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Necessary and Sufficient Tauberian Conditions for Logarithmic Summable Sequences in Two-Normed Spaces

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ABSTRACT: Our aim in this paper is to make a novel interpretation of the relation between the logarithmic summability method and convergence under the coverage of 2-normed spaces. In line with this aim, we introduce a necessary and sufficient Tauberian condition of Móricz-type for logarithmic summable sequences in these kinds of spaces. Following this, we investigate whether conditions designed as O-type such as the slow oscillation and Hardy-type conditions with respect to summability $(\ell, 1)$ due to the non-existence of relation of "order" in 2-normed spaces are the conditions needed for logarithmic summable sequences to be convergent.

Key Words: Tauberian theorems, logarithmic summability method, 2-normed spaces, slowly oscillating sequences with respect to summability $(\ell, 1)$, two-sided Tauberian conditions.

Contents

1. Introduction

One of the fundamental problems of functional analysis is to introduce such a metric on the space formed by specific objects that the convergence of a sequence of these objects corresponds to a given notion of convergence in a given Euclidean space. Despite the fact that the notion of distance confronting researchers at this point makes it possible to solve the mentioned problem for many spaces, there are still spaces which cannot be "distanciés" in the sense of Fréchet. As a solution proposal, Vulich [24] introduced a metric being more complicated than the distance and improved the theory of K-normed spaces via this metric. However, this theory did not attract so much attention as the mathematical structure which was put forward by Gähler [4] in order to obtain a suitable generalization of normed linear spaces. Within the scope of this structure, Gähler constructed a bivariate real-valued function on a linear space X satisfying specific properties and indicated that the space X equipped with the structure called as 2-norm is a locally convex topological linear space. In addition, Gähler presented an example of a 2-normed linear space of uncountable dimension which is not metrizable and so does not have any norm. Following this research, White [25] defined not only convergent and Cauchy sequences but also completeness and bilinear functionals in 2-normed spaces. In 1976, Iséki [9] committed to paper a brief survey including some outstanding results in studies of major contributors to the area. Finally, Gunawan and Mashadi [5] stated that any finite-dimensional 2-normed space with basis $\{u_1, \ldots, u_d\}$ is a normed space with the topology which squares with that generated by the norm $\|x\|_\infty = \max\{\|x, u_i\| : 1 \leq i \leq d\}$.

Being proceeded a long way in studies in the field of functional analysis paved the way for researchers getting into the act in the summability theory, as well. Initially, Şahiner et al. [21] presented $\mathcal{I}$-convergence in 2-normed spaces and investigated some new sequence spaces via 2-norm. Defining the concepts of statistical convergence and statistical Cauchy sequence in 2-normed spaces, Gürdal and Pehlivan [6]

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pointed out that some properties related to statistical convergence of sequences of real numbers are also valid for sequences in these spaces. Besides, they obtained a criterion for a sequence in 2-normed spaces to be a statistical Cauchy sequence. On the other side, Dutta [3] scrutinized the difference sequence spaces defined via 2-norm under the head of statistical convergence. Various convergence results have been also investigated in random and fuzzy 2-normed spaces. For example, Savaş [17] explored λ-statistical convergence, λ-statistical Cauchy sequences, and λ-statistical completeness in random 2-normed spaces. Similarly, Mohiuddine and Aiyub [12] examined lacunary statistical convergence, while Mohiuddine et al. [13] focused on ideal convergence in random 2-normed spaces. Furthermore, Mursaleen et al. [16] analyzed nonlinear operators and Fréchet derivatives in fuzzy 2-normed spaces, offering significant insights into their behavior. In parallel with the work of Savaş [17], Hazarika [8], focusing on the concepts of λ-statistical convergence, statistical λ-convergence and their characterizations in 2-normed spaces, is concerned with finding necessary and sufficient conditions to establish a relation between statistical convergence and λ-statistical convergence. Moreover, Hazarika proved that every bounded and λ-statistically convergent sequence is also statistically λ-convergent in 2-normed spaces. Finally, Belen and Yildirim [1] considered the concepts of the A-statistical convergence being an extension of that of statistical and the $A^{\mathcal{I}}$-statistical convergence formed by the usage of ideals in 2-normed spaces.

As it is seen, the whole studies executed in 2-normed spaces so far serve to be defined new types of concepts or sequence spaces. Herein, the purpose is to reflect the relationship among these concepts or sequence spaces in general. Unlike the studies mentioned above, Savaş and Sezer [18] presented some Tauberian theorems in 2-normed spaces that convergence follows from the Cesàro summability under Hardy-type condition $n\Delta s_n = O(1)$, Schmidt-type slow oscillation condition and more general condition involving the notion of deferred mean. In addition, they extended these classical Tauberian results to Hölder summability method. Following this research, Çanak et al. [2] introduced the weighted mean method of summability in 2-normed spaces and they revealed necessary and sufficient conditions and some conditions controlling O-oscillatory behaviour of a sequence to retrieve its convergence from its weighted mean method of summability. This paper, which has characteristics of being extensionality of the last two studies mentioned is constructed by considering the following headings for the logarithmic summability method in 2-normed spaces:

(a) Finding a necessary and sufficient condition for the logarithmic summability of a sequence.

