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Some Curvature Conditions Provided by the Projective Curvature Tensor on the Almost $C(\alpha)$-Manifold

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ABSTRACT: In this article, the projective curvature tensor on a $C(\alpha)$-manifold is discussed. Some special curvature conditions provided by the projective curvature tensor on the Riemann, Ricci, concircular curvature tensors have been investigated. As a result of the curvature conditions given with the projective curvature tensor on a $C(\alpha)$-manifold, cases such as the $(2n+1)-$dimensional manifold being Einstein and $\eta-$Einstein manifolds have been characterized.

Key Words: Projective curvature tensor, $C(\alpha)$-manifold, Riemann curvature tensor.

Contents

1. Introduction

Curvatures and the concepts of determining the manifold on which the curvatures are defined the most important concepts of differential geometry. The answers to these interesting questions were sought in Riemannian geometry. Two important curvature properties are flatness and symmetry. Semi-symmetric spaces, which are a generalization of locally symmetric spaces, have recently started to attract attention [1]. Curvatures have been studied in symmetrical, semi-symmetrical spaces ([2], [3]). The projective curvature tensor is one of the tensors that characterizes the important properties of the manifold on which it is defined [4]. Again, many geometers investigated the properties of manifolds on different curvature tensors ([5]- [12]).

In this article, some special curvature conditions are investigated for the projective curvature tensor on a $(2n+1)$-dimensional almost $C(\alpha)-$manifold. The special relationship between the projective curvature tensor and Riemann, Ricci, the concircular curvature tensors and the effect of the projective curvature tensor on itself are discussed. Under these special curvature conditions, some important properties of the almost $C(\alpha)-$manifold are obtained.

2. Preliminary

Let's take an $(2n+1)-$dimensional differentiable M manifold. If the R Riemann curvature tensor of the M almost contact metric manifold satisfies the condition

$$\begin{aligned} R(\varpi_1,\varpi_2,\varpi_3,\varpi_4) = \; & R(\varpi_1,\varpi_2,\phi\varpi_3,\phi\varpi_4)+\alpha\{-g(\varpi_1,\varpi_3)g(\varpi_2,\varpi_4) \\ & +g(\varpi_1,\varpi_4)g(\varpi_2,\varpi_3)+g(\varpi_1,\phi\varpi_3)g(\varpi_2,\phi\varpi_4) \\ & -g(\varpi_1,\phi\varpi_4)g(\varpi_2,\phi\varpi_3)\} \end{aligned}$$

for all $\varpi_1,\varpi_2,\varpi_3,\varpi_4 \in \chi(M)$,for $\alpha \in \mathbb{R}$, then M is called the almost $C(\alpha)$-manifold where ϕ is $(1,1)-$type tensor field. Also, the Riemann curvature tensor of a almost $C(\alpha)$-manifold with $c-$constant

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sectional curvature is given by

$$\begin{aligned} R(\varpi_1,\varpi_2)\varpi_3 &= \left(\frac{c+3\alpha}{4}\right)\{g(\varpi_2,\varpi_3)\varpi_1 - g(\varpi_1,\varpi_3)\varpi_2\} \\ &+\left(\frac{c-\alpha}{4}\right)\{g(\varpi_1,\phi\varpi_3)\phi\varpi_2 - g(\varpi_2,\phi\varpi_3)\phi\varpi_1 \\ &+2g(\varpi_1,\phi\varpi_2)\phi\varpi_3 + \eta(\varpi_2)\eta(\varpi_3)\varpi_1 \\ &+g(\varpi_1,\varpi_3)\eta(\varpi_2)\xi - g(\varpi_2,\varpi_3)\eta(\varpi_1)\xi\}. \end{aligned} \tag{1}$$

If putting $\varpi_1=\xi, \varpi_2=\xi$ and $\varpi_3=\xi$ in (1), the following relations are obtained.

$$R(\xi,\varpi_2)\varpi_3 = \alpha\left[g(\varpi_2,\varpi_3)\xi - \eta(\varpi_3)\varpi_2\right], \tag{2}$$

$$R(\varpi_1,\xi)\varpi_3 = \alpha\left[-g(\varpi_1,\varpi_3)\xi + \eta(\varpi_3)\varpi_1\right], \tag{3}$$

$$R(\varpi_1,\varpi_2)\xi = \alpha\left[\eta(\varpi_2)\varpi_1 - \eta(\varpi_1)\varpi_2\right]. \tag{4}$$

Also, if we take the inner product of both sides of equation (1) with the vector $\xi \in \chi(M)$, we have

$$\eta(R(\varpi_1,\varpi_2)\varpi_3) = \alpha\left[g(\varpi_2,\varpi_3)\eta(\varpi_1) - g(\varpi_1,\varpi_3)\eta(\varpi_2)\right]. \tag{5}$$

