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Geodetic number and domination number of $\Gamma(\mathbf{R}(+)\mathbf{M})$

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ABSTRACT: This article investigates two important parameters of graph theory, the geodetic number and the domination number, in terms of zero divisor graphs for idealization rings $R(+)M$: the study offers a systematic analysis of these parameters, considering cases where R is an entirely integral domain and, in contrast, it is not. For the geodetic number, we give explicit formulas in several cases, including instances such as when R is an entirely consistent domain but also to certain constructions of $Z_N(+)Z_M$ where N and M are related as prime powers. Similarly, for the domination number, we obtain exact values under various algebraic conditions. These results reveal interesting ties between the algebraic structure of the idealization ring and its associated zero divisor graph's geometric properties.

Key Words: The idealization rings **R**, Domination number, Geodetic number, Zero-divisor graph.

Contents

1. Introduction

The paper points out new aspects and makes a significant contribution to the field of research for the relationship of number theory and the graphs associated with some algebraic rings.
Graph-theoretic representations for algebraic structures have in recent years proven invaluable tools to grasp their properties and relationships. One area that has been particularly fruitful is the study of zero-divisor graphs associated with rings, including their extensions. Here we take up the idealization. Ring $R(+)M$, where R is a commutative ring with 1 and M an $R-$module.
The idealized set $R(+)M$ is defined by all points (r_1, n_1) that r_1 belong to R and n_1 belong to M. Their addition is defined as $(r_1, n_1) + (r_2, n_2) = (r_1 + r_2, n_1 + n_2)$, and their multiplication as $(r_1, n_1)(r_2, n_2) = (r_1r_2, r_1n_2 + r_2n_1)$. In terms of these two operations, when it $R(+)M$ is realized as a commutative ring with unity. The zero-divisor graph of $R(+)M$, denoted by $\Gamma(R(+)M)$, is the set of all non-zero zero-divisors; that is to say, it consists only of vertices. In addition, if two different points (not both zero) (a, b), (c, d) are joined by an edge, their product is $(0, 0)$.
This article considers all graphs $\mathbf{G}$ as undirected simple graphs whose vertices are called $V(\mathbf{G})$.Let v_1

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and v_2 be two vertices in $V(\mathbf{G})$. Along with the $v_1 - v_2$ geodesic which is the shortest possible path between these two vertices, we define $I_G[v_1,\ v_2]$ to be the set of all vertices in $V(\mathbf{G})$ lying on the $v_1 - v_2$ geodesic and containing v_1 and v_2. The geodetic number $\mathbf{G}$ is any subset S of G which is such that if $I[S] = V(\mathbf{G})$ where $I[S] = \bigcup_{v_1,v_2 \in S} I_G[v_1,\ v_2]$. To each graph $\mathbf{G}$ we assign a number, $g(\mathbf{G})$ which is known as the geodetic number $\mathbf{G}$ and is the minimum number in a geodetic set for $\mathbf{G}$.
We exemplify two fundamental graphic parameters in this paper: the geodetic number and the domination number. If there exists a set S of smallest cardinality such that every vertex will be connected by a S-shortest path to S, the geodetic number of a graph, it is defined as this number. Conversely, if the smallest set whose boundaries include all vertices in the graph consists of $\emptyset$, it is called the domination number for that set. These two numbers provide important structural information about graphs and for algebraic constructions based upon them.
We can trace the evolution of zero-divisor graph theory from Beck's [10] work on ring coloring in 1988 to its present formulation in a straightforward manner.and in 1988 he did not innovate content In 1993, Anderson Neseer [8] expanded the concepts of Beck"s ring coloring even further. And in 1999 Anderson Livingston [9] made the crucial modification of focusing solely on nonzero zero-divisors as vertices. Another advance in this theory was made in 2006 by Axtell and Stickles [7]. There they gave complete characterizations of zero-divisor graphs for commutative rings, establishing important structural properties and relations between the algebraic properties of rings on one hand and their corresponding graph structures on the other. Their collective work made zero-divisor graphs into an essential tool for studying ring structure, showing how ring properties can be deduced from graph-theoretic methods.