(b) Determining certain subsets T_X of sequence space in 2-normed space X having the property: "A sequence $(s_n) \in T_X$ being logarithmic summable to $l \in X$ is also convergent to the same value." The theorem making such a situation possible is called a *Tauberian theorem,* and the condition allowing to be defined T_X is called a *Tauberian condition* (see [22]).

When it comes to the logarithmic summability method, it is firstly (to our knowledge) encountered with Hardy [7]. Drawing inspiration from Hardy's works in 1963, Ishiguro [10] proved that if a sequence (s_n) is logarithmic summable to l and $\omega_n^{(0)}(s) = o(1)$, where

$$\omega_n^{(0)}(s) = (n+1)\ell_{n-1}(s_n - s_{n-1}) \sim n \log n (s_n - s_{n-1}),$$

then it also converges to same value. In sequel, Kwee [11] demonstrated that necessary condition for convergence of a sequence which is logarithmic summable is

$$\liminf_{m\to\infty}(s_n - s_m) = 0 \quad \text{whenever } n > m \to \infty \text{ and } \frac{\log n}{\log m} \to 1. \tag{1.1}$$

Introducing the concept of slow oscillation of a sequence with respect to summability $(\ell, 1)$, being equivalent to condition (1.1), Móricz [14] established two results dealing with implication from the statistical logarithmic summability to convergence under conditions controlling O-oscillatory behaviour of sequence. Totur and Okur [23] investigated the logarithmic summability methods of numerical sequences and their applications such as Tauberian theorems. Using the sequence $(\omega_n^{(r)}(s))$ defined recursively instead of $(\omega_n^{(0)}(s))$, Sezer and Çanak [20] generated some Tauberian results based on this sequence.

In spite of the fact that the mentioned researches are milestones in this area, the main thing that urges us to do this research is the idea of carrying the result obtained by Móricz [15] a step further by extending to 2-normed spaces. Móricz formulated this result as follows:

Theorem 1.1 *If a sequence (s_n) of complex numbers is logarithmic summable to some $\zeta \in \mathbb{C}$ and slowly oscillating with respect to summability $(\ell, 1)$, then (s_n) is convergent to ζ.*

In this paper, we firstly indicate that Abelian theorems such as

$$s_n \to l \text{ implies } s_n \to l \ (\ell, 1) \quad \text{or} \quad s_n \to l \ (C, 1) \text{ implies } s_n \to l \ (\ell, 1)$$

hold true in 2-normed spaces, as well. We emphasize the existence of sequences which is logarithmic summable, but not Cesàro summable or convergent, independently of each other. Following these, we are interested in relations between the logarithmic summability method and ordinary convergence for sequences in 2-normed spaces. In accordance with this purpose, we derive some Tauberian conditions controlling O-oscillatory behaviour of a sequence in 2-normed spaces.

2. Preliminaries

In this section, we preface with basic definitions and notations in 2- normed spaces that will be needed throughout this paper. Subsequent to these definings, we present several examples of 2-normed spaces. We indicate the existence of implication from convergence to the logarithmic summability, viz. its regularity, in 2-normed spaces. In addition, we point out that the set of all logarithmic summable sequences includes that of Cesàro summable. We construct two examples to show that the reverse of asserted propositions is not true in general.

Let X be a real linear space of dimension greater than 1 and $\|\cdot,\cdot\|$ be a real-valued function on $X \times X$ satisfying the following properties

(P_1) $\|x, y\| = 0$ if and only if x and y are linearly dependent,

(P_2) $\|x, y\| = \|y, x\|$,

(P_3) $\|\gamma x, y\| = |\gamma|\|x, y\|$,

(P_4) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$

for every $x, y, z \in X$ and $\gamma \in \mathbb{R}$.

$\|\cdot,\cdot\|$ is said to be a 2-norm on X and the pair $(X, \|\cdot,\cdot\|)$ a linear 2-normed space.

It is immediately follows from the properties $(P_1) - (P_4)$ that the 2-norms confirm the properties $\|x, y\| \geq 0$ and $\|x, y + \gamma x\| = \|x, y\|$ for every $x, y \in X$ and $\gamma \in \mathbb{R}$, as well.

Some standart examples for linear 2-normed spaces most common in the literature are listed as follows.