For M is a $(2n+1)$-dimensional Riemann manifold, the projective curvature tensor P defined as

$$P(\varpi_1,\varpi_2)\varpi_3 = R(\varpi_1,\varpi_2)\varpi_3 - \frac{1}{2n}\left[S(\varpi_2,\varpi_3)\varpi_1 - S(\varpi_1,\varpi_3)\varpi_2\right], \tag{6}$$

for each $\varpi_1,\varpi_2,\varpi_3 \in \chi(M)$, [13]. If $\varpi_1=\xi, \varpi_2=\xi$ and $\varpi_3=\xi$ are selected respectively in (6), the following relations are obtained.

$$P(\xi,\varpi_2)\varpi_3 = \alpha g(\varpi_2,\varpi_3)\xi - \frac{1}{2n}S(\varpi_2,\varpi_3)\xi, \tag{7}$$

$$P(\varpi_1,\xi)\varpi_3 = -\alpha g(\varpi_1,\varpi_3)\xi + \frac{1}{2n}S(\varpi_1,\varpi_3)\xi, \tag{8}$$

$$P(\varpi_1,\varpi_2)\xi = 0. \tag{9}$$

Also, if we take the inner product of both sides of (6) with the vector $\xi \in \chi(M)$, we have

$$\begin{aligned} \eta(P(\varpi_1,\varpi_2)\varpi_3) &= \eta(\varpi_1)\left[\alpha g(\varpi_2,\varpi_3) - \tfrac{1}{2n}S(\varpi_2,\varpi_3)\right] \\ &-\eta(\varpi_2)\left[\alpha g(\varpi_1,\varpi_3) - \tfrac{1}{2n}S(\varpi_1,\varpi_3)\right]. \end{aligned} \tag{10}$$

As M is a $(2n+1)$-dimensional Riemann manifold, the tensor $\tilde{Z}$ defined as

$$\begin{aligned} \tilde{Z}(\varpi_1,\varpi_2)\varpi_3 &= R(\varpi_1,\varpi_2)\varpi_3 - \tfrac{r}{2n(2n+1)}\left[g(\varpi_2,\varpi_3)\varpi_1\right. \\ &\left.-g(\varpi_1,\varpi_3)\varpi_2\right] \end{aligned} \tag{11}$$

for each $\varpi_1,\varpi_2,\varpi_3 \in \chi(M)$, is called the concircular curvature tensor [14]. If $\varpi_1=\xi, \varpi_2=\xi$ and putting $\varpi_3=\xi$ in (11), the following relations are obtained.

$$\tilde{Z}(\xi,\varpi_2)\varpi_3 = \left[\alpha - \frac{r}{2n(2n+1)}\right]\left[g(\varpi_2,\varpi_3)\xi - \eta(\varpi_3)\varpi_2\right], \tag{12}$$

$$\tilde{Z}(\varpi_1,\xi)\varpi_3=\left[\alpha-\frac{r}{2n\,(2n+1)}\right]\left[-g\left(\varpi_1,\varpi_3\right)\xi+\eta\left(\varpi_3\right)\varpi_1\right], \tag{13}$$

$$\tilde{Z}(\varpi_1,\varpi_2)\xi=\left[\alpha-\frac{r}{2n\,(2n+1)}\right]\left[\eta\left(\varpi_2\right)\varpi_1-\eta\left(\varpi_1\right)\varpi_2\right]. \tag{14}$$

Also, if we take the inner product of both sides of equation (11) with the vector $\xi\in\chi\left(M\right)$, we have

$$\begin{aligned}\eta\left(\tilde{Z}\left(\varpi_1,\varpi_2\right)\varpi_3\right) &= \left[\alpha-\tfrac{r}{2n(2n+1)}\right]\left[g\left(\varpi_2,\varpi_3\right)\eta\left(\varpi_1\right)\right.\\ &\quad \left.-g\left(\varpi_1,\varpi_3\right)\eta\left(\varpi_2\right)\right].\end{aligned} \tag{15}$$

For a $(2n+1)-$dimensional M almost $C\left(\alpha\right)-$manifold, the following equations hold.

$$\begin{aligned}S\left(\varpi_1,\varpi_2\right) &= \left[\tfrac{\alpha(3n-1)+c(n+1)}{2}\right]g\left(\varpi_1,\varpi_2\right)\\ &\quad +\tfrac{(\alpha-c)(n+1)}{2}\eta\left(\varpi_1\right)\eta\left(\varpi_2\right),\end{aligned} \tag{16}$$

$$S\left(\varpi_1,\xi\right)=2n\alpha\eta\left(\varpi_1\right), \tag{17}$$

$$Q\varpi_1=\left[\frac{\alpha\left(3n-1\right)+c\left(n+1\right)}{2}\right]\varpi_1+\frac{\left(\alpha-c\right)\left(n+1\right)}{2}\eta\left(\varpi_1\right)\xi \tag{18}$$

$$Q\xi=2n\alpha\xi \tag{19}$$

for each $\varpi_1,\varpi_2,\in\chi\left(M\right)$, where Q and S are the Ricci operator and Ricci tensor of manifold M, respectively.

3. Projective Curvature Tensor on the Almost $C(\alpha)$-Manifold

Let M be a $(2n+1)-$dimensional almost $C(\alpha)-$manifold. Let us first examine a special curvature condition established between the projective curvature tensor and the Riemann curvature tensor. Let us state and prove the following theorem.

