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The sharp bound of certain second Hankel determinants for inverse functions of the class of convex functions with respect to symmetric points

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ABSTRACT: In this paper, we focus on the sharp upper bounds for specific second-order Hankel determinants pertaining to the inverse function, denoted as f^{-1}. Our investigation is centered on scenarios where f is a member of the class of convex functions with respect to symmetric points. By delving into this particular class, we aim to uncover and establish sharp bounds for these second Hankel determinants, providing valuable insights to extremal functions within this context.

Key Words: Analytic function, Upper bound, Hankel determinant, Carathéodory function.

Contents

1. Introduction

Let $\mathcal{A}$ represent the family of all analytic normalized mappings f of the form

$$f(z) = z + \sum_{t=2}^{\infty} a_t z^t \tag{1.1}$$

in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{S}$ is the subfamily of $\mathcal{A}$, consisting of univalent (schlicht) mappings. Pommerenke [12] characterized the r^{th}- Hankel determinant of order n, for f with $r,\ n \in \mathbb{N} = \{1, 2, 3, ...\}$, namely

$$H_{r,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+r-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+r} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+r-1} & a_{n+r} & \cdots & a_{n+2r-2} \end{vmatrix} (a_1 = 1). \tag{1.2}$$

The Fekete-Szegö functional is obtained for $r = 2$ and $n = 1$ in (1.2), denoted by $H_{2,1}(f)$. Further, for $r = n = 2$ and $r = 2,\ n = 3$ in (1.2), we obtain certain second order Hankel determinants, respectively given by

$$H_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} \tag{1.3}$$

and

$$H_{2,3}(f) = \begin{vmatrix} a_3 & a_4 \\ a_4 & a_5 \end{vmatrix}. \tag{1.4}$$

The exact estimate of $|H_{2,2}(f)|$ for the sub families of $\mathcal{S}$, were proved by Janteng et al. [7,8]. Zaparwa et al. [30] obtained sharp bound of $|H_{2,3}(f)|$ for the class of bounded turning, starlike and convex functions

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by taking $a_2 = 0$. For the class of convex functions with respect to symmetric points, introduced by Das and Singh [2], denoted as $\mathcal{K}_s$, satisfying the analytic condition

$$\operatorname{Re}\left\{\frac{2\{zf'(z)\}'}{\{f(z)-f(-z)\}'}\right\} > 0, \tag{1.5}$$

Ramreddy et al. [13] shown that $|H_{2,2}(f)| \leq 1/9$.

For $f \in \mathcal{K}_s$ Hern et al. [6] proved that $|H_{2,3}(f)| \leq 17/240$, and $|H_{2,3}(f)| \leq 1/15$ which is sharp, by assuming $a_2 = 0$.

For $f \in \mathcal{S}$, has an inverse f^{-1} given by

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} t_n w^n, \quad |w| < r_o(f); \left(r_o(f) \geq \frac{1}{4}\right). \tag{1.6}$$

Ali [1] determined sharp bounds on the first four coefficients and sharp estimate for the Fekete-Szegö coefficient functional of the inverse functions belong to the class of strongly starlike functions. Recently Sim et al. [22] obtained sharp bound of $|H_{2,2}(f^{-1})|$ for the class of strongly Ozaki functions.

A function $f \in \mathcal{A}$ is said to be biunivalent in $\mathbb{D}$ if both f and f^{-1} are univalent in $\mathbb{D}$ [23].

Motivated by the results obtained by the authors mentioned above, in this paper, we are making an attempt to estimate sharp bound for coefficient functionals namely $|H_{2,1}(f^{-1})|$, $|H_{2,2}(f^{-1})|$ and $|H_{2,3}(f^{-1})|$, when f belongs to the class $\mathcal{K}_s$.