(i) Let $X = \mathbb{R}^2$ be equipped with the 2-norm defined by

$$\|x, y\| = |x_1 y_2 - x_2 y_1|,$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Then, $(\mathbb{R}^2, \|\cdot,\cdot\|)$ is a 2-normed space. Herein, the 2-norm function geometrically represents the area of the usual parallelogram spanned by the two associated vectors.

(ii) Let $X = \mathbb{R}^2$ be equipped with the 2-norm defined by

$$\|x, y\| = |x_1 y_1 + x_2 y_2|,$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Then, $(\mathbb{R}^2, \|\cdot,\cdot\|)$ is a 2-normed space.

(iii) Let $X = \mathbb{R}^3$ be equipped with the 2-norm defined by

$$\|x, y\| = \max\{|x_1 y_2 - x_2 y_1|, |x_1 y_3 - x_3 y_1|, |x_2 y_3 - x_3 y_2|\},$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. Then, $(\mathbb{R}^3, \|\cdot,\cdot\|)$ is a 2-normed space.

(iv) Let $X = l^\infty$, the space of bounded sequences of real numbers, be equipped with the 2-norm defined by

$$\|x, y\| = \sup_{i\in\mathbb{N}} \sup_{j\in\mathbb{N}} |x_i y_j - x_j y_i|,$$

where $x = (x_1, x_2, x_3, \dots)$ and $y = (y_1, y_2, y_3, \dots)$. Then, $(l^\infty, \|\cdot,\cdot\|)$ is a 2-normed space.

(v) Let E^3 denote the three-dimensional Euclidean vector space and let $X = E^3$ be equipped with the 2-norm defined by

$$\|x, y\| = |x \times y| = abs \begin{vmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \left|(x_2y_3 - x_3y_2)^2 i + (x_3y_1 - x_1y_3)^2 j + (x_1y_2 - y_1x_2)^2 k\right|^{\frac{1}{2}}$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. Then, $(E^3, \|\cdot,\cdot\|)$ is a 2-normed space.

(vi) Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space of $\dim(X) \geq 2$ and let X be equipped with the 2-norm defined by

$$\|x, y\| = \begin{vmatrix} \langle x, x\rangle & \langle x, y\rangle \\ \langle y, x\rangle & \langle y, y\rangle \end{vmatrix}^{\frac{1}{2}}.$$

Then, $(X, \|\cdot,\cdot\|)$ is a 2-normed space.

A sequence (s_n) in a 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be convergent to $l \in X$ and denoted by $s_n \xrightarrow{\|\cdot,\cdot\|_X} l$ provided that for all $y \in X$

$$\lim_{n\to\infty} \|s_n - l, y\| = 0.$$

A sequence (s_n) in a 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be bounded provided that there exists a $C > 0$ such that for all $y \in X$

$$\|s_n, y\| \leq C, \quad n = 1, 2, \dots.$$

Let (s_n) be a sequence in 2-normed space $(X, \|\cdot,\cdot\|)$. The Cesàro means *(or called arithmetic means)* of (s_n) are defined by

$$\sigma_n = \frac{1}{n}\sum_{k=1}^{n} s_k, \quad n = 1, 2, \dots.$$

A sequence (s_n) is said to be Cesàro summable to $l \in X$ and denoted by $s_n \xrightarrow{\|\cdot,\cdot\|_X} l \ (C, 1)$ provided that (σ_n) is convergent to l, i.e., for all $y \in X$

$$\lim_{n\to\infty} \|\sigma_n - l, y\| = 0.$$

The logarithmic means *(or called the harmonic means)* of (s_n) are defined by

$$\tau_n = \frac{1}{\ell_n}\sum_{k=1}^{n} \frac{s_k}{k}, \quad \text{where } \ell_n = \sum_{k=1}^{n} \frac{1}{k} \sim \log n, \ n = 1, 2, \dots.$$

A sequence (s_n) is said to be logarithmic summable to $l \in X$ and denoted by $s_n \xrightarrow{\|\cdot,\cdot\|_X} l \ (\ell, 1)$ provided that (τ_n) is convergent to l, i.e., for all $y \in X$

$$\lim_{n\to\infty} \|\tau_n - l, y\| = 0.$$

The following theorem indicates the regularity of the logarithmic summability method in 2-normed space $(X, \|\cdot,\cdot\|)$.