Theorem 3.1 *If a $(2n+1)-$ dimensional almost $C(\alpha)-$manifold satisfies the curvature condition*

$$P\left(\varpi_1,\varpi_2\right).R=\lambda_1Q\left(g,R\right),$$

then the almost $C(\alpha)-$manifold is an $\eta-$Einstein manifold.

Proof: Let's assume that a $(2n+1)-$dimensional almost $C(\alpha)-$manifold satisfies the curvature condition

$$\left(P\left(\varpi_1,\varpi_2\right).R\right)\left(\varpi_5,\varpi_4,\varpi_3\right)=\lambda_1Q\left(g,R\right)\left(\varpi_5,\varpi_4,\varpi_3;\varpi_1,\varpi_2\right),$$

for each $\varpi_1,\varpi_2,\varpi_5,\varpi_4,\varpi_3\in\chi\left(M\right)$. In this situation, we can write

$$\begin{aligned}&P\left(\varpi_1,\varpi_2\right)R\left(\varpi_5,\varpi_4\right)\varpi_3-R\left(P\left(\varpi_1,\varpi_2\right)\varpi_5,\varpi_4\right)\varpi_3\\ &-R\left(\varpi_5,P\left(\varpi_1,\varpi_2\right)\varpi_4\right)\varpi_3-R\left(\varpi_5,\varpi_4\right)P\left(\varpi_1,\varpi_2\right)\varpi_3\\ &=-\lambda_1\left\{g\left(\varpi_2,\varpi_5\right)R\left(\varpi_1,\varpi_4\right)\varpi_3-g\left(\varpi_1,\varpi_5\right)R\left(\varpi_2,\varpi_4\right)\varpi_3\right.\\ &+g\left(\varpi_2,\varpi_4\right)R\left(\varpi_5,\varpi_1\right)\varpi_3-g\left(\varpi_1,\varpi_4\right)R\left(\varpi_5,\varpi_2\right)\varpi_3\\ &\left.+g\left(\varpi_2,\varpi_3\right)R\left(\varpi_5,\varpi_4\right)\varpi_1-g\left(\varpi_1,\varpi_3\right)R\left(\varpi_5,\varpi_4\right)\varpi_2\right\}.\end{aligned} \tag{20}$$

If we choose $\varpi_1 = \xi$ in (20) and make use of (2), (3), (4), (7), we obtain

$$\begin{aligned}
&\alpha g(\varpi_2, R(\varpi_5, \varpi_4)\varpi_3)\xi - \tfrac{1}{2n} S(\varpi_2, R(\varpi_5, \varpi_4)\varpi_3)\xi \\
&+\alpha^2 g(\varpi_2, \varpi_5) g(\varpi_4, \varpi_3)\xi + \alpha^2 g(\varpi_2, \varpi_5)\eta(\varpi_3)\varpi_4 \\
&+\tfrac{1}{2n}\alpha S(\varpi_2, \varpi_5) g(\varpi_4, \varpi_3)\xi - \tfrac{1}{2n}\alpha S(\varpi_2, \varpi_5)\eta(\varpi_3)\varpi_4 \\
&-\alpha^2 g(\varpi_2, \varpi_4)\eta(\varpi_3)\varpi_5 + \alpha^2 g(\varpi_2, \varpi_4) g(\varpi_5, \varpi_3)\xi \\
&+\tfrac{1}{2n}\alpha S(\varpi_2, \varpi_4)\eta(\varpi_3)\varpi_5 - \tfrac{1}{2n}\alpha S(\varpi_2, \varpi_4) g(\varpi_5, \varpi_3)\xi \\
&-\alpha^2 g(\varpi_2, \varpi_3)\eta(\varpi_4)\varpi_5 + \alpha^2 g(\varpi_2, \varpi_3)\eta(\varpi_5)\varpi_4 \\
&= -\lambda_1 \{\alpha g(\varpi_2, \varpi_5) g(\varpi_4, \varpi_3)\xi - \alpha g(\varpi_2, \varpi_5)\eta(\varpi_3)\varpi_4 \\
&-\eta(\varpi_5) R(\varpi_2, \varpi_4)\varpi_3 - \alpha g(\varpi_2, \varpi_4) g(\varpi_5, \varpi_3)\xi \\
&+\alpha g(\varpi_2, \varpi_4)\eta(\varpi_3)\varpi_5 - \eta(\varpi_4) R(\varpi_5, \varpi_2)\varpi_3 \\
&+\alpha g(\varpi_2, \varpi_3)\eta(\varpi_4)\varpi_5 - \alpha g(\varpi_2, \varpi_3)\eta(\varpi_5)\varpi_4 \\
&-\eta(\varpi_3) R(\varpi_5, \varpi_4)\varpi_2\}.
\end{aligned} \tag{21}$$