The collection $\mathcal{P}$, of all functions p, each one called as Carathéodory function [3] of the form,

$$p(z) = 1 + \sum_{t=1}^{\infty} c_t z^t, \tag{1.7}$$

having a positive real part in $\mathbb{D}$. In view of (1.5) and (1.7), the coefficients of functions in $\mathcal{K}_s$ have suitable representation expressed by coefficients of functions in $\mathcal{P}$. Hence, to estimate the upper bound of $|H_{2,n}(f^{-1})|$, for $n \in \{1, 2, 3\}$, we build our computing on the well known formulas on coefficients c_2 (see, [11, p. 166]), c_3 (see [9,10]) and c_4 can be found in [15].

The foundation for proofs of our main results are the following lemmas and we adopt the procedure framed through Libera and Zlotkiewicz [10].

Lemma 1.1 *([5]) If $p \in \mathcal{P}$, then $|c_i - \mu c_j c_{i-j}| \leq 2$, satisfies for the values $i, j \in \mathbb{N}$, with $i > j$ and $\mu \in [0,1]$, which is same as $|c_{n+k} - \mu c_n c_k| \leq 2$, for $n, k \in \mathbb{N}$, with $\mu \in [0,1]$.*

Lemma 1.2 *([11]) For $p \in \mathcal{P}$, then $|c_t| \leq 2$, for $t \in \mathbb{N}$, equality occurs for the function $p_o = \frac{1+z}{1-z}$, $z \in \mathbb{D}$.*

Lemma 1.3 *([15]) If $p \in \mathcal{P}$, is of the form (1.7) with $c_1 \geq 0$, then*

$$2c_2 = c_1^2 + t\zeta,$$
$$4c_3 = c_1^3 + 2c_1 t\zeta - c_1 t\zeta^2 + 2t\left(1 - |\zeta|^2\right)\eta,$$

and

$$8c_4 = c_1^4 + 3c_1^2 t\zeta + \left(4 - 3c_1^2\right)t\zeta^2 + c_1^2 t\zeta^3 + 4t\left(1 - |\zeta|^2\right)\left(1 - |\eta|^2\right)\xi + 4t\left(1 - |\zeta|^2\right)\left(c_1\eta - c_1\zeta\eta - \bar{\zeta}\eta^2\right),$$

where $t := 4 - c_1^2$, for some ζ, η and ξ such that $c_1 \in [0,2]$, $|\zeta| \leq 1$, $|\eta| \leq 1$ and $|\xi| \leq 1$.

2. Main Results

Theorem 2.1 *If $f \in \mathcal{K}_s$ then*

$$\left|H_{2,1}(f^{-1})\right| \leq \frac{1}{3},$$

and the result is sharp for $f_o = \log\{\sqrt{(1+z)/(1-z)}\}$.

Proof: For $f \in \mathcal{K}_s$, in view of 1.5, there exists an analytic function $p(z)$, in $\mathbb{D}$, with Re $p(z) > 0$, such that

$$\frac{2\{zf'(z)\}'}{\{f(z)-f(-z)\}'} = p(z) \quad \Leftrightarrow \quad 2\{f'(z) + zf''(z)\} = \{f'(z)+f'(-z)\}\, p(z). \tag{2.1}$$

Using the series representation for $f''(z)$, $f'(-z)$, $f'(z)$ and $p(z)$ in (2.1) after simplifying, upon equating the coefficients on both sides, we obtain

$$a_2 = \frac{c_1}{4};\ a_3 = \frac{c_2}{6};\ a_4 = \frac{(2c_3 + c_1c_2)}{32};\quad a_5 = \frac{(2c_4 + c_2^2)}{40}. \tag{2.2}$$

Since $f \in \mathcal{K}_s$, using the definition of inverse function of f, we have

$$w = f(f^{-1}) = f^{-1}(w) + \sum_{n=2}^{\infty} a_n (f^{-1}(w))^n. \tag{2.3}$$

Further, we have

$$w = f(f^{-1}) = w + \sum_{n=2}^{\infty} t_n w^n + \sum_{k=2}^{\infty} a_k (w + \sum_{n=2}^{\infty} t_n w^n)^k. \tag{2.4}$$