Theorem 2.1 *Let (s_n) in $(X, \|\cdot,\cdot\|)$ be convergent to $l \in X$. Then, (τ_n) of its logarithmic means is also convergent to l.*

Proof: Assume the convergence of (s_n) to $l \in X$. Then for each $\epsilon > 0$ and $y \in X$, there exists an integer $n_0 > 0$ such that $\|s_n - l, y\| \leq \epsilon/2$ whenever $n > n_0$ and $C > 0$ such that $\|s_n - l, y\| \leq C$ whenever $n \leq n_0$. Therefore, for each $y \in X$, we obtain

$$\begin{aligned}
\|\tau_n - l, y\| &= \left\| \frac{1}{\ell_n} \sum_{k=1}^{n} \frac{s_k}{k} - \frac{1}{\ell_n} \sum_{k=1}^{n} \frac{l}{k}, y \right\| \\
&= \left\| \frac{1}{\ell_n} \sum_{k=1}^{n} \frac{s_k - l}{k}, y \right\| \\
&\leq \frac{1}{\ell_n} \sum_{k=1}^{n} \frac{1}{k} \|s_k - l, y\| \\
&= \frac{1}{\ell_n} \sum_{k=1}^{n_0} \frac{1}{k} \|s_k - l, y\| + \frac{1}{\ell_n} \sum_{k=n_0+1}^{n} \frac{1}{k} \|s_k - l, y\| \\
&\leq \frac{C\ell_{n_0}}{\ell_n} + \frac{\epsilon}{2}.
\end{aligned}$$

Since $\frac{\ell_{n_0}}{\ell_n} \to 0$ as $n \to \infty$, there exists an integer $n_1 > 0$ satisfying $\frac{C\ell_{n_0}}{\ell_n} \leq \frac{\epsilon}{2}$ whenever $n > n_1$. In conclusion, we reach $\|\tau_n - l, y\| \leq \epsilon$ whenever $n > \max\{n_0, n_1\}$, which means that (s_n) is logarithmic summable to l. □

In spite of the fact that convergence of a sequence in a 2-normed space implies its logarithmic summability to the same number, the opposite of this proposition is not always true. Now, we construct an example indicating this situation.

Example 2.1 *Let $X = \mathbb{R}^2$ be equipped with the 2-norm defined by $\|x, y\| = |x_1 y_2 - x_2 y_1|$ where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Consider the sequence $(s_n) \subset X$ given by*

$$s_n = \left(\sum_{p=0}^{n} (-1)^{p+1} p, 0 \right), \quad n = 1, 2, \ldots.$$

Then, the sequence (τ_n) of the logarithmic means of (s_n) is as follows:

$$\tau_n = \begin{cases} \left(\frac{1}{4\ell_n} \log\left(2(n-1)\right), 0 \right), & \text{if } n \text{ is even}, \\ \left(\frac{1}{4\ell_n} \log\left(2(n-2)\right) + \frac{n+1}{2n\ell_n}, 0 \right), & \text{if } n \text{ is odd}. \end{cases}$$

Case I. *Let n be even. Then, we obtain for every $y = (y_1, y_2) \in \mathbb{R}^2$*

$$\begin{aligned}
\lim_{n\to\infty} \|\tau_n - l, y\| &= \lim_{n\to\infty} \left\| \left(\frac{1}{4\ell_n} \log\left(2(n-1)\right), 0 \right) - \left(\frac{1}{4}, 0 \right), (y_1, y_2) \right\| \\
&= \lim_{n\to\infty} \left\| \left(\frac{1}{4\ell_n} \log\left(2(n-1)\right) - \frac{1}{4}, 0 \right), (y_1, y_2) \right\| \\
&= \lim_{n\to\infty} \left| \frac{y_2}{4\ell_n} \log\left(2(n-1)\right) - \frac{y_2}{4} \right| = 0.
\end{aligned}$$

Case II. *Let n be odd. Then, we obtain for all $y = (y_1, y_2) \in \mathbb{R}^2$*

$$\begin{aligned}
\lim_{n\to\infty} \|\tau_n - l, y\| &= \lim_{n\to\infty} \left\| \left(\frac{1}{4\ell_n} \log\left(2(n-2)\right) + \frac{n+1}{2n\ell_n}, 0 \right) - \left(\frac{1}{4}, 0 \right), (y_1, y_2) \right\| \\
&= \lim_{n\to\infty} \left\| \left(\frac{1}{4\ell_n} \log\left(2(n-2)\right) + \frac{n+1}{2n\ell_n} - \frac{1}{4}, 0 \right), (y_1, y_2) \right\| \\
&= \lim_{n\to\infty} \left| \frac{y_2}{4\ell_n} \log\left(2(n-2)\right) + \frac{(n+1)y_2}{2n\ell_n} - \frac{y_2}{4} \right| = 0.
\end{aligned}$$

Therefore, it follows from the definition of logarithmic summability that (s_n) *is logarithmic summable to* $(\frac{1}{4},0)$. *On the other hand, one can easily check that the sequence* (s_n) *is*

$$s_n = \begin{cases} \left((-1)^{n+1}\frac{n}{2},0\right), & \text{if } n \text{ is even,} \\ \left((-1)^{n+1}\frac{n+1}{2},0\right), & \text{if } n \text{ is odd.} \end{cases}$$