If we choose $\varpi_5 = \xi$ in (21) and from (2), we get

$$\begin{aligned}
&\tfrac{1}{2n} S(\varpi_2, \varpi_4)\eta(\varpi_3)\xi + \alpha g(\varpi_4, \varpi_3)\eta(\varpi_2)\xi \\
&-\alpha\eta(\varpi_3)\eta(\varpi_2)\varpi_4 - \alpha g(\varpi_2, \varpi_4)\eta(\varpi_3)\xi \\
&-\alpha g(\varpi_2, \varpi_3)\eta(\varpi_4)\xi + \alpha g(\varpi_2, \varpi_3)\varpi_4 \\
&+\tfrac{1}{2n} S(\varpi_2, \varpi_3)\eta(\varpi_4)\xi - \tfrac{1}{2n} S(\varpi_2, \varpi_3)\varpi_4 \\
&= -\lambda_1 \{\alpha g(\varpi_4, \varpi_3)\eta(\varpi_2)\xi - \alpha g(\varpi_4, \varpi_2)\eta(\varpi_3)\xi \\
&-R(\varpi_2, \varpi_4)\varpi_3 + \alpha\eta(\varpi_4)\eta(\varpi_3)\varpi_2 \\
&-\alpha g(\varpi_2, \varpi_3)\varpi_4\}.
\end{aligned} \tag{22}$$

If we take the inner product of both sides of equation (22) by $\xi \in \chi(M)$ and make the necessary adjustments, we get

$$\begin{aligned}
&\tfrac{1}{2n} S(\varpi_2, \varpi_4)\eta(\varpi_3) + \alpha g(\varpi_4, \varpi_3)\eta(\varpi_2) \\
&-\alpha\eta(\varpi_3)\eta(\varpi_2)\eta(\varpi_4) - \alpha g(\varpi_2, \varpi_4)\eta(\varpi_3) \\
&= -\lambda_1 \{-\alpha g(\varpi_4, \varpi_2)\eta(\varpi_3) + \alpha\eta(\varpi_3)\eta(\varpi_2)\eta(\varpi_4)\}.
\end{aligned} \tag{23}$$

If we choose $\varpi_3 = \xi$ in equation (23), we obtain

$$S(\varpi_2, \varpi_4) = 2n\alpha(1 - \lambda_1) g(\varpi_2, \varpi_4) - 2n\alpha\lambda_1\eta(\varpi_2)\eta(\varpi_4).$$

This completes the proof. □

Corollary 3.1 *Let M be $(2n+1)$ dimensional almost $C(\alpha)-$manifold provided*

$$P(\varpi_1,\varpi_2).R=\lambda_1 Q(g,R).$$

Then M is an Einstein manifold if and only if $\lambda_1=0$.

Corollary 3.2 *If $\lambda_1=0$, then the $(2n+1)-$dimensional almost $C(\alpha)-$manifold, which satisfies the $P(\varpi_1,\varpi_2).R=0$ curvature condition, is a real space form with $c=\alpha$.*

Theorem 3.2 *If a $(2n+1)-$ dimensional almost $C(\alpha)-$manifold satisfies the curvature condition*

$$P(\varpi_1,\varpi_2).R=\lambda_2 Q(S,R),$$

then the almost $C(\alpha)-$manifold is an $\eta-$Einstein manifold.

Corollary 3.3 *Let M be $(2n+1)$ dimensional almost $C(\alpha)-$manifold provided*

$$P(\varpi_1,\varpi_2).R=\lambda_2 Q(S,R).$$

Then M is an Einstein manifold if and only if $\lambda_2=0$.

Corollary 3.4 *If $\lambda_2=0$, then the $(2n+1)-$dimensional almost $C(\alpha)-$manifold, which satisfies the $P(\varpi_1,\varpi_2).R=0$ curvature condition, is a real space form with $c=\alpha$.*

Let us secondly examine a special curvature condition established between the projective curvature tensor and again the projective curvature tensor. Let us state and prove the following theorem.

Theorem 3.3 *If a $(2n+1)-$ dimensional almost $C(\alpha)-$manifold satisfies the curvature condition*

$$P(\varpi_1,\varpi_2).P=\lambda_3 Q(g,R),$$

then $\lambda_3=0$ or scalar curvature of the almost $C(\alpha)-$manifold $r=2n\alpha(2n+1)$.

Proof: Let's assume that a $(2n+1)-$dimensional almost $C(\alpha)-$manifold satisfies the curvature condition

$$(P(\varpi_1,\varpi_2).P)(\varpi_3,\varpi_5,\varpi_4)=\lambda_3 Q(g,P)(\varpi_3,\varpi_5,\varpi_4;\varpi_1,\varpi_2),$$

for each $\varpi_1,\varpi_2,\varpi_5,\varpi_4,\varpi_3\in\chi(M)$. In this situation, we can write

$$\begin{aligned}
&P(\varpi_1,\varpi_2)P(\varpi_3,\varpi_5)\varpi_4-P(P(\varpi_1,\varpi_2)\varpi_3,\varpi_5)\varpi_4\\
&-R(\varpi_3,P(\varpi_1,\varpi_2)\varpi_5)\varpi_4-P(\varpi_3,\varpi_5)P(\varpi_1,\varpi_2)\varpi_4\\
&=-\lambda_3\{g(\varpi_2,\varpi_3)P(\varpi_1,\varpi_5)\varpi_4-g(\varpi_1,\varpi_3)P(\varpi_2,\varpi_5)\varpi_4\\
&+g(\varpi_2,\varpi_5)P(\varpi_3,\varpi_1)\varpi_4-g(\varpi_1,\varpi_5)P(\varpi_3,\varpi_2)\varpi_4\\
&+g(\varpi_2,\varpi_4)P(\varpi_3,\varpi_5)\varpi_1-g(\varpi_1,\varpi_4)P(\varpi_3,\varpi_5)\varpi_2\}.
\end{aligned}\tag{24}$$