Upon simplification, we obtain

$$\begin{aligned}(t_2 + a_2)w^2 + (t_3 + 2a_2t_2 + a_3)w^3 + (t_4 + 2a_2t_3 + a_2t_2^2 + 3a_3t_2 + a_4)w^4 \\ +(t_5 + 2a_2t_4 + 2a_2t_2t_3 + 3a_3t_3 + 3a_3t_2^2 + 4a_4t_2 + a_5)w^5 + ... = 0.\end{aligned} \tag{2.5}$$

Equating the coefficients of like powers in (2.5), upon simplification, we obtain

$$\begin{aligned}t_2 = -a_2; t_3 = -a_3 + 2a_2^2; t_4 = -a_4 + 5a_2a_3 - 5a_2^3; \\ t_5 = -a_5 + 6a_2a_4 - 21a_2^2a_3 + 3a_3^2 + 14a_2^4.\end{aligned} \tag{2.6}$$

From (2.2) and (2.6), after simplifying, we get

$$\begin{aligned} t_2 = -\frac{c_1}{4},\ t_3 = \frac{c_1^2}{8} - \frac{c_2}{6},\ t_4 = \frac{-15c_1^3 + 34c_1c_2 - 12c_3}{192} \\ \text{and } t_5 = \frac{105c_1^4 - 330c_1^2c_2 + 112c_2^2 + 180c_1c_3 - 96c_4}{1920}. \end{aligned} \tag{2.7}$$

Now,

$$H_{2,1}(f^{-1}) = \begin{vmatrix} 1 & t_2 \\ t_2 & t_3 \end{vmatrix} = t_3 - t_2^2, \tag{2.8}$$

Substituting the values of $t_j (j = 2, 3)$ from (2.7) in (2.8), we obtain

$$\begin{aligned} H_{2,1}(f^{-1}) &= \frac{c_1^2}{16} - \frac{c_2}{6} \\ &= -\frac{1}{6}\left(c_2 - \frac{3}{8}c_1^2\right) \end{aligned} \tag{2.9}$$

By an application of modulus on either side of the above relation and execution of Lemma 1.1, we obtain

$$\begin{aligned} |H_{2,1}(f^{-1})| &\leq \frac{1}{6}\left|c_2 - \frac{3}{8}c_1^2\right| \\ &= \frac{1}{3}. \end{aligned} \tag{2.10}$$

From f_o we obtain $a_1 = 1$, $a_2 = a_4 = 0$, $a_3 = 1/3$ and $a_5 = 1/5$, further $t_2 = 0$ and $t_3 = -\frac{1}{3}$, which follows the result. □

Theorem 2.2 *If $f \in \mathcal{K}_s$ then*

$$\left|H_{2,2}(f^{-1})\right| \leq \frac{1}{9},$$

and the result is sharp for the same function mentioned in Theorem 2.1.

Proof: For $f \in \mathcal{K}_s$, inview of (1.3), we have

$$H_{2,2}(f^{-1}) = \begin{vmatrix} t_2 & t_3 \\ t_3 & t_4 \end{vmatrix}, \tag{2.11}$$

Using the values of $t_j,\ (j = 2, 3, 4)$ from (2.7) in (2.11), after simplifying, we get

$$H_{2,2}(f^{-1}) = \frac{1}{2304}\Big(9c_1^4 - 6c_1^2c_2 - 64c_2 + 36c_1c_3\Big). \tag{2.12}$$

Substituting the values of c_2 and c_3 from Lemma 1.3 in (2.12), upon simplification, we obtain

$$H_{2,2}(f^{-1}) = \frac{1}{2304}\Big\{c_1^4 + t\left[-17c_1^2\zeta - 16t\zeta^2 - 9c_1^2\zeta^2 + 18c_1(1 - |\zeta|^2)\eta\right]\Big\} \tag{2.13}$$

