^{p-\mu_k,\varrho;p,\Xi}\mathbb{Q}\left(\zeta_k, \mathcal{A}(\zeta_k), \mathcal{A}\left(\frac{\zeta_k}{\eta}\right)\right)\right), t \in J.
\end{aligned}$$

Note that $\mathcal{T} = \mathcal{G} + \mathcal{H}$. For any $\mathcal{A}, \overline{\mathcal{A}} \in B_{r_2}$, we have

$$\begin{aligned}
\left|\mathcal{G}\mathcal{A}(t) + \mathcal{H}\overline{\mathcal{A}}(t)\right| \leq& \sup_{t\in J}\left\{\mathcal{I}_{0^+}^{p,\varrho,\Xi}\left|\mathbb{Q}\left(t, \mathcal{A}(t), \mathcal{A}\left(\frac{t}{\eta}\right)\right)\right| + |\wedge|\, x(t,\vartheta-1)\right.\\
&\times\left(|\varpi| + \sum_{i=1}^{m}|\delta_i|\,\mathcal{I}_{0^+}^{p+j,\varrho,\Xi}\left|\mathbb{Q}\left(\alpha_i, \overline{\mathcal{A}}(\alpha_i), \overline{\mathcal{A}}\left(\frac{\alpha_i}{\eta}\right)\right)\right|\right.\\
&+\sum_{j=1}^{n}|\omega_j|\,\mathcal{I}_{0^+}^{p+\beta_j;\varrho,\Xi}\left|\mathbb{Q}\left(\theta_j, \overline{\mathcal{A}}(\theta_j), \overline{\mathcal{A}}\left(\frac{\theta_j}{\eta}\right)\right)\right|\\
&\left.\left.+\sum_{k=1}^{r}|\lambda_k|\,\mathcal{I}_{0^+}^{p-\mu_k,\varrho;p,\Xi}\mathbb{Q}\left(\zeta_k, \overline{\mathcal{A}}(\zeta_k), \overline{\mathcal{A}}\left(\frac{\zeta_k}{\eta}\right)\right)\right)\right\},\\
\leq& \|\sigma\|\left\{x(T,p) + |\wedge|\, x(t,\vartheta-1) + \left(\sum_{i=1}^{m}|\delta_i|\, x(\alpha_i,p)\right.\right.\\
&\left.\left.+\sum_{j=1}^{n}|\omega_j|\, x(\theta_j, p+\beta_j) + \sum_{k=1}^{r}|\lambda_k|\, x(\zeta_k, p-\mu_k)\right)\right\} + |\wedge|\,|\varpi|\, x(T,\vartheta-1)\\
\leq& \|\sigma\|\,\Omega_1 + |\wedge|\,|\varpi|\, x(T,\vartheta-1) \leq r_2.
\end{aligned}$$

This implies that $\mathcal{G}\mathcal{A} + \mathcal{H}\mathcal{A} \in B_{r_2}$, which satisfies the first assumption of Theorem 2.5.

We show that the second assumption of Theorem 2.5 is satisfied.
Let $\mathcal{A}_n$ be a sequence such that $\mathcal{A}_n \to \mathcal{A}$ in C. Then for each $t \in J$, we have

$$\begin{aligned}
|\mathcal{G}\mathcal{A}_n(t) - \mathcal{G}\mathcal{A}(t)| \leq& \mathcal{I}_{0^+}^{p,\varrho,\Xi}\left|\mathbb{Q}\left(t, \mathcal{A}_n(t), \mathcal{A}_n\left(\frac{t}{\eta}\right)\right) - \mathbb{Q}\left(t, \mathcal{A}(t), \mathcal{A}\left(\frac{t}{\eta}\right)\right)\right|\\
\leq& x(T,p)\left\|\mathbb{Q}\left(\cdot, \mathcal{A}_n(\cdot), \mathcal{A}_n\left(\frac{\cdot}{\eta}\right)\right) - \mathbb{Q}\left(\cdot, \mathcal{A}(\cdot), \mathcal{A}\left(\frac{\cdot}{\eta}\right)\right)\right\|.
\end{aligned}$$

Since $\mathbb{Q}$ is continuous, this implies that $\mathbb{Q}\left(\cdot, \mathcal{A}(\cdot), \mathcal{A}\left(\frac{\cdot}{\eta}\right)\right)$ is also continuous. Hence, we obtain $\left\|\mathbb{Q}\left(\cdot, \mathcal{A}_n(\cdot), \mathcal{A}_n\left(\frac{\cdot}{\eta}\right)\right) - \mathbb{Q}\left(\cdot, \mathcal{A}(\cdot), \mathcal{A}\left(\frac{\cdot}{\eta}\right)\right)\right\| \to 0$ as $n \to \infty$.
Thus this shows that $\mathcal{G}\mathcal{A}$ is continuous. Also the set $\mathcal{G}B_{r_2}$ is uniformly bounded on B_{r_2} as $\|\mathcal{G}\mathcal{A}\| \leq x(T,p)\,\|\sigma\|$.