2. Preliminaries

In this section, we present some basic results and lemmas that are needed in this article to study the geodetic number.
Al-Labadi in [1-6], she studied properties of the zero-divisor graph of the idealization ring. $\mathbf{Z}_N(+)\mathbf{Z}_M$. Now, the elements of the zero-divisor graph of the idealization ring are $\mathbf{Z}_N(+)\mathbf{Z}_M = \{(0,t) : t \in \mathbf{Z}_M^*\} \bigcup \{(a,l) : a \in \mathbf{Z}_N^*, l \in \mathbf{Z}_M,\ gcd(a,N) \neq 1 \ or \ gcd(a,M) \neq 1\}$.
The following notation set is used to simplify the proof in the rest of the paper.

If it N is a power of a prime, i.e., $N = p^r$ and $M = p$ where it p is a prime number.
$L_0 = \{(0,1),(0,2),...,(0,p-1)\}$ and $|L_0| = M - 1$.
$L_s = \{(kp^s,t) : t \in \mathbf{Z}_M,\ gcd(k,p^{r-s}) = 1\}$ for $1 \leq s \leq r-1$ and $|L_s| = M\phi(p^{r-s})$, where ϕ is an Euler function.

If N is product of distinct two different primes such that $N = p_1p_2$ where p_1 and p_2 are distinct prime numbers, $p_1 < p_2$ and $M = p_1$.
$L_{0^*} = \{(0,1),(0,2),...,(0,p_1-1)\}$ and $|L_{0^*}| = p_1 - 1$.
$L_{1^*} = \{(kp_1,t) : t \in \mathbf{Z}_{p_1},\ gcd(k,p_2) = 1\}$ and $|L_{1^*}| = p_1\phi(p_2) = p_1(p_2-1)$.
$L_{1^{**}} = \{(kp_2,t) : t \in \mathbf{Z}_{p_1},\ gcd(k,p_1) = 1\}$ and $|L_1^{**}| = p_1\phi(p_1) = p_1(p_1-1)$.

3. Simple Example of Idealization Ring $R(+)M$

To obtain a rudimentary understanding of ring idealizations, we are best served by looking at some concrete examples. For example, consider the simple case where $R = \mathbf{Z}$ (the ring of all integers) and $M = \mathbf{Z}_6$ (the ring of all integers modulo 6). The idealization ring $R(+)M = \mathbf{Z}(+)\mathbf{Z}_6$ consists of ordered pairs (a,m) where $a \in \mathbf{Z}$ and $m \in \mathbf{Z}_6$. This construction presents rich algebraic structure through its two basic operations: addition, and multiplication. These operations keep intact the ring structure of $\mathbf{Z}$ and the module structure of $\mathbf{Z}_6$, while at the same time generating new algebraic properties that are characteristic of such constructions. Because there are zero-divisors here and the interplay between elements of R, as a ring, and elements of M, as a module, is so pronounced, this example is particularly instructive for getting a grip on the more general notion of idealization rings.
Let's consider the following example:

Let $R = \mathbf{Z}$ and $M = \mathbf{Z}_6$. Then $R(+)M = \mathbf{Z}(+)\mathbf{Z}_6 = \{(a,m) : a \in \mathbf{Z}, m \in \mathbf{Z}_6\}$.

1. Addition:

$$(a_1,m_1) + (a_2,m_2) = (a_1 + a_2, m_1 + m_2).$$

2. Multiplication:

$$(a_1, m_1)(a_2, m_2) = (a_1a_2, a_1m_2 + a_2m_1).$$

3. Notation:

1. Zero element for the ring $\mathbf{Z}(+)\mathbf{Z}_6$: $(0, 0)$

2. Unit element for the ring $\mathbf{Z}(+)\mathbf{Z}_6$: $(1, 0)$

3. Additive inverse of (a, m) the ring $\mathbf{Z}(+)\mathbf{Z}_6$ is $(-a, 6 - m)$

4. The elements of the zero-divisor graph $\mathbf{Z}(+)\mathbf{Z}_6$ have the form:

$$\mathbf{Z}(+)\mathbf{Z}_6 = \{(0, m) : m = 0, 1, 2, 3, 4, 5\}$$

$$= \{(0, 1), (0, 2), (0, 3), (0, 4), (0, 5)\}.$$

4. Geodetic Sets and Geodetic Number of Zero-Divisor Graph $\Gamma(\mathbf{R}(+)\mathbf{M})$ where R is an integral domain