Taking modulus on both sides, applying the triangle inequality by taking $c_1 := c \in [0, 2],\ t = 4 - c^2$ and $|\zeta| := x$, using $|\eta| \leq 1$ in (2.13), we have

$$\begin{aligned}
\left|H_{2,2}(f^{-1})\right| &\leq \frac{1}{2304}\Big\{c^4 + (4 - c^2)\left[17c^2x + 16(4 - c^2)x^2 + 9c^2x^2 + 18c(1 - x^2)\right]\Big\} \\
&= \frac{1}{2304}\Big\{c^4 + (4 - c^2)\left[18c + 17c^2x + (2 - c)(32 + 7c)x^2\right]\Big\} \\
&\leq \frac{1}{2304}\Big\{c^4 + (4 - c^2)\left[18c + 17c^2 + (2 - c)(32 + 7c)\right]\Big\} \\
&= \frac{1}{2304}\Big\{256 - 24c^2 - 9c^4\Big\} \\
&= \frac{1}{9} - \frac{c^2}{96} - \frac{c^4}{256} \leq \frac{1}{9}.
\end{aligned}$$

From $f_o(z)$, we obtain $t_2 = 0$, $t_3 = -1/3$ and $t_4 = 0$, hence the result. □

Theorem 2.3 *If $f \in \mathcal{K}_s$ then*

$$\left|H_{2,3}(f^{-1})\right| \leq \frac{2}{45},$$

and the result is sharp for the same function mentioned in Theorem 2.1.

Proof: For $f \in \mathcal{K}_s$, based on (1.4), we have

$$H_{2,3}(f^{-1}) = \begin{vmatrix} t_3 & t_4 \\ t_4 & t_5 \end{vmatrix}, \tag{2.14}$$

Using the values of $t_j,\ (j = 3, 4, 5)$ from (2.7) in (2.14), it simplifies to

$$\begin{aligned}
H_{3,1}(f^{-1}) = \frac{1}{184320}\Big(&135c_1^6 - 540c_1^4c_2 + 844c_1^2c_2^2 - 1792c_2^3 + 360c_1^3c_3 + 1200c_1c_2c_3 \\
&- 720c_3^2 - 1152c_1^2c_4 + 1536c_2c_4\Big).
\end{aligned} \tag{2.15}$$

Substituting the values of c_2, c_3 and c_4 from Lemma 1.3 in (2.15), upon simplification, we obtain

$$\begin{aligned} H_{3,1}(f^{-1}) = \frac{1}{184320}\Bigg\{ & - c_1^6 + t\Bigg(- 118c_1^4\zeta - 6c_1^4\zeta^2 - 192c_1^2\zeta^2 - 48c_1^4\zeta^3 \\ & + t\left[-53c_1^2\zeta^2 - 258c_1^2\zeta^3 + 51c_1^2\zeta^4 + 384\zeta^3 - 224t\zeta^3\right] \\ & + 12\left[(9+16\zeta)c_1^3 + c_1t\zeta(27-17\zeta)\right](1-|\zeta|^2)\eta \\ & + 12\left[16c_1^2\bar{\zeta} - t(9+17|\zeta|^2)\right](1-|\zeta|^2)\eta^2 \\ & + 192\left[2t\zeta - c_1^2\right](1-|\zeta|^2)(1-|\eta|^2)\xi\Bigg)\Bigg\} \end{aligned} \tag{2.16}$$

For $c_1 := c$ and $t := 4 - c^2$ in (2.16), it takes the form

$$\begin{aligned} H_{3,1}(f^{-1}) = \frac{1}{184320}\Bigg\{ & - c^6 + (4-c^2)\Bigg(- 118c^4\zeta - (404-47c^2)c^2\zeta^2 \\ & - 2(1024 - 188c^2 + 7c^4)\zeta^3 + (204 - 51c^2)c^2\zeta^4 \\ & + 12\left[(9+16\zeta)c^3 + c(4-c^2)\zeta(27-17\zeta)\right](1-|\zeta|^2)\eta \\ & + 12\left[16c^2\bar{\zeta} - (4-c^2)(9+17|\zeta|^2)\right](1-|\zeta|^2)\eta^2 \\ & + 192\left[2(4-c^2)\zeta - c^2\right](1-|\zeta|^2)(1-|\eta|^2)\xi\Bigg)\Bigg\} \end{aligned} \tag{2.17}$$