Next we show that the compactness of the operator $\mathcal{G}$. Let $\sup_{(t,u,v)\in J\times B_{r_2}^2}|\mathbb{Q}(t,u,v)| = \tilde{\mathbb{Q}} < \infty$, then for each $t_1, t_2 \in J$ with $0 \le t_1 \le t_2 \le T$, we obtain

$$\begin{aligned}|\mathcal{G}\mathcal{A}(t_2) - \mathcal{G}\mathcal{A}(t_1)| =& \frac{1}{\varrho^p \Gamma(p)} \Bigg| \int_0^{t_1} \Xi'(s) \Big[e^{\frac{\varrho-1}{\varrho}(\Xi(t_2)-\Xi(s))} (\Xi(t_2)-\Xi(s))^{p-1} \\ & - e^{\frac{\varrho-1}{\varrho}(\Xi(t_1)-\Xi(s))} (\Xi(t_1)-\Xi(s))^{p-1} \Big] \mathbb{Q}\left(s, \mathcal{A}(s), \mathcal{A}\left(\frac{s}{\eta}\right)\right) ds \\ & + \int_{t_1}^{t_2} \Xi'(s) e^{\frac{\varrho-1}{\varrho}(\Xi(t_2)-\Xi(s))} (\Xi(t_2)-\Xi(s))^{p-1} \mathbb{Q}\left(s, \mathcal{A}(s), \mathcal{A}\left(\frac{s}{\eta}\right)\right) ds \Bigg| \\ \le & \frac{\tilde{\mathbb{Q}}}{\varrho^{p+1}\Gamma(p+1)} \Big[2 e^{\frac{\varrho-1}{\varrho}(\Xi(t_2)-\Xi(t_1))} (\Xi(t_2)-\Xi(t_1))^p \\ & + \left| e^{\frac{\varrho-1}{\varrho}(\Xi(t_2)-\Xi(0))} (\Xi(t_2)-\Xi(0))^p - e^{\frac{\varrho-1}{\varrho}(\Xi(t_1)-\Xi(0))} (\Xi(t_1)-\Xi(0))^p \right| \Big].\end{aligned}$$

Obviously, the right hand side of in the above inequality is independent of $\mathcal{A}$ and tends to zero as $t_2 \to t_1$. Therefore $\mathcal{G}$is equicontinuous. So $\mathcal{G}$ is relatively compact on B_{r_2}. Then by the Arzela-Ascoli theorem, $\mathcal{G}$ is compact on B_{r_2}.

Moreover, it is easy to prove using the condition (3.7), the operator $\mathcal{H}$ is a contraction and thus the third assumption of Theorem 2.5 holds.

Hence all the assumptions of Theorem 2.5 are satisfied. So the conclusion the Krasnoselskii's fixed point theorem implies that the problem (1.1) has atleast one solution on J. The proof is completed. □

4. Stability Theory

Next, we are developing Ulam-Hyers-Rassias (UHR) stable for the proposed problem (1.1).

Let $\epsilon > 0$ be a positive real number and $\Theta : J \to \mathbb{R}^+$ be a continuous function. We consider the following inequality

$$\left| \mathcal{D}_{0^+}^{p,q,\varrho,\Xi} \mathcal{A}(t) - \mathbb{Q}\left(t, \mathcal{A}(t), \mathcal{A}\left(\frac{t}{\eta}\right)\right) \right| \le \epsilon \Theta(t), \tag{4.1}$$

$$\left| \mathcal{D}_{0^+}^{p,q,\varrho,\Xi} \mathcal{A}(t) - \mathbb{Q}\left(t, \mathcal{A}(t), \mathcal{A}\left(\frac{t}{\eta}\right)\right) \right| \le \Theta(t), \tag{4.2}$$

and the following condition,

(H_3) : There exist an increasing function $\Theta \in C(J, \mathbb{R}^+)$ and there exists $n_\Theta > 0$, such that for any $t \in J$, the following integral inequality

$$\mathcal{I}_{0^+}^{p,\varrho,\Xi} \Theta(t) \le n_\Theta \Theta(t). \tag{4.3}$$