If we choose $\varpi_1=\xi$ in (24) and make use of $(7),(8),(9)$, we obtain

$$\begin{aligned}
&\alpha g(\varpi_2,P(\varpi_3,\varpi_5)\varpi_4)\xi-\alpha g(\varpi_2,\varpi_3)P(\xi,\varpi_5)\varpi_4\\
&-\alpha g(\varpi_2,\varpi_5)P(\varpi_3,\xi)\varpi_4-\alpha g(\varpi_2,\varpi_4)P(\varpi_3,\varpi_5)\xi\\
&-\tfrac{1}{2n}S(\varpi_2,P(\varpi_3,\varpi_5)\varpi_4)\xi+\tfrac{1}{2n}S(\varpi_2,\varpi_3)P(\xi,\varpi_5)\varpi_4\\
&+\tfrac{1}{2n}S(\varpi_2,\varpi_5)P(\varpi_3,\xi)\varpi_4+\tfrac{1}{2n}S(\varpi_2,\varpi_4)P(\varpi_3,\varpi_5)\xi\\
&=-\lambda_3\left\{\alpha g(\varpi_2,\varpi_3)g(\varpi_5,\varpi_4)\xi-\tfrac{1}{2n}g(\varpi_2,\varpi_3)S(\varpi_5,\varpi_4)\xi\right.\\
&-\eta(\varpi_3)P(\varpi_2,\varpi_5)\varpi_4-\alpha g(\varpi_2,\varpi_5)g(\varpi_3,\varpi_4)\xi\\
&+\tfrac{1}{2n}g(\varpi_2,\varpi_5)S(\varpi_3,\varpi_4)\xi-\eta(\varpi_5)P(\varpi_3,\varpi_2)\varpi_4\\
&\left.-\eta(\varpi_4)P(\varpi_3,\varpi_5)\varpi_2\right\}.
\end{aligned}\tag{25}$$

From (7) in (25), we have

$$\begin{aligned}
&\alpha g(\varpi_2, P(\varpi_3,\varpi_5)\varpi_4)\xi - \alpha^2 g(\varpi_2,\varpi_3) g(\varpi_5,\varpi_4)\xi \\
&+\tfrac{1}{2n}\alpha g(\varpi_2,\varpi_3) S(\varpi_5,\varpi_4)\xi - \tfrac{1}{2n}\alpha g(\varpi_2,\varpi_5) S(\varpi_3,\varpi_4)\xi \\
&+\alpha^2 g(\varpi_2,\varpi_5) g(\varpi_4,\varpi_3)\xi - \tfrac{1}{2n} S(\varpi_2, P(\varpi_3,\varpi_5)\varpi_4)\xi \\
&+\alpha g(\varpi_5,\varpi_4) S(\varpi_2,\varpi_3)\xi - \tfrac{1}{2n} S(\varpi_2,\varpi_3) S(\varpi_5,\varpi_4) \\
&+\tfrac{1}{2n} S(\varpi_2,\varpi_5) S(\varpi_3,\varpi_4)\xi - \alpha S(\varpi_2,\varpi_5) g(\varpi_4,\varpi_3)\xi \\
&= -\lambda_3 \left\{\alpha g(\varpi_2,\varpi_3) g(\varpi_5,\varpi_4)\xi - \tfrac{1}{2n} g(\varpi_2,\varpi_3) S(\varpi_5,\varpi_4)\xi \right.\\
&-\eta(\varpi_3) P(\varpi_2,\varpi_5)\varpi_4 - \alpha g(\varpi_2,\varpi_5) g(\varpi_3,\varpi_4)\xi \\
&+\tfrac{1}{2n} g(\varpi_2,\varpi_5) S(\varpi_3,\varpi_4)\xi - \eta(\varpi_5) P(\varpi_3,\varpi_2)\varpi_4 \\
&\left.-\eta(\varpi_4) P(\varpi_3,\varpi_5)\varpi_2\right\}.
\end{aligned} \tag{26}$$