Taking modulus on both sides in (2.17), with $|\zeta| := x \in [0,1]$, $|\eta| := y \in [0,1]$, $c_1 := c \in [0,2]$ and $|\xi| \le 1$, we obtain

$$\left|H_{3,1}(f^{-1})\right| \le \frac{\Phi(c,x,y)}{184320} \tag{2.18}$$

where $\Phi(c,x,y) : \mathbb{R}^3 \to \mathbb{R}$ is defined as

$$\begin{aligned} \Phi(c,x,y) = c^6 + (4-c^2)\Bigg(& 118c^4x + (404-47c^2)c^2x^2 \\ & + 2(1024 - 188c^2 + 7c^4)x^3 + (204 - 51c^2)c^2x^4 \\ & + 12\left[(9+16x)c^3 + c(4-c^2)x(27+17x)\right](1-x^2)y \\ & + 12\left[16c^2x + (4-c^2)(9+17x^2)\right](1-x^2)y^2 \\ & + 192\left[2(4-c^2)x + c^2\right](1-x^2)(1-y^2)\Bigg) \end{aligned} \tag{2.19}$$

Now, we will maximize the function $\Phi(c,x,y)$ in the region of the parallelepiped formed by $[0,2] \times [0,1] \times [0,1]$.

A. On the vertices of the parallelepiped, we obtain

$$\begin{aligned} & \Phi(2,0,0) = \Phi(2,1,0) = \Phi(2,1,1) = \Phi(2,0,0) = 64, \\ & \Phi(0,0,1) = 2880,\ \Phi(0,1,0) = \Phi(0,1,1) = 8192,\ \Phi(0,0,0) = 0. \end{aligned}$$

B. Now, considering the eight edges of the parallelepiped, with respect to (2.19), discussed as follows.

(i) For $c = 0,\ x = 0$, we have

$$\Phi(0,0,y) = 2880y^2 \le 2880,\ y \in (0,1).$$

(ii) For $c = 0,\ x = 1, 0 < y < 1$

$$\Phi(0,1,y) = 8192.$$

(iii) For $c = 0,\ y = 0$,

$$\Phi(0,x,0) = 6144x + 2048x^3 \le 8192,\ x \in (0,1).$$

(iv) For $c = 0,\ y = 1, 0 < x < 1$,

$$\Phi\left(0, x, 1\right) = 2880 + 384x^2 + 8192x^3 - 3264x^4 \leq 8192.$$

(v) For $x = 0,\ y = 0, 0 < c < 2$,

$$\Phi\left(c, 0, 0\right) = 768c^2 - 192c^4 + c^6 \leq 768c^2 + c^6 \leq 3136.$$

(vi) For $x = 0,\ y = 1, 0 < c < 2$,

$$\Phi\left(c, 0, 1\right) = 2880 - 1440c^2 + 432c^3 + 180c^4 - 108c^5 + c^6 \leq 2880.$$

(vii) For $c = 2,\ x = 0, 0 < y < 1$; $c = 2,\ x = 1, 0 < y < 1$;
$c = 2,\ y = 0, 0 < x < 1$; $c = 2,\ y = 1, 0 < x < 1$ in (2.19), we have

$$\Phi\left(2, x, y\right) = 64.$$

(viii) For $x = 1,\ y = 0, 0 < c < 2$; $x = 1,\ y = 1, 0 < c < 2$, in (2.19), we get

$$\Phi\left(c, 1, y\right) = 8192 - 1120c^2 - 96c^4 - 33c^6 \leq 8192.$$

C. Now, we consider the six faces of the parallelepiped, by considering expression in (2.19)
(i) For $c = 2$, we obtain $\Phi\left(2, x, y\right) = 64$ for $x, y \in (0, 1)$.
(ii) If $c = 0$, then