Definition 4.1 *[23] The problem (1.1) is said to be UHR stable with respect to $\Theta \in C(J, \mathbb{R}^+)$ if there exists a real number $M_{\mathbb{Q},\Theta} > 0$ such that for each $\epsilon > 0$ and for each solution $z \in C$ of the inequality (4.1), there exist a solution $\mathcal{A} \in C$ of the problem (1.1) with*

$$|z(t) - \mathcal{A}(t)| \le M_{\mathbb{Q},\Theta} \epsilon \Theta(t), \quad t \in J. \tag{4.4}$$

Definition 4.2 *[23] The problem (1.1) is said to be Generalized Ulam Hyers Rassias (GUHR) stable with respect to $\Theta \in C(J, \mathbb{R}^+)$ if there exists a real number $M_{\mathbb{Q},\Theta} > 0$ such that for each solution $z \in C$ of the inequality (4.1), there exist a solution $\mathcal{A} \in C$ of the problem (1.1) with*

$$|z(t) - \mathcal{A}(t)| \le M_{\mathbb{Q},\Theta} \Theta(t), \quad t \in J. \tag{4.5}$$

Remark 4.1 *[23] A function $z \in C(J, \mathbb{R})$ is a solution of the inequality (4.1) if and only if there exist a function $\nu \in C$(which depends on z) such that*

(i) $|\nu(t)| \le \epsilon \Theta(t), \quad \forall t \in J.$

(ii) $\mathcal{D}_{0^+}^{p,q,\varrho,\Xi} z(t) = \mathbb{Q}\left(t, z(t), z\left(\frac{t}{\eta}\right)\right) + \nu(t), \quad t \in J.$

Lemma 4.1 *Let $p \in (1,2], q \in [0,1]$. If $z \in C$ is a solution of the inequality (4.1), then z is a solution of the following inequality*

$$\left| z(t) - \mathcal{R}_z - \mathcal{I}_{0^+}^{p,\varrho,\Xi}\mathbb{Q}(t, z(t), z\left(\frac{t}{\eta}\right)) \right| \leq \Omega_2 \epsilon n_\Theta \Theta(t), \tag{4.6}$$

where,

$$\Omega_2 = 1 + x(T,\vartheta - 1)\,|\wedge| \left(\sum_{i=1}^{m} |\delta_i| + \sum_{j=1}^{n} |\omega_j| + \sum_{k=1}^{r} |\lambda_k| \right), \tag{4.7}$$

and

$$\begin{aligned} \mathcal{R}_z =& x(t,\vartheta-1) \wedge \Bigg(\varpi - \sum_{i=1}^{m} \delta_i \mathcal{I}_{0^+}^{p,\varrho,\Xi}\mathbb{Q}\left(\alpha_i, z(\alpha_i), z\left(\frac{\alpha_i}{\eta}\right)\right) \\ & - \sum_{j=1}^{n} \omega_j \mathcal{I}_{0^+}^{p+\beta_j;\varrho,\Xi}\mathbb{Q}\left(\theta_j, z(\theta_j), z\left(\frac{\theta_j}{\eta}\right)\right) - \sum_{k=1}^{r} \lambda_k \mathcal{I}_{0^+}^{p-\mu_k,\varrho;p,\Xi}\mathbb{Q}\left(\zeta_k, z(\zeta_k), z\left(\frac{\zeta_k}{\eta}\right)\right) \Bigg). \end{aligned}$$

Proof: Let z be a solution of the inequality (4.1). So in the view of Remark 4.1 (ii) and Lemma 2.3, we have

$$\begin{cases} \mathcal{D}_{0^+}^{p,q,\varrho,\Xi} z(t) = \mathbb{Q}\left(t, z(t), z\left(\frac{t}{\eta}\right)\right) + \nu(t), \quad t \in J, \quad \eta > 1, \\ z(0) = 0, \quad \sum_{i=1}^{m} \delta_i z(\alpha_i) + \sum_{j=1}^{n} \omega_j \mathcal{I}_{0^+}^{\beta_j,\varrho,\Xi} z(\theta_j) + \sum_{k=1}^{r} \lambda_k \mathcal{D}_{0^+}^{\mu_k,\varrho;\Xi} z(\zeta_k) = \varpi. \end{cases} \tag{4.8}$$