If (s_n) *is convergent, then its limit should be* $(\frac{1}{4},0)$. *However, for* $y=(0,1)\in\mathbb{R}^2$, *we find*

$$\begin{aligned} \lim_{n\to\infty}\|s_n-l,y\| &= \lim_{n\to\infty}\left\|\left((-1)^{n+1}\frac{n+1}{2}-\frac{1}{4},0\right),(y_1,y_2)\right\| \\ &= \lim_{n\to\infty}\left|(-1)^{n+1}\frac{(n+1)y_2}{2}-\frac{y_2}{4}\right| \\ &= \lim_{n\to\infty}\left|(-1)^{n+1}\frac{n+1}{2}-\frac{1}{4}\right| \\ &= \infty\neq 0, \end{aligned}$$

which means that (s_n) *is not convergent to any value in* $\mathbb{R}^2$.

Comparing the Cesàro summability method with the logarithmic one, we reach the following theorem.

Theorem 2.2 *If a sequence* (s_n) *in* $(X,\|\cdot,\cdot\|)$ *is Cesàro summable to* $l\in X$, *then it is also logarithmic summable to* l.

Proof: Assume that (s_n) is Cesàro summable to $l\in X$. For each $n\in\mathbb{N}$, we have

$$\tau_n=\frac{1}{\ell_n}\sigma_n+\frac{1}{\ell_n}\sum_{k=1}^{n}\frac{\sigma_{k-1}}{k}. \tag{2.1}$$

Since the first term on the right-hand side of (2.1) converges to zero and the second term converges to l by virtue of Theorem 2.1, the sequence (τ_n) converges to l. □

The following example demonstrate that there is a sequence in $(X,\|\cdot,\cdot\|)$ being logarithmic summable, but not Cesàro summable.

Example 2.2 *Let* $X=\mathbb{R}^2$ *be equipped with the 2-norm defined by* $\|x,y\|=|x_1y_1+x_2y_2|$ *where* $x=(x_1,x_2)$ *and* $y=(y_1,y_2)$. *Consider the sequence* $(s_n)\subset X$ *given by*

$$s_n=\left(1+(-1)^n(2n+3),3+(-1)^{n+1}(n+7)\right),\quad n=1,2,\ldots.$$

Then, the sequence (τ_n) *of the logarithmic means of* (s_n) *is as follows:*

$$\tau_n=\begin{cases}\left(1-\frac{3}{2\ell_n}\log\frac{4(n-1)}{n},3+\frac{7}{2\ell_n}\log\frac{4(n-1)}{n}\right), & \text{if } n \text{ is even,}\\ \left(1+\frac{-2n-3}{n\ell_n}-\frac{3}{2\ell_n}\log\frac{4(n-2)}{n-1},3+\frac{n+7}{n\ell_n}+\frac{7}{2\ell_n}\log\frac{4(n-2)}{n-1}\right), & \text{if } n \text{ is odd.}\end{cases}$$

Case I. *Let* n *be even. Then, we obtain for every* $y=(y_1,y_2)\in\mathbb{R}^2$

$$\begin{aligned} \lim_{n\to\infty}\|\tau_n-l,y\| &= \lim_{n\to\infty}\left\|\left(1-\frac{3}{2\ell_n}\log\frac{4(n-1)}{n},3+\frac{7}{2\ell_n}\log\frac{4(n-1)}{n}\right)-(1,3),(y_1,y_2)\right\| \\ &= \lim_{n\to\infty}\left\|\left(-\frac{3}{2\ell_n}\log\frac{4(n-1)}{n},\frac{7}{2\ell_n}\log\frac{4(n-1)}{n}\right),(y_1,y_2)\right\| \\ &= \lim_{n\to\infty}\left|-\frac{3y_1}{2\ell_n}\log\frac{4(n-1)}{n}+\frac{7y_2}{2\ell_n}\log\frac{4(n-1)}{n}\right| \\ &= \lim_{n\to\infty}\left|\frac{-3y_1+7y_2}{2\ell_n}\log\frac{4(n-1)}{n}\right|=0. \end{aligned}$$

Case II. *Let n be odd. Then, we obtain for all $y = (y_1, y_2) \in \mathbb{R}^2$*

$$\begin{aligned} \lim_{n\to\infty} \|\tau_n - l, y\| &= \lim_{n\to\infty} \left\| \left(\frac{-2n-3}{n\ell_n} - \frac{3}{2\ell_n} \log \frac{4(n-2)}{n-1}, \frac{n+7}{n\ell_n} + \frac{7}{2\ell_n} \log \frac{4(n-2)}{n-1} \right), (y_1, y_2) \right\| \\ &= \lim_{n\to\infty} \left| \frac{(-2n-3)y_1}{n\ell_n} - \frac{3y_1}{2\ell_n} \log \frac{4(n-2)}{n-1} + \frac{(n+7)y_2}{n\ell_n} + \frac{7y_2}{2\ell_n} \log \frac{4(n-2)}{n-1} \right| \\ &= \lim_{n\to\infty} \left| \frac{-2y_1 + y_2}{\ell_n} + \frac{7y_2 - 3y_1}{n\ell_n} + \frac{7y_2 - 3y_1}{2\ell_n} \log \frac{4(n-2)}{n-1} \right| = 0. \end{aligned}$$