The study of geodesic sets and geodesic numbers for zero divisors gives important information on the underlying algebraic structure of idealization rings. When R is an integral domain, the zero divisor graph $\Gamma(R(+)M)$ has special properties, which make the study of these geodesic sets particularly interesting. A geodesic set $\Gamma(R(+)M)$ is a subset S of vertices such that every vertex in the graph lies on some shortest path between two vertices in S; and the geodesic number is the minimal cardinality of such a set. This investigation is significant in that it links the graph-theoretic property of geodesic sets to the algebraic structure of zero divisors in $R(+)M$. Under the condition that it R is an integral domain, all zero divisors in $R(+)M$ are formed by the module structure of M, and so this study of geodesic sets can be guided along easier paths. Our investigation reveals how the module structure of induced shortest paths, and in turn what the number and makeup of least geodesic sets are, are influenced by the module structure and formation of these graphs. However, as this relationship becomes clear, it can not only deepen our understanding of zero divisor graphs but also afford new tools to explore the structure of idealization rings through such properties on their graphs.

Definition 4.1 *The annihilator of $\boldsymbol{M}$ is denoted by $ann(\boldsymbol{M})$. Which is the set of all elements $a \in \boldsymbol{R}$ such that $am = 0$ for all $m \in \boldsymbol{M}$.*

Now, we first study the geodetic number when the ring $\mathbf{R}$ is an integral domain.

Theorem 4.1 *Let $\boldsymbol{M}$ be an $\boldsymbol{R}-$module and let $\boldsymbol{R}$ be an integral domain. Then we have the following:*

- ***Case 1:*** *If $ann(\boldsymbol{M}) = 0$, then the geodetic number of $\Gamma(\boldsymbol{R}(+)\boldsymbol{M})$ is $g(\Gamma(\boldsymbol{R}(+)\boldsymbol{M})) = |\boldsymbol{M}| - 1$.*

- ***Case 2:*** *If $ann(\boldsymbol{M}) \neq 0$, then the geodetic number of $\Gamma(\boldsymbol{R}(+)\boldsymbol{M})$ is $g(\Gamma(\boldsymbol{R}(+)\boldsymbol{M})) = (|ann(\boldsymbol{M})| - 1)|\boldsymbol{M}|$.*

Proof.

- **Case 1**: If $ann(\mathbf{M}) = 0$, then $Z^*(\mathbf{R}(+)\mathbf{M}) = \{(0, m) : m \in \mathbf{M}^*\}$ which is a complete. Thus the geodetic set is $\{(0, m) : m \in M^*\}$ where the geodetic number is $g(\Gamma(\mathbf{R}(+)\mathbf{M})) = |\mathbf{M}^*| = |\mathbf{M}| - 1$, see the following figure 1.

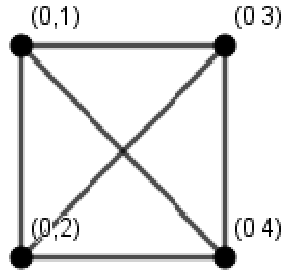

Figure 1: $\Gamma(\mathbf{Z}_5(+)\mathbf{Z}_5)$.

- **Case 2**: If $ann(\mathbf{M}) \neq 0$, then $Z^*(\mathbf{R}(+)\mathbf{M}) = \{(0, m_1), (a_i, m_2) : \ a_i \in \ ann(\mathbf{M}) \setminus \{0\}, \ m_1 \in \mathbf{M} \setminus \{0\} \ , m_2 \in \mathbf{M}\}$. So, the geodetic set is $\{(a, m) : \ a \in \ ann(\mathbf{M}) \setminus \{0\}, \ m \in M\}$. Where the geodetic number is $g(\Gamma(\mathbf{R}(+)\mathbf{M})) = |\mathbf{M}|(|ann(\mathbf{M})| - 1)$.
  See, the following figure 2, when $|\mathbf{M}| = 4$ and $|ann(\mathbf{M}) \setminus \{0\}| = 1$.