Thus, a solution of the problem (4.8) can be written by

$$\begin{aligned} z(t) =& \mathcal{I}_{0^+}^{p,\varrho,\Xi}\mathbb{Q}\left(t, z(t), z\left(\frac{t}{\eta}\right)\right) + x(t,\vartheta-1) \wedge \Bigg(\varpi - \sum_{i=1}^{m} \delta_i \mathcal{I}_{0^+}^{p,\varrho,\Xi}\mathbb{Q}\left(\alpha_i, z(\alpha_i), z\left(\frac{\alpha_i}{\eta}\right)\right) \\ & - \sum_{j=1}^{n} \omega_j \mathcal{I}_{0^+}^{p+\beta_j;\varrho,\Xi}\mathbb{Q}\left(\theta_j, z(\theta_j), z\left(\frac{\theta_j}{\eta}\right)\right) - \sum_{k=1}^{r} \lambda_k \mathcal{I}_{0^+}^{p-\mu_k,\varrho;p,\Xi}\mathbb{Q}\left(\zeta_k, z(\zeta_k), z\left(\frac{\zeta_k}{\eta}\right)\right) \Bigg) \\ & + \mathcal{I}_{0^+}^{p,\varrho,\Xi}\mathbb{Q}\left(t, \nu(t), \nu\left(\frac{t}{\eta}\right)\right) - x(t,\vartheta-1) \wedge \Bigg(\varpi - \sum_{i=1}^{m} \delta_i \mathcal{I}_{0^+}^{p,\varrho,\Xi}\mathbb{Q}\left(\alpha_i, \nu(\alpha_i), \nu\left(\frac{\alpha_i}{\eta}\right)\right) \\ & - \sum_{j=1}^{n} \omega_j \mathcal{I}_{0^+}^{p+\beta_j;\varrho,\Xi}\mathbb{Q}\left(\theta_j, \nu(\theta_j), \nu\left(\frac{\theta_j}{\eta}\right)\right) - \sum_{k=1}^{r} \lambda_k \mathcal{I}_{0^+}^{p-\mu_k,\varrho;p,\Xi}\mathbb{Q}\left(\zeta_k, \nu(\zeta_k), \nu\left(\frac{\zeta_k}{\eta}\right)\right) \Bigg). \end{aligned}$$

Then by using Remark 4.1 (ii), it follows that

$$\begin{aligned} \left| z(t) - \mathcal{R}_z - \mathcal{I}_{0^+}^{p,\varrho,\Xi}\mathbb{Q}\left(t, z(t), z\left(\frac{t}{\eta}\right)\right) \right| =& \Bigg| \mathcal{I}_{0^+}^{p,\varrho,\Xi}\mathbb{Q}\left(t, \nu(t), \nu\left(\frac{t}{\eta}\right)\right) - x(t,\vartheta-1) \wedge \\ & \times \Bigg(\sum_{i=1}^{m} \delta_i \mathcal{I}_{0^+}^{p,\varrho,\Xi}\mathbb{Q}\left(\alpha_i, \nu(\alpha_i), \nu\left(\frac{\alpha_i}{\eta}\right)\right) \\ & + \sum_{j=1}^{n} \omega_j \mathcal{I}_{0^+}^{p+\beta_j;\varrho,\Xi}\mathbb{Q}\left(\theta_j, \nu(\theta_j), \nu\left(\frac{\theta_j}{\eta}\right)\right) \\ & + \sum_{k=1}^{r} \lambda_k \mathcal{I}_{0^+}^{p-\mu_k,\varrho;p,\Xi}\mathbb{Q}\left(\zeta_k, \nu(\zeta_k), \nu\left(\frac{\zeta_k}{\eta}\right)\right) \Bigg) \Bigg| \\ \leq& \left(1 + x(T,\vartheta-1)\,|\wedge| \left(\sum_{i=1}^{m} |\delta_i| + \sum_{j=1}^{n} |\omega_j| + \sum_{k=1}^{r} |\lambda_k| \right) \right) \epsilon n_\Theta \Theta(t) \\ \leq& \Omega_2 \epsilon n_\Theta \Theta(t). \end{aligned}$$

From which inequality (4.6) is obtained. The proof is completed. □

Theorem 4.1 *Assume that the function $\mathbb{Q} : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and (H_1) holds then the problem (1.1) is UHR stable on J.*