Therefore, it follows from the definition of logarithmic summability that (s_n) is logarithmic summable to $(1, 3)$. On the other hand, one can easily check that the sequence (σ_n) of the arithmetic means of (s_n) is

$$\sigma_n = \left(1 + (-1)^n + \frac{2}{n} \left((-1)^n - 1 \right), -\frac{1}{2}(-1)^n + \frac{15}{4n} \left(1 - (-1)^n \right) + 3 \right)$$

for all $n \in \mathbb{N}$. In the alternative to this representation, we can write

$$\sigma_n = \begin{cases} \left(2, \frac{5}{2}\right), & \text{if } n \text{ is even,} \\ \left(-\frac{4}{n}, \frac{7n+15}{2n}\right), & \text{if } n \text{ is odd.} \end{cases}$$

If (σ_n) is convergent, then its limit should be $(1, 3)$. However, for $y = (0, 2) \in \mathbb{R}^2$, we have

$$\begin{aligned} \lim_{n\to\infty} \|\sigma_n - l, y\| &= \lim_{n\to\infty} \left\| \left(-\frac{n+4}{n}, \frac{n+15}{2n} \right), (y_1, y_2) \right| \\ &= \lim_{n\to\infty} \left| -\frac{(n+4)y_1}{n} + \frac{(n+15)y_2}{2n} \right| \\ &= \lim_{n\to\infty} \left| \frac{2(n+15)}{2n} \right| \\ &= 1 \neq 0, \end{aligned}$$

which means that (σ_n) is not convergent and hence (s_n) is not Cesàro summable to any value in $\mathbb{R}^2$.

In this paper, our purpose is to put a novel interpretation of the relation between the logarithmic summability method and convergence under the coverage of 2-normed spaces. In line with this purpose, we introduce a necessary and sufficient Tauberian condition of Móricz-type for logarithmic summable sequences in these kinds of spaces. Although the process in Tauberian theory is based on weakening conditions imposed on sequences, since there is no relation of "order" in 2-normed spaces, we design here our conditions as two-sided Tauberian conditions rather than one-sided Tauberian conditions associated with Ishiguro and Kwee for the logarithmic summability method. After defining the concepts of slow oscillation and Hardy-type conditions with respect to the summability $(\ell, 1)$ in 2-normed spaces, we investigate whether these are the conditions needed for logarithmic summable sequences to be convergent.

3. Main Results

In this section, we derive a Tauberian theorem dealing with the implication from the logarithmic summability to convergence in 2-normed spaces. As special cases of our main theorem, we establish some classical Tauberian results in 2-normed spaces. The proofs of the theorem and corollaries are included in the next section for completeness.

Theorem 3.1 *Let (s_n) be a sequence in $(X, \|\cdot, \cdot\|)$ that is logarithmic summable to $l \in X$. Then (s_n) is convergent to l if and only if one of the following two conditions is satisfied for every $y \in X$*

$$\inf_{\lambda>1} \limsup_{n\to\infty} \left\| \frac{1}{\ell_{[n^\lambda]} - \ell_n} \sum_{k=n+1}^{[n^\lambda]} \frac{s_k - s_n}{k}, y \right\| = 0, \tag{3.1}$$

$$\inf_{0<\lambda<1} \limsup_{n\to\infty} \left\| \frac{1}{\ell_n - \ell_{[n^\lambda]}} \sum_{k=[n^\lambda]+1}^{n} \frac{s_n - s_k}{k}, y \right\| = 0 \tag{3.2}$$

where $[n^\lambda]$ *denotes the integer part of* n^λ.

Remark 3.1 Following [19], we say that (s_n) is slowly oscillating with respect to summability $(\ell, 1)$ in 2-norm provided that for all $y \in X$,

$$\inf_{\lambda>1} \limsup_{n\to\infty} \max_{n<k\leq[n^\lambda]} \|s_k - s_n, y\| = 0. \tag{3.3}$$

Using ϵ's and λ's this is: For given $y \in X$ and $\epsilon > 0$, there exist $n_0 > 0$ and $\lambda > 1$ such that $\|s_k - s_n, y\| \leq \epsilon$ whenever $n_0 < n < k \leq [n^\lambda]$.
An equivalent reformulation of (3.3) is as follows:

$$\inf_{0<\lambda<1} \limsup_{n\to\infty} \max_{[n^\lambda]<k\leq n} \|s_n - s_k, y\| = 0. \tag{3.4}$$

Using ϵ's and λ's this is: For given $y \in X$ and $\epsilon > 0$, there exist $n_0 > 0$ and $\lambda > 1$ such that $\|s_n - s_k, y\| \leq \epsilon$ whenever $n_0 < [n^\lambda] < k \leq n$.