$$\begin{aligned}
\Phi\left(0, x, y\right) &= 4(2048x^3 + (1 - x^2)(720 + 816x^2)y^2 + 1536x(1 - x^2)(1 - y^2)) \\
&= 6144x + 2048x^3 + (2880 - 6144x + 384x^2 + 6144x^3 - 3264x^4)y^2 \\
&= 6144x + 2048x^3 + 192(-1 + x)^2(1 + x)(15 - 17x)y^2 \\
&\leq 6144x + 2048x^3 + 192(-1 + x)^2(1 + x)(15 - 15x)y^2 \\
&:= G_1(x, y) \text{ for, } \ x \in (0, 1) \text{ and } y \in (0, 1).
\end{aligned}$$

As $G_1(x, y)$ is an increasing function of y and hence

$$G_1(x, y) \leq G_1(x, 1) = 2880 + 384x + 7808x^3 - 2880x^4 := G_2(x),$$

which is an increasing function of x, therefore

$$\Phi\left(0, x, y\right) \leq G_2(x) \leq G_2(1) = 8192.$$

(iii) If $x = 0$ in (2.19), then

$$\begin{aligned}
\Phi\left(c, 0, y\right) &= c^6 + (4 - c^2)\left[108c^3 y + 180(4 - c^2)y^2 + 192c^2(1 - y^2)\right] \\
&= c^6 + (4 - c^2)\left[720y^2 + c^2(192 - 372y^2) + 108c^3 y\right] \\
&\leq c^6 + (4 - c^2)\left[720y^2 + 192c^2 + 108c^3 y\right] \\
&\leq c^6 + (4 - c^2)\left[720 + 192c^2 + 108c^3\right] \\
&= 2880 + 48c^2 + 432c^3 - 192c^4 - 108c^5 + c^6 \\
&\leq 2880 + 48c^2 + 432c^3 + c^6 = 6592.
\end{aligned}$$

(iv) On the face $x = 1$, $c \in (0, 2)$, $y \in (0, 1)$, from (2.19), we observe that the function $\Phi(c, 1, y)$ is independent of y, from **B**(viii), we have

$$\Phi(c, 1, y) \leq 8192.$$

(v) On the face $y = 0$, $c \in (0, 2)$, $x \in (0, 1)$ from (2.19), we have

$$\begin{aligned}\Phi(c, x, 0) = & c^6 + (4 - c^2)\Big(118c^4x + (404 - 47c^2)c^2x^2 \\ & + 2(1024 - 188c^2 + 7c^4)x^3 + (204 - 51c^2)c^2x^4 \\ & + 192\left[2(4 - c^2)x + c^2\right](1 - x^2)\Big) \\ = & c^6 + (4 - c^2)\Big(1536x + 512x^3 + c^2(192 - 384x + 212x^2 + 8x^3 + 204x^4) \\ & + c^4(118x - 47x^2 + 14x^3 - 51x^4)\Big) \\ \le & c^6 + (4 - c^2)\Big(2048 + 232c^2 + 53c^4\Big) \\ = & 8192 - 1120\,c^2 - 20c^4 - 52c^6 \le 8192.\end{aligned}$$

(vi) On the face $y = 1$, $c \in (0, 2)$, $x \in (0, 1)$ from (2.19), we have

$$\begin{aligned}\Phi(c, x, 1) = & c^6 + (4 - c^2)\Big\{118c^4x + (404 - 47c^2)c^2x^2 \\ & + 2(1024 - 188c^2 + 7c^4)x^3 + (204 - 51c^2)c^2x^4 \\ & + 12\left[(9 + 16x)c^3 + c(4 - c^2)x(27 + 17x)\right](1 - x^2) \\ & + 12\left[16c^2x + (4 - c^2)(9 + 17x^2)\right](1 - x^2)\Big\} \\ := & g(c, x), \text{with } c \in (0, 2) \text{ and } x \in (0, 1)\end{aligned}$$