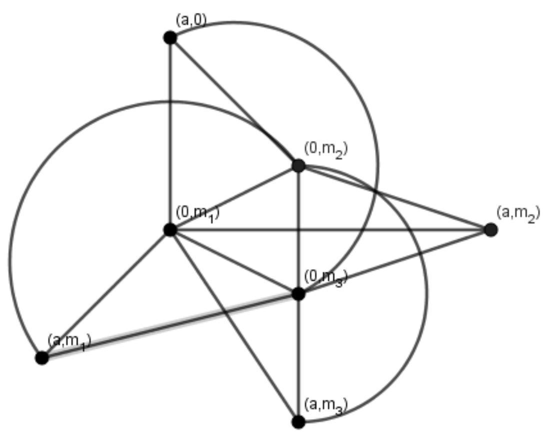

Figure 2: $\Gamma(\mathbf{R}(+)\mathbf{M})$.

5. Geodetic Number of Zero-Divisor Graph $\Gamma(\mathbf{Z}_N(+)\mathbf{Z}_M)$

In this section, the geodetic number of the zero-divisor graph provides crucial insights into the underlying architecture of idealization rings. This parameter, representing the minimum cardinality of a set S that meets the condition that every vertex in the graph is on some shortest path between two in S, gives important details about how zero-divisors are distributed within the idealization ring. Geodetic number for the specific case of $\Gamma(\mathbf{Z}_N(+)\mathbf{Z}_M)$ depends crucially on number-theoretic presentational best decomposition of the N and M. It reflects how zero-divisors are distributed in the idealizatio ring. If N is either a prime power p^r or the product of distinct primes $p_1 p_2$,

First, we find the geodetic number of the zero-divisor graph of the idealization ring $\mathbf{Z}_N(+)\mathbf{Z}_M$ where $N = p^2$ and $M = p$.

Theorem 5.1 *Let $M = p$ and $N = p^2$ where p is a prime number. Then $g(\Gamma(\boldsymbol{Z}_{p^2}(+)\boldsymbol{Z}_p)) = p^2 - 1$.*

Proof Assume that $M = p$ and $N = p^2$. Then the elements of $\Gamma(Z_{p^2}(+)Z_p)$ are $\{(0, 1), \cdots, (0, p - 2), (0, p - 1)\} \cup \{(kp, t) : t \in \mathbf{Z}_p, \ gcd(k, p) = 1\}$.
Then the graph $\Gamma(Z_{p^2}(+)Z_p)$ is a complete graph. Hence, the geodetic set is $\{(0, 1), \cdots, (0, p-2), (0, p-$

$1)\}\cup\{(kp,t): t\in \mathbf{Z}_p,\ gcd(k,p)=1\}$. Therefore, the geodetic number $g(\Gamma(\mathbf{Z}_{p^2}(+)\mathbf{Z}_p)) = p-1+p\phi(p) = p-1+p(p-1) = p^2-1$.

Theorem 5.2 *If $M = p$ and $N = p^r$ where p is a prime number, $r > 2$, then*
$$g(\Gamma(\boldsymbol{Z}_{p_1p_2}(+)\boldsymbol{Z}_{p_1})) = \begin{cases} \sum_{i=1}^{\frac{r}{2}} p\phi(p^{r-i}) & \text{if } r \text{ is even number} \\ \sum_{i=1}^{\lfloor \frac{r}{2} \rfloor} p\phi(p^{r-i}) & \text{if } r \text{ is odd number} \end{cases}.$$

Proof Assume that $M = p$ and $N = p^r$. Then the elements of the zero divisor graph $\Gamma(Z_{p^r}(+)Z_p)$ are $\{(0,1),\cdots,(0,p-2),(0,p-1)\}\cup\{(kp^i,0),(kp^i,1),\cdots,(kp^i,p-2),(kp^i,p-1),\ gcd(k,p^{r-i})=1\}$ where $1\le i\le r-1$.