Considering Remark 3.1, we attain 2-normed analogues of classical Tauberian conditions in [19] and [7], respectively.

Corollary 3.1 *If a sequence* (s_n) *in* $(X, \|\cdot,\cdot\|)$ *is logarithmic summable to* $l \in X$ *and slowly oscillating with respect to summability* $(\ell, 1)$ *in* 2*-norm, then it is convergent to* l.

Corollary 3.2 *If a sequence* (s_n) *in* $(X, \|\cdot,\cdot\|)$ *is logarithmic summable to* $l \in X$ *and if for all* $y \in X$ *there exists* $C > 0$ *such that*

$$n \log n \|s_n - s_{n-1}, y\| \leq C, \tag{3.5}$$

then (s_n) *is convergent to* l.

4. An Auxiliary Result

In this section, we give a lemma used in the proofs of the main results.

Lemma 4.1 *If a sequence* (s_n) *in* $(X, \|\cdot,\cdot\|)$ *is logarithmic summable to* $l \in X$*, then for each* $y \in X$

$$\lim_{n\to\infty} \left\| \frac{1}{\ell_{[n^\lambda]} - \ell_n} \sum_{k=n+1}^{[n^\lambda]} \frac{s_k}{k} - l, y \right\| = 0 \quad \textit{for all} \quad \lambda > 1 \tag{4.1}$$

and

$$\lim_{n\to\infty} \left\| \frac{1}{\ell_n - \ell_{[n^\lambda]}} \sum_{k=[n^\lambda]+1}^{n} \frac{s_k}{k} - l, y \right\| = 0 \quad \textit{for all} \quad 0 < \lambda < 1. \tag{4.2}$$

Proof: Denote the sequences $(\eta_n^>)$ for $\lambda > 1$ and $(\eta_n^<)$ for $0 < \lambda < 1$ of moving logarithmic averages of (s_n) by

$$\eta_n^> = \frac{1}{\ell_{[n^\lambda]} - \ell_n} \sum_{k=n+1}^{[n^\lambda]} \frac{s_k}{k} \quad \text{and} \quad \eta_n^< = \frac{1}{\ell_n - \ell_{[n^\lambda]}} \sum_{k=[n^\lambda]+1}^{n} \frac{s_k}{k},$$

respectively. Consider the case $\lambda > 1$. Then, we obtain for each $y \in X$

$$\begin{aligned}
\left\|\eta_n^{>} - l, y\right\| &= \left\|\eta_n^{>} + \tau_{[n^\lambda]} - \tau_{[n^\lambda]} - l, y\right\| \\
&= \left\| \frac{1}{\ell_{[n^\lambda]} - \ell_n} \sum_{k=1}^{[n^\lambda]} \frac{s_k}{k} - \frac{1}{\ell_{[n^\lambda]} - \ell_n} \sum_{k=1}^{n} \frac{s_k}{k} + \frac{1}{\ell_{[n^\lambda]}} \sum_{k=1}^{[n^\lambda]} \frac{s_k}{k} - \frac{1}{\ell_{[n^\lambda]}} \sum_{k=1}^{[n^\lambda]} \frac{s_k}{k} - l, y \right\| \\
&= \left\| \frac{\ell_n}{\ell_{[n^\lambda]} - \ell_n} \frac{1}{\ell_{[n^\lambda]}} \sum_{k=1}^{[n^\lambda]} \frac{s_k}{k} - \frac{\ell_n}{\ell_{[n^\lambda]} - \ell_n} \frac{1}{\ell_n} \sum_{k=1}^{n} \frac{s_k}{k} + \frac{1}{\ell_{[n^\lambda]}} \sum_{k=1}^{[n^\lambda]} \frac{s_k}{k} - l, y \right\| \\
&\leq \frac{\ell_n}{\ell_{[n^\lambda]} - \ell_n} \left\|\tau_{[n^\lambda]} - \tau_n, y\right\| + \left\|\tau_{[n^\lambda]} - l, y\right\|.
\end{aligned}$$

Therefore, we arrive

$$\left\|\eta_n^{>} - l, y\right\| \leq \frac{\ell_n}{\ell_{[n^\lambda]} - \ell_n} \left\|\tau_{[n^\lambda]} - \tau_n, y\right\| + \left\|\tau_{[n^\lambda]} - l, y\right\|. \tag{4.3}$$

Because we have

$$\frac{\ell_n}{\ell_{[n^\lambda]} - \ell_n} \leq \frac{2}{\lambda - 1} \tag{4.4}$$

for large enough n, (4.1) follows from (4.3), (4.4) and the logarithmic summability of (s_n) to l. The proof of (4.2) can be obtained similarly as the proof of (4.1). □