If the expression (6) is used in (26) , we get

$$\begin{aligned}
&\alpha g(\varpi_2, R(\varpi_3,\varpi_5)\varpi_4)\xi - \alpha^2 g(\varpi_2,\varpi_3) g(\varpi_5,\varpi_4)\xi \\
&+\alpha^2 g(\varpi_2,\varpi_5) g(\varpi_4,\varpi_3)\xi - \tfrac{1}{2n} S(\varpi_2, R(\varpi_3,\varpi_5)\varpi_4)\xi \\
&+\tfrac{1}{2n}\alpha S(\varpi_2,\varpi_3) g(\varpi_5,\varpi_4)\xi - \tfrac{1}{2n}\alpha S(\varpi_2,\varpi_5) g(\varpi_4,\varpi_3)\xi \\
&= -\lambda_3 \left\{\alpha g(\varpi_2,\varpi_3) g(\varpi_5,\varpi_4)\xi - \tfrac{1}{2n} g(\varpi_2,\varpi_3) S(\varpi_5,\varpi_4)\xi \right.\\
&-\eta(\varpi_3) R(\varpi_2,\varpi_5)\varpi_4 + \tfrac{1}{2n} S(\varpi_5,\varpi_4)\eta(\varpi_3)\varpi_2 \\
&-\tfrac{1}{2n} S(\varpi_2,\varpi_4)\eta(\varpi_3)\varpi_5 - \alpha g(\varpi_2,\varpi_5) g(\varpi_3,\varpi_4)\xi \\
&+\tfrac{1}{2n} g(\varpi_2,\varpi_5) S(\varpi_3,\varpi_4)\xi - \eta(\varpi_5) R(\varpi_3,\varpi_2)\varpi_4 \\
&+\tfrac{1}{2n} S(\varpi_2,\varpi_4)\eta(\varpi_5)\varpi_3 - \tfrac{1}{2n} S(\varpi_3,\varpi_4)\eta(\varpi_5)\varpi_2 \\
&-\eta(\varpi_4) R(\varpi_3,\varpi_5)\varpi_2 + \tfrac{1}{2n} S(\varpi_5,\varpi_2)\eta(\varpi_4)\varpi_3 \\
&\left.-\tfrac{1}{2n} S(\varpi_3,\varpi_2)\eta(\varpi_4)\varpi_5\right\}.
\end{aligned} \tag{27}$$

If (27) is put in (16) and we take the inner product of both sides of the equation by $\xi \in \chi(M)$, we get

$$\begin{aligned}
&\left(\tfrac{\alpha-c}{4n}\right)\left[R(\varpi_3,\varpi_5)\varpi_4 - \alpha\left(g(\varpi_5,\varpi_4)\varpi_3 - g(\varpi_4,\varpi_3)\varpi_5\right)\right] \\
&= -\lambda_3 \tfrac{(\alpha-c)(n+1)}{4n}\left\{g(\varpi_2,\varpi_3) g(\varpi_5,\varpi_4)\right. \\
&-g(\varpi_5,\varpi_4)\eta(\varpi_2)\eta(\varpi_3) \\
&-g(\varpi_2,\varpi_5) g(\varpi_3,\varpi_4) \\
&\left.+g(\varpi_3,\varpi_4)\eta(\varpi_5)\eta(\varpi_2)\right\}.
\end{aligned} \tag{28}$$

If setting $\varpi_5 = \xi$ in (28) and necessary arrangements are made,

$$-\lambda_3 \frac{(\alpha - c)(n+1)}{4n} \left[\eta(\varpi_4)\left(g(\varpi_2, \varpi_3) - \eta(\varpi_2)\eta(\varpi_3)\right)\right] = 0.$$

This implies that

$$\lambda_3 = 0 \text{ or } r = 2n\alpha(2n+1).$$

This completes the proof. □

Corollary 3.5 *Let M be $(2n+1)$ dimensional almost $C(\alpha)-$manifold provided*

$$P(\varpi_1, \varpi_2).P = \lambda_3 Q(g, P).$$

Then M is a real space form with $\alpha = c$ if and only if $\lambda_3 = 0$.

Let us as the third examine a special curvature condition established between the projective curvature tensor and the concircular curvature tensor. Let us state and prove the following theorem.

Theorem 3.4 *If a $(2n+1)-$ dimensional almost $C(\alpha)-$manifold satisfies the curvature condition*

$$P(\varpi_1, \varpi_2).\tilde{Z} = \lambda_4 Q\left(g, \tilde{Z}\right),$$

then the almost $C(\alpha)-$manifold is an $\eta-$Einstein manifold.

Proof: Let's assume that a $(2n+1)-$dimensional almost $C(\alpha)-$manifold satisfies the curvature condition

$$\left(P(\varpi_1, \varpi_2).\tilde{Z}\right)(\varpi_5, \varpi_4, \varpi_3) = \lambda_4 Q\left(g, \tilde{Z}\right)(\varpi_5, \varpi_4, \varpi_3; \varpi_1, \varpi_2),$$

for each $\varpi_1, \varpi_2, \varpi_5, \varpi_4, \varpi_3 \in \chi(M)$. In this situation, we can write