- **Case 1:** If it r is an even number, then the induced subgraph $L_0\bigcup L_{\frac{r}{2}}\bigcup L_{\frac{r}{2}+1}\cdots\bigcup L_{r-1}$ is a complete graph. And each vertex in these sets $L_1,\ L_2,\cdots,L_{\frac{r}{2}-1}$ is not adjacent to each other. Then the geodetic set is $S=\{L_1\bigcup L_2\bigcup\cdots\bigcup L_{\frac{r}{2}-1}\}$ and $I[S]=V(\mathbf{G})$. So, the geodetic number $g(\Gamma(\mathbf{Z}_{p^r}(+)\mathbf{Z}_p)) = \sum_{i=1}^{\frac{r}{2}-1} p\phi(p^{r-i})$.
- **Case 2:** If it r is an odd number, then the induced subgraph $L_0\bigcup L_{\lceil\frac{r}{2}\rceil}\bigcup L_{\lceil\frac{r}{2}\rceil+1}\cdots\bigcup L_{r-1}$ is a complete graph. And each vertex in these sets $L_1,\ L_2,\cdots,L_{\lfloor\frac{r}{2}\rfloor}$ is not adjacent to each other. Then the geodetic set is $S=\{L_1\bigcup L_2\bigcup\cdots\bigcup L_{\lfloor\frac{r}{2}\rfloor}\}$ and $I[S]=V(\mathbf{G})$. So, the geodetic number $g(\Gamma(\mathbf{Z}_{p^r}(+)\mathbf{Z}_p)) = \sum_{i=1}^{\lfloor\frac{r}{2}\rfloor} p\phi(p^{r-i})$.

Now, we investigate the geodetic set and the geodetic number of the idealization ring $\mathbf{Z}_N(+)\mathbf{Z}_M$ when $N = p_1p_2$ and $M = p_1$, the structural can be determined for geodetic sets in $\Gamma(\mathbf{Z}_N(+)\mathbf{Z}_M)$ when $N = p_1p_2$ and $M = p_1$ according to the way different primes with respect to their interaction. This vertex set partitions schematically into three fundamental layers, L_{0^*} containing elements of form $(0,t)$, L_{1^*}, $L_{1^{**}}$ for points divisible by p_2 and p_1 respectively. The geodetic number will depend crucially on the way in which these layers are connected by shortest paths: thus, most emphasis in this study falls on connections between elements from different layers. This structure enables us to characterize minimal geodetic sets by selecting vertices carefully from which all possible shortest paths through the graph can be reached.

Theorem 5.3 *If $M = p_1$ and $N = p_1p_2$ where p_i is a prime number for all $i = 1,2$, then the geodetic number of the idealization ring $\Gamma(\boldsymbol{Z}_{p_1p_2}(+)\boldsymbol{Z}_{p_1})$ is* $g(\Gamma(\boldsymbol{Z}_{p_1p_2}(+)\boldsymbol{Z}_{p_1})) = \begin{cases} \phi(p_2)p_1 & \text{if } p_1 > p_2 \\ \phi(p_1)p_1 + p_1 - 1 & \text{if } p_2 > p_1 \end{cases}$.

Proof. Assume that $M = p_1$ and $N = p_1p_2$. Then the elements of the zero divisor graph of the ring $Z_{p_1p_2}(+)Z_{p_1}$ are $L_{0^*}\bigcup L_{1^*}\bigcup L_{1^{**}}$.
The induced subgraph of $\Gamma(Z_{p_1p_2}(+)Z_{p_1})$ over the set of vertices L_{0^*} is the complete graphs. And each element in L_{1^*} is adjacent to L_{0^*}. Moreover every element of the form $(kp_1,0)$ in L_{1^*} is adjacent to each element in $L_{1^{**}}$. So, the geodetic set is $S=\{(kp_1,t):\ t\in\mathbf{Z}_p,\ gcd(k,p_2)=1\}$ or is $S=\{(kp_2,t):\ t\in\mathbf{Z}_p,\ gcd(k,p_1)=1\}\bigcup\{(0,t)\in\mathbf{Z}_{p_1}^*\}$. Then the geodetic number of the idealization ring is dependent the the number p_1 and the number p_2. So, $g(\Gamma(\mathbf{Z}_{p_1p_2}(+)\mathbf{Z}_{p_1})) = \begin{cases} \phi(p_2)p_1 & \text{if } p_1 > p_2 \\ \phi(p_1)p_1 + p_1 - 1 & \text{if } p_2 > p_1 \end{cases}$.

6. Domination Number of Zero-Divisor Graph $\Gamma(\mathbf{R}(+)\mathbf{M})$ where R is an integral domain

For integral domain rings R, the study of domination numbers in zero-divisor graphs of idealization rings $\Gamma(\mathbf{R}(+)\mathbf{M})$ reveals essential properties about zero-divisors. In this context, a dominating set is a subset S of vertices such that every vertex not in S is adjacent to at least one vertex in S. The domination number is the minimum number of such a set, denoted by $\gamma(\mathbf{G})$. When a domain arrives as R is high-tech field of maths then lacks zero divisors while those from affluent structures are more trying to have some very rigid classifications of dominant sets. This serves to bring to naught or deny the parameter of patience in waiting for any results on zero-divisor graphs, whose diagonals describe offshoots and transfers between research agendas. This property also significantly influences thLhorizontally adjacent neighbors' heights in the graph and therefore also in ques of minimum dominating sets, whether to be formallized or not, giving us a way to contemplate rotation operations that are also defined directly at zero divisors.