5. Proofs of Main Results

Proof of Theorem 3.1: *Necessity.* Suppose that (s_n) converges to l. For any $\lambda > 1$, by virtue of Lemma 4.1, we attain for every $y \in X$

$$\lim_{n\to\infty} \left\| \frac{1}{\ell_{[n^\lambda]} - \ell_n} \sum_{k=n+1}^{[n^\lambda]} \frac{s_k - s_n}{k}, y \right\| \leq \lim_{n\to\infty} \left\| \frac{1}{\ell_{[n^\lambda]} - \ell_n} \sum_{k=n+1}^{[n^\lambda]} \frac{s_k}{k} - l, y \right\| + \lim_{n\to\infty} \left\|s_n - l, y\right\| = 0.$$

In other words, we obtain a condition that is even stronger than (3.1). In the same manner, given any $0 < \lambda < 1$, we find

$$\lim_{n\to\infty} \left\| \frac{1}{\ell_n - \ell_{[n^\lambda]}} \sum_{k=[n^\lambda]+1}^{n} \frac{s_n - s_k}{k}, y \right\| = 0,$$

that is stronger than (3.2).

Sufficiency. Assume that (3.1) is satisfied, and that $y \in X$ be arbitrarily fixed. Hence, for all $\epsilon > 0$ there exists $\lambda > 0$ satisfying

$$\limsup_{n\to\infty} \left\| \frac{1}{\ell_{[n^\lambda]} - \ell_n} \sum_{k=n+1}^{[n^\lambda]} \frac{s_k - s_n}{k}, y \right\| \leq \epsilon. \tag{5.1}$$

By virtue of (5.1), the logarithmic summability of (s_n) and Lemma 4.1, we obtain for all $y \in X$

$$\limsup_{n\to\infty} \left\|s_n - l, y\right\| \leq \limsup_{n\to\infty} \left\| \frac{1}{\ell_{[n^\lambda]} - \ell_n} \sum_{k=n+1}^{[n^\lambda]} \frac{s_k}{k} - l, y \right\| + \limsup_{n\to\infty} \left\| \frac{1}{\ell_{[n^\lambda]} - \ell_n} \sum_{k=n+1}^{[n^\lambda]} \frac{s_k - s_n}{k}, y \right\| \leq \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we reach the convergence of (s_n) to l.
A similar proof can be done in the case (3.2) holds. □

Proof of Corollary 3.1: Assume that (s_n) is slowly oscillating with respect to summability $(\ell,1)$ in 2-norm. Then, we have for any given $y\in X$

$$\left\|\frac{1}{\ell_{[n^\lambda]}-\ell_n}\sum_{k=n+1}^{[n^\lambda]}\frac{s_k-s_n}{k},y\right\|\le\frac{1}{\ell_{[n^\lambda]}-\ell_n}\sum_{k=n+1}^{[n^\lambda]}\frac{1}{k}\left\|s_k-s_n,y\right\|\le\max_{n<k\le[n^\lambda]}\left\|s_k-s_n,y\right\|.$$

Taking the lim sup of both sides of the inequality above as $n\to\infty$, we obtain for every $y\in X$

$$\limsup_{n\to\infty}\left\|\frac{1}{\ell_{[n^\lambda]}-\ell_n}\sum_{k=n+1}^{[n^\lambda]}\frac{s_k-s_n}{k},y\right\|\le\limsup_{n\to\infty}\max_{n<k\le[n^\lambda]}\left\|s_k-s_n,y\right\|.$$

Therefore, we conclude

$$\inf_{\lambda>1}\limsup_{n\to\infty}\left\|\frac{1}{\ell_{[n^\lambda]}-\ell_n}\sum_{k=n+1}^{[n^\lambda]}\frac{s_k-s_n}{k},y\right\|=0.$$

The proof follows from Theorem 3.1. □

Proof of Corollary 3.2: Let (3.5) be satisfied for large enough n, say $n>N$. For any $\epsilon>0$, choose $\lambda=\exp(\epsilon/C)$. If $N<n<k\le[n^\lambda]$ or equivalently $\log N<\log n<\log k\le\lambda\log n$, then we have

$$\begin{aligned}\left\|s_k-s_n,y\right\| &= \left\|\sum_{j=n+1}^{k}(s_j-s_{j-1}),y\right\|\\ &\le \sum_{j=n+1}^{k}\left\|s_j-s_{j-1},y\right\|\\ &\le C\sum_{j=n+1}^{k}\frac{1}{j\log j}\\ &\le C\log\left(\frac{\log k}{\log n}\right)\le C\log\lambda=\epsilon.\end{aligned}$$

In conclusion, the proof follows from Corollary 3.1. □

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