$$\begin{aligned}
&P(\varpi_1, \varpi_2)\tilde{Z}(\varpi_5, \varpi_4)\varpi_3 - \tilde{Z}(P(\varpi_1, \varpi_2)\varpi_5, \varpi_4)\varpi_3 \\
&-\tilde{Z}(\varpi_5, P(\varpi_1, \varpi_2)\varpi_4)\varpi_3 - \tilde{Z}(\varpi_5, \varpi_4)P(\varpi_1, \varpi_2)\varpi_3 \\
&= -\lambda_4 \Big\{g(\varpi_2, \varpi_5)\tilde{Z}(\varpi_1, \varpi_4)\varpi_3 - g(\varpi_1, \varpi_5)\tilde{Z}(\varpi_2, \varpi_4)\varpi_3 \\
&+g(\varpi_2, \varpi_4)\tilde{Z}(\varpi_5, \varpi_1)\varpi_3 - g(\varpi_1, \varpi_4)\tilde{Z}(\varpi_5, \varpi_2)\varpi_3 \\
&+g(\varpi_2, Z)\tilde{Z}(\varpi_5, \varpi_4)\varpi_1 - g(\varpi_1, \varpi_3)\tilde{Z}(\varpi_5, \varpi_4)\varpi_2\Big\}.
\end{aligned} \tag{29}$$

If we choose $\varpi_1 = \xi$ in (29) and make use of $(7), (12), (13), (14)$, we obtain

$$\begin{aligned}
&\alpha g\left(\varpi_2, \tilde{Z}(\varpi_5, \varpi_4)\varpi_3\right)\xi - \tfrac{1}{2n}S\left(\varpi_2, \tilde{Z}(\varpi_5, \varpi_4)\varpi_3\right)\xi \\
&-\alpha g(\varpi_2, \varpi_5)\tilde{Z}(\xi, \varpi_4)\varpi_3 + \tfrac{1}{2n}S(\varpi_2, \varpi_5)\tilde{Z}(\xi, \varpi_4)\varpi_3 \\
&-\alpha g(\varpi_2, \varpi_4)\tilde{Z}(\varpi_5, \xi)\varpi_3 + \tfrac{1}{2n}S(\varpi_2, \varpi_4)\tilde{Z}(\varpi_5, \xi)\varpi_3 \\
&-\alpha g(\varpi_2, \varpi_3)\tilde{Z}(\varpi_5, \varpi_4)\xi + \tfrac{1}{2n}S(\varpi_2, \varpi_3)\tilde{Z}(\varpi_5, \varpi_4)\xi \\
&= -\lambda_4\Big\{\left(\alpha - \tfrac{r}{2n(2n+1)}\right)g(\varpi_2, \varpi_5)g(\varpi_4, \varpi_3)\xi \\
&-\left(\alpha - \tfrac{r}{2n(2n+1)}\right)g(\varpi_2, \varpi_5)\eta(\varpi_3)\varpi_4 \\
&-\eta(\varpi_5)\tilde{Z}(\varpi_2, \varpi_4)\varpi_3 - \eta(\varpi_4)\tilde{Z}(\varpi_5, \varpi_2)\varpi_3 \\
&-\left(\alpha - \tfrac{r}{2n(2n+1)}\right)g(\varpi_2, \varpi_4)g(\varpi_5, \varpi_3)\xi \\
&+\left(\alpha - \tfrac{r}{2n(2n+1)}\right)g(\varpi_2, \varpi_4)\eta(\varpi_3)\varpi_5 \\
&+\left(\alpha - \tfrac{r}{2n(2n+1)}\right)g(\varpi_2, \varpi_3)\eta(\varpi_4)\varpi_5 \\
&-\left(\alpha - \tfrac{r}{2n(2n+1)}\right)g(\varpi_2, \varpi_3)\eta(\varpi_5)\varpi_4 \\
&-\eta(\varpi_3)\tilde{Z}(\varpi_5, \varpi_4)\varpi_2\Big\}.
\end{aligned} \tag{30}$$

If we choose $\varpi_5 = \xi$ in (30) and make use of $(12), (13)$, we have

$$\begin{aligned}
&\left(\alpha - \tfrac{r}{2n(2n+1)}\right)[\alpha g(\varpi_4, \varpi_3)\eta(\varpi_2)\xi - \alpha g(\varpi_2, \varpi_4)\eta(\varpi_3)\xi \\
&-\tfrac{1}{2n}g(\varpi_4, \varpi_3)S(\varpi_2, \xi)\xi + \tfrac{1}{2n}S(\varpi_2, \varpi_4)\eta(\varpi_3)\xi \\
&-\alpha g(\varpi_2, \varpi_3)\eta(\varpi_4)\xi + \alpha g(\varpi_2, \varpi_3)\varpi_4 \\
&+\tfrac{1}{2n}S(\varpi_2, \varpi_3)\eta(\varpi_4)\xi - \tfrac{1}{2n}S(\varpi_2, \varpi_3)\varpi_4] \\
&= -\lambda_4\Big\{\left(\alpha - \tfrac{r}{2n(2n+1)}\right)[g(\varpi_4, \varpi_3)\eta(\varpi_2)\xi - \eta(\varpi_2)\eta(\varpi_3)\varpi_4 \\
&-g(\varpi_2, \varpi_3)\eta(\varpi_4)\xi + \eta(\varpi_3)\eta(\varpi_4)\varpi_2 \\
&-g(\varpi_2, \varpi_4)\eta(\varpi_3)\xi + g(\varpi_2, \varpi_4)\eta(\varpi_3)\xi \\
&+g(\varpi_2, \varpi_3)\eta(\varpi_4)\xi - g(\varpi_2, \varpi_3)\varpi_4 \\
&-g(\varpi_4, \varpi_2)\eta(\varpi_3)\xi + \eta(\varpi_3)\eta(\varpi_2)\varpi_4] \\
&-\tilde{Z}(\varpi_2, \varpi_4)\varpi_3\Big\}.
\end{aligned} \tag{31}$$