Theorem 6.1 *Let $\boldsymbol{M}$ be an $\boldsymbol{R}-$module and let $\boldsymbol{R}$ be an integral domain. Then $\gamma(\Gamma(\boldsymbol{R}(+)\boldsymbol{M})) = 1$.*

Proof.

- **Case 1**: If $ann(\mathbf{M}) = 0$, then the elements of the zero-divisor graph of idealization ring $\mathbf{R}(+)\mathbf{M}$ are $\{(0,m) : m \in \mathbf{M}^*\}$ which is a complete graph so, the domination number of the zero-divisor graph of idealization ring $\mathbf{R}(+)\mathbf{M}$ is $\gamma(\Gamma(\mathbf{R}(+)\mathbf{M})) = 1$. See, the following figure 3.

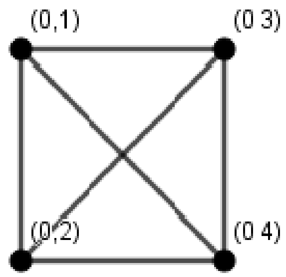

Figure 3: $\Gamma(\mathbf{Z}_5(+)\mathbf{Z}_5)$.

- **Case 2**: If $ann(\mathbf{M}) \neq 0$, then the elements of the zero-divisor graph of idealization ring $\mathbf{R}(+)\mathbf{M}$ is $\{(0,m_1),(a_i,m_2) : a_i \in ann(\mathbf{M}) \setminus \{0\}, m_1 \in \mathbf{M} \setminus \{0\}, m_2 \in \mathbf{M}\}$. So, the induced subgraph $\{(0,m_1) : m_1 \in \mathbf{M}^*\}$ is a complete graph and each this vertex is adjacent to each vertex in $ann(\mathbf{M})$. Then the domination number of the zero-divisor graph of idealization ring $\mathbf{R}(+)\mathbf{M}$ is $\gamma(\Gamma(\mathbf{R}(+)\mathbf{M})) = 1$.
  See, the following figure 4, when $|\mathbf{M}| = 4$ and $|ann(\mathbf{M}) \setminus \{0\}| = 1$.

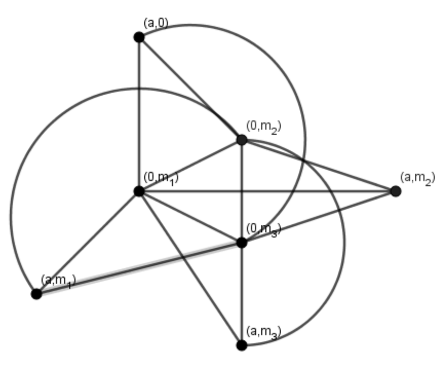

Figure 4: $\Gamma(\mathbf{R}(+)\mathbf{M})$.

7. Domination Number of Zero-Divisor Graph $\Gamma(\mathbf{Z}_N(+)\mathbf{Z}_M)$

In this section, we determining the domination numbers for zero-divisor graphs of idealization rings $\Gamma(\mathbf{Z}_N(+)\mathbf{Z}_M)$ requires a careful examination of how elements in the ring $\mathbf{Z}_N$ interact with those in the module $\mathbf{Z}_M$.

Theorem 7.1 *Let $M = p$ and $N = p^r$ where p is a prime number and $r \geq 2$. Then the domination number of the zero-divisor graph of idealization ring is $\gamma(\Gamma(\boldsymbol{Z}_N(+)\boldsymbol{Z}_M)) = 1$.*

Proof. Assume that $M = p$ and $N = p^r$, $r \geq 2$. Then the elements of the zero divisor graph of $\mathbf{Z}_{p^r}(+)\mathbf{Z}_p$ are $L_0 \bigcup L_1 \bigcup ... \bigcup L_{r-1}$.
The induced subgraph of $\Gamma(Z_{p^r}(+)Z_p)$ over the set of vertices L_0 and L_i for all $i \geq \lceil \frac{r}{2} \rceil$ is the complete graph and for all $0 < i < \lceil \frac{r}{2} \rceil$ each element in each set L_i is not adjacent. Then there exist an element $(p^{r-1}, 0)$ is an adjacent to all element of the sets $L_0 \bigcup L_1 \bigcup ... \bigcup L_{r-1}$. So, $\gamma(\Gamma(\mathbf{Z}_{p^r}(+)\mathbf{Z}_p)) = 1$.