Further,

$$\begin{aligned}\frac{\partial g}{\partial c} = & -2880c + 1296c^2 + 720c^3 - 540c^4 + 6c^5 + 5184x + 1536cx - 5472c^2x \\ & + 1120c^3x + 660c^4x - 708c^5x + 3264x^2 + 2848cx^2 - 6192c^2x^2 - 2272c^3x^2 \\ & + 1560c^4x^2 + 282c^5x^2 - 5184x^3 - 8640cx^3 + 5472c^2x^3 + 2496c^3x^3 - 660c^4x^3 \\ & - 84c^5x^3 - 3264x^4 + 4896cx^4 + 4896c^2x^4 - 2448c^3x^4 - 1020c^4x^4 + 306c^5x^4\end{aligned}$$

$$\begin{aligned}\frac{\partial g}{\partial x} = & 5184c + 768c^2 - 1824c^3 + 280c^4 + 132c^5 - 118c^6 + 768x + 6528cx + 2848c^2x \\ & - 4128c^3x - 1136c^4x + 624c^5x + 94c^6x + 24576x^2 - 15552cx^2 - 12960c^2x^2 \\ & + 5472c^3x^2 + 1872c^4x^2 - 396c^5x^2 - 42c^6x^2 - 13056x^3 \\ & - 13056cx^3 + 9792c^2x^3 + 6528c^3x^3 - 2448c^4x^3 - 816c^5x^3 + 204c^6x^3\end{aligned}$$

Upon solving the expressions $\frac{\partial g}{\partial c} = 0$ and $\frac{\partial g}{\partial x} = 0$ (by using MATHEMATICA 12.0), we obtain no critical in $(0, 2) \times (0, 1)$.

Therefore, there is no critical point for $y = 1$, $(c, x) \in (0, 2) \times (0, 1)$

D. Now considering the interior region of the parallelepiped i.e. $(0, 2) \times (0, 1) \times (0, 1)$.

Similiarly, upon solving $\frac{\partial \Phi}{\partial c} = 0$, $\frac{\partial \Phi}{\partial x} = 0$, $\frac{\partial \Phi}{\partial y} = 0$ (by using MATHEMATICA 12.0), we obtain no critical point in the region $(0, 2) \times (0, 1) \times (0, 1)$.

Therefore, $\Phi(c, x, y)$ has no critical point in the interior of the parallelepiped.

In review of cases **A**, **B**, **C** and **D**, we obtain

$$\max\Big\{\Phi(c, x, y) : c \in [0, 2], x \in [0, 1],\ y \in [0, 1]\Big\} = 8192. \tag{2.20}$$

From expression (2.18) and (2.20), we obtain

$$\left|H_{2,3}(f^{-1})\right| \le \frac{2}{45}. \tag{2.21}$$

Form $f_o(z)$, we obtain $= t_3 = -1/3, t_4 = 0, t_5 = 2/15$ and it follows the result. □

3. Concluding Remarks and Observations

The challenge of determining sharp estimates for second-order Hankel determinants, particularly when $r = 2$ and $n = 3$ in (1.2), defining $H_{r,n}(f)$, proves to be a formidable task, even more so than the case of $r = n = 2$. In our present investigation, we have successfully addressed this challenge by deriving sharp bounds for the second-order Hankel determinants $H_{2,3}(f^{-1})$, as well as $H_{2,2}(f^{-1})$ and $H_{2,1}(f^{-1})$, for functions f^{-1} within a specific subclass of normalized analytic functions residing in the open unit disk $\mathbb{D}$. These functions are the inverses of those belonging to the class of convex functions with respect to symmetric points.

In conclusion, with a view to motivating further researches along the lines of our present investigation, we have chosen to refer the interested reader to several related recent developments (see, for example, [4], [14], [18], [19],[16], [17], [20], [21], [24], [25], [26], [27], [28] and [29]) on bounds for the Hankel determinants $H_{r,n}(f)$ for many different values of the positive integers r and n.

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