Proof: Let $\epsilon > o$ and $z \in C$ be the solution of the inequality (4.1). Let $\mathcal{A} \in C$ be the unique solution of the problem (1.1). By the Lemma 2.3, we obtain

$$\mathcal{A}(t) = \mathcal{R}_{\mathcal{A}} + \mathcal{I}_{0^+}^{p,\varrho,\Xi}\mathbb{Q}\left(t, \mathcal{A}\left(t\right), \mathcal{A}\left(\frac{t}{\eta}\right)\right),$$

where,

$$\begin{aligned}\mathcal{R}_{\mathcal{A}} =& x(t,\vartheta-1)\wedge\left(\varpi - \sum_{i=1}^{m}\delta_i\mathcal{I}_{0^+}^{p,\varrho,\Xi}\mathbb{Q}\left(\alpha_i, \mathcal{A}\left(\alpha_i\right), \mathcal{A}\left(\frac{\alpha_i}{\eta}\right)\right) - \sum_{j=1}^{n}\omega_j\mathcal{I}_{0^+}^{p+\beta_j;\varrho,\Xi}\mathbb{Q}\left(\theta_j, \mathcal{A}\left(\theta_j\right), \mathcal{A}\left(\frac{\theta_j}{\eta}\right)\right)\right.\\ &\left.- \sum_{k=1}^{r}\lambda_k\mathcal{I}_{0^+}^{p-\mu_k,\varrho;p,\Xi}\mathbb{Q}\left(\zeta_k, \mathcal{A}\left(\zeta_k\right), \mathcal{A}\left(\frac{\zeta_k}{\eta}\right)\right)\right).\end{aligned}$$

On the other hand, if $\mathcal{A}(0) = z(0), \mathcal{A}\left(\alpha_i\right) = z\left(\alpha_i\right), \mathcal{I}_{0^+}^{\beta_j,\varrho,\Xi}\mathcal{A}(\theta_j) = \mathcal{I}_{0^+}^{\beta_j,\varrho,\Xi}z(\theta_j)$ and $\mathcal{D}_{0^+}^{\mu_k,\varrho;\Xi}\mathcal{A}(\zeta_k) = \mathcal{D}_{0^+}^{\mu_k,\varrho;\Xi}z(\zeta_k)$, then it is easy to see that $\mathcal{R}_{\mathcal{A}} = \mathcal{R}_z$.

Now applying $|u-v| \le |u| + |v|$ and Lemma 4.1, for any $t \in J$, we have

$$\begin{aligned}|z(t) - \mathcal{A}(t)| \le& \left|z(t) - \mathcal{R}_{\mathcal{A}} - \mathcal{I}_{0^+}^{p,\varrho,\Xi}\mathbb{Q}\left(t, \mathcal{A}\left(t\right), \mathcal{A}\left(\frac{t}{\eta}\right)\right)\right|\\ \le& \left|z(t) - \mathcal{R}_z - \mathcal{I}_{0^+}^{p,\varrho,\Xi}\mathbb{Q}\left(t, z\left(t\right), z\left(\frac{t}{\eta}\right)\right)\right|\\ &+ \mathcal{I}_{0^+}^{p,\varrho,\Xi}\left|\mathbb{Q}\left(t, z\left(t\right), z\left(\frac{t}{\eta}\right)\right) - \mathbb{Q}\left(t, \mathcal{A}\left(t\right), \mathcal{A}\left(\frac{t}{\eta}\right)\right)\right| + |\mathcal{R}_z - \mathcal{R}_{\mathcal{A}}|\\ \le& \Omega_2\epsilon n_\Theta\Theta(t) + \Omega_1 x(T,p)L_1\left|z(t) - \mathcal{A}(t)\right|.\end{aligned}$$

This implies that

$$|z(t) - \mathcal{A}(t)| \le \frac{\Omega_2 n_\Theta}{1 - \Omega_1 x(T,p)L_1}\epsilon\Theta(t).$$

By setting

$$M_{\mathbb{Q},\Theta} = \frac{\Omega_2 n_\Theta}{1 - \Omega_1 x(T,p)L_1},$$

we obtain

$$|z(t) - \mathcal{A}(t)| \le M_{\mathbb{Q},\Theta}\epsilon\Theta(t).$$

Therefore, the problem (1.1) is UHR stable. Further it is easy to check that the solution of the problem (1.1) is GHUR stable. This completes the proof. □