If we take inner product both sides of (31) by $\xi \in \chi(M)$ and then choose $\varpi_3 = \xi$, we obtain

$$S(\varpi_2, \varpi_4) = 2n(\alpha + \lambda_4) g(\varpi_2, \varpi_4) - 2n\lambda_4 \eta(\varpi_2) \eta(\varpi_4).$$

This completes the proof. □

Corollary 3.6 *Let M be $(2n+1)$ dimensional almost $C(\alpha)-$manifold provided*

$$P(\varpi_1, \varpi_2).\tilde{Z} = \lambda_4 Q\left(g, \tilde{Z}\right).$$

Then M is an Einstein manifold if and only if $\lambda_4 = 0$.

Corollary 3.7 *If $\lambda_4 = 0$, then the $(2n+1)-$dimensional almost $C(\alpha)-$manifold, which satisfies the $P(\varpi_1, \varpi_2).\tilde{Z} = 0$ curvature condition, is a real space form with $c = \alpha$.*

Theorem 3.5 *If a $(2n+1)-$ dimensional almost $C(\alpha)-$manifold satisfies the curvature condition*

$$P(\varpi_1, \varpi_2).\tilde{Z} = \lambda_5 Q\left(S, \tilde{Z}\right),$$

then the almost $C(\alpha)-$manifold is an $\eta-$Einstein manifold.

Corollary 3.8 *Let M be $(2n+1)$ dimensional almost $C(\alpha)-$manifold provided*

$$P(\varpi_1, \varpi_2).\tilde{Z} = \lambda_5 Q\left(S, \tilde{Z}\right).$$

Then M is an Einstein manifold if and only if $\lambda_5 = 0$.

Corollary 3.9 *If $\lambda_5 = 0$, then the $(2n+1)-$dimensional almost $C(\alpha)-$manifold, which satisfies the $P(\varpi_1, \varpi_2).\tilde{Z} = 0$ curvature condition, is a real space form with $c = \alpha$.*

Let us as the finally examine a special curvature condition established between the projective curvature tensor and the Ricci curvature tensor. Let us state and prove the following theorem.

Theorem 3.6 *If a $(2n+1)-$ dimensional almost $C(\alpha)-$manifold satisfies the curvature condition*

$$P(\varpi_1, \varpi_2).S = \lambda_6 Q(g, S),$$

then the almost $C(\alpha)-$manifold is an Einstein manifold.

Proof: Let's assume that a $(2n+1)-$dimensional almost $C(\alpha)-$manifold satisfies the curvature condition

$$(P(\varpi_1, \varpi_2).S)(\varpi_5, \varpi_4) = \lambda_6 Q(g, S)(\varpi_5, \varpi_4; \varpi_1, \varpi_2),$$

for each $\varpi_1, \varpi_2, \varpi_5, \varpi_4 \in \chi(M)$. This mean

$$\begin{aligned} &-S(P(\varpi_1, \varpi_2)\varpi_5, \varpi_4) - S(\varpi_5, P(\varpi_1, \varpi_2)\varpi_4) \\ &= -\lambda_6 \{g(\varpi_2, \varpi_5) S(\varpi_1, \varpi_4) - g(\varpi_1, \varpi_5) S(\varpi_2, \varpi_4) \\ &+ g(\varpi_2, \varpi_4) S(\varpi_5, \varpi_1) - g(\varpi_1, \varpi_4) S(\varpi_5, \varpi_2)\}. \end{aligned} \tag{32}$$

If we choose $\varpi_1 = \xi$ in (32) and make use of (7), we have

$$\begin{aligned} &\alpha g(\varpi_2, \varpi_4)\xi + \alpha g(\varpi_2, \varpi_5)\xi - \tfrac{1}{2n} S(\varpi_2, \varpi_4)\xi - \tfrac{1}{2n} S(\varpi_2, \varpi_5)\xi \\ &= \lambda_6 \{2n\alpha g(\varpi_2, \varpi_5)\eta(\varpi_4) + 2n\alpha g(\varpi_2, \varpi_4)\eta(\varpi_5) \\ &-\eta(\varpi_5) S(\varpi_2, \varpi_4) - \eta(\varpi_4) S(\varpi_5, \varpi_2)\}. \end{aligned} \tag{33}$$

If we choose $\varpi_5 = \xi$ in (33), we obtain

$$S(\varpi_2, \varpi_4) = 2n\alpha g(\varpi_2, \varpi_4).$$

This completes the proof. □

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