Now, we investigate the domination number when $M = p_1$ and $N = p_1p_2$ where p_i for all $i = 1, 2$ is a prime number.

Theorem 7.2 *If $M = p_1$ and $N = p_1p_2$ where p_i for all $i = 1, 2$ is a prime number, then the graph $\Gamma(\boldsymbol{Z}_N(+)\boldsymbol{Z}_M))$ has $\gamma(\Gamma(\boldsymbol{Z}_N(+)\boldsymbol{Z}_M)) = 2$.*

Proof. If $M = p_1$ and $N = p_1p_2$, then the elements of the zero divisor graph of idealization ring $\mathbf{Z}_{p_1p_2}(+)\mathbf{Z}_{p_1}$ are $L_{0^*} \bigcup L_{1^*} \bigcup L_{1^{**}}$.

8. Conclusion

The induced subgraph L_{0^*} of the graph $\Gamma(\mathbf{Z}_{p_1p_2}(+)\mathbf{Z}_{p_1})$ is a complete graph. Moreover, each element L_1^* is adjacent to L_{0^*}. Moreover, the vertex $(p_1, 0)$ in L_1^* is adjacent to each vertex in L_1** and adjacent to each vertex in L_{0^*}. The vertex $(p_2, 0)$ in $L_{1^{**}}$ is adjacent to each vertex in L_{1^*}. So, $\gamma(\Gamma(Z_{p_1p_2}(+)Z_{p_1})) = 2$.
This article looks at geodetic sets and domination numbers of zero-divisor graphs representing the idealization ring $\mathbf{Z}_N(+)\mathbf{Z}_M$. It focuses on two main cases: when N is a prime power p^r and when N is a product of distinct prime numbers p_1p_2, with M usually taken as being the smaller of them. The vertex set structure has distinct layers L_s, $L_{\{0^*\}}$, $L_{\{1^*\}}$ and $L_{\{1^{**}\}}$ depending on the prime factorization of N. For geodetic sets, the analysis shows how shortest paths between vertices determine minimal sets for covering all vertices. Domination number investigations consider both integral and non-integral domains, showing how the structure of the ring determines adjacency patterns and minimal dominating sets. As a result, explicit formulas are provided for both geodetic numbers and domination numbers in terms of prime factors, which establish relationships between core algebraic properties and graph theory invariants.

Corollary 8.1 *Let M be an R-module and R be an integral domain. Then:*

1. *If $ann(M) = 0$, then $g(\Gamma(R(+)M)) = |M| - 1$.*
2. *If $ann(M) \neq 0$, then $g(\Gamma(R(+)M)) = (|ann(M)| - 1)|M|$.*

Corollary 8.2 *For $N = p^r$ where p is prime and $M = p$, we have:*

$$g(\Gamma(Z_N(+)Z_M)) = \begin{cases} p^2 - 1 & \text{if } r = 2 \\ \sum_{i=1}^{\lfloor r/2 \rfloor} p\phi(p^{r-i}) & \text{if } r > 2 \text{ and } r \text{ is an odd number} \\ \sum_{i=1}^{\frac{r}{2}} p\phi(p^{r-i}) & \text{if } r > 2 \text{ and } r \text{ is even number} \end{cases}.$$

Corollary 8.3 *For any idealization ring $R(+)M$ where R is an integral domain:*

$$\gamma(\Gamma(R(+)M)) = 1.$$

Corollary 8.4 *Let $M = p$ and $N = p_1p_2$ where p_1, p_2 are distinct primes. Then:*

$$\gamma(\Gamma(Z_N(+)Z_M)) = \begin{cases} 1 & \text{if } M = p \text{ and } N = p^r, r \geq 2 \\ 2 & \text{if } M = p_1 \text{ and } N = p_1p_2 \end{cases}.$$

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