5. An Example

Here to support our findings, we present the following example. Let consider the following FDE with Ξ-Hilfer generalized PFD as

$$\begin{cases}\mathcal{D}_{0^+}^{\frac{1}{3},\frac{1}{7},\frac{2}{3},\Xi}\mathcal{A}(t) = \mathcal{A}(t) - \frac{1}{9}\mathcal{A}\left(\frac{t}{9}\right), \quad t \in [0,1],\\ \mathcal{A}(0) = 0, \quad \sum_{i=1}^{3}\left(\frac{-i}{i+5}\right)^{i+1}\mathcal{A}\left(\frac{i}{3}\right) + \sum_{j=1}^{2}\left(\frac{j+1}{j+2}\right)\mathcal{I}^{\frac{j}{3},\frac{2}{3}}\mathcal{A}\left(\frac{j}{2}\right) + \sum_{k=1}^{4}\frac{-k}{k+2}\mathcal{D}_{0^+}^{\frac{1}{3},\frac{1}{7},\frac{2}{3},\Xi}\mathcal{A}\left(\frac{k}{4}\right) = \frac{1}{2}.\end{cases} \tag{5.1}$$

Now comparing Eq. (5.1) with our proposed problem (1.1), we get
$p = \frac{1}{9}, \quad q = \frac{1}{7}, \quad \varrho = \frac{2}{3}, \quad \vartheta = \frac{3}{7}, \quad a = 0, \quad b = 1, \quad \varpi = \frac{1}{2} \quad \delta_i = \left(\frac{-i}{i+5}\right)^{i+1}, \omega_j = \left(\frac{j+1}{j+2}\right), \lambda_k = \frac{-k}{k+2}$
are in $\mathbb{R}$ are given constants, the points $\alpha_i = \frac{i}{3}, \theta_j = \frac{j}{2}, \zeta_k = \frac{k}{4} \in [0,1] = J, i = 1,2,3, \quad j = 1,2, \quad k = 1,2,3,4$.
Also, $\mathbb{Q} : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. is a function defined by

$$\mathbb{Q}\left(t, \mathcal{A}(t), \mathcal{A}\left(\frac{t}{\eta}\right)\right) = \mathcal{A}(t) - \frac{1}{9}\mathcal{A}\left(\frac{t}{9}\right).$$

Clearly, $\mathbb{Q}$ is continuous function. For $\mathcal{A}_1(t), \mathcal{A}_2(t), \mathcal{A}_1\left(\frac{t}{\eta}\right), \mathcal{A}_2\left(\frac{t}{\eta}\right) \in \mathbb{R}$ and $t \in [0,1]$, we have

$$\left|\mathbb{Q}\left(t, \mathcal{A}_1(t), \mathcal{A}_1\left(\frac{t}{\eta}\right)\right) - \mathbb{Q}\left(t, \mathcal{A}_2(t), \mathcal{A}_2\left(\frac{t}{\eta}\right)\right)\right| \leq \frac{1}{9}\left\{|\mathcal{A}_1 - \mathcal{A}_2| + \left|\mathcal{A}_1\left(\frac{t}{\eta}\right) - \mathcal{A}_2\left(\frac{t}{\eta}\right)\right|\right\}.$$

Hence the hypothesis (H_1) holds with $\varpi = \frac{1}{9}$.
Now choose $\Xi(t) = e^t$, then it implies that $\Xi(t)$ is positive increasing and continuous in $[0,1]$.
Next substituting the values that we mentioned and find the value of $|\wedge|, \Omega_0, \Omega_1$ and Ω_2

$$|\wedge| = |0.99152994855| < 1,$$
$$\Omega_1 = 2.5158265895,$$
$$\Omega_2 = 2.45428339635.$$

Hence

$$\Omega_1 L_1 \approx 0.27954 < 1,$$

since all of the assumptions of Theorem 3.1 are satisfied, then the problem (1.1) has a unique solution on $[0,1]$.
In addition, by setting $\Theta(t) = \Xi(t) - \Xi(0)$ it is easy to calculate the value of $\mathcal{I}_{0^+}^{p,\varrho,\Xi}\Theta(t)$.

$$\mathcal{I}_{0^+}^{p,\varrho,\Xi}\Theta(t) \approx 0.2841\Theta(t).$$

Thus (H_3) satisfied with $n_\Theta = 0.2841 > 0$

$$M_{\mathbb{Q},\Theta} = \frac{\Omega_2 n_\Theta}{1 - \Omega_1 x(T,p) L_1} \approx 1.0821369 > 0.$$

Hence by the Theorem 4.1, the problem (1.1) is UHR stable on J. And also GUHR stable on J.

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