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# Existence results for some nonlinear and noncoercive anisotropic elliptic equations with Neumann boundary conditions

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ABSTRACT: The aim of this work is to prove the existence of renormalized solutions for the following anisotropic elliptic problem with degenerate coercivity and Fourier boundary conditions

$$\begin{cases} -\sum_{i=1}^{N} D^i a_i(x,u,\nabla u) + g(x,u,\nabla u) + \alpha(x)|u|^{r-1}u = f & \text{in } \Omega, \\ \sum_{i=1}^{N} a_i(x,u,\nabla u).n_i + \lambda u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega$ is an open bounded subset of $I\!R^N$ $(N \geq 2)$, the data $f$ belong to $L^1(\Omega)$ and the Carathéodory functions $a_i(x,s,\xi)$ and $g(x,s,\xi)$ verify some nonstandard conditions.

Key Words: Anisotropic Sobolev spaces, renormalized solutions, non-coercive elliptic Neumann problem, Fourier boundary conditions.

## Contents



## 1. Introduction

In the last years, researchers have shown great interest in studying anisotropic elliptic and parabolic problems. This is mainly due to the fact that these problems modeling physical processes in an anisotropic continuous medium. (see [5,33]). The concept of anisotropic Sobolev spaces was first introduced by Nikolskii [32] and Troisi [35]. Later, Trudinger [36] developed it further in the context of Orlicz spaces. Note that the anisotropic Sobolev spaces are the spaces where the regularity vary in different directions. This means that we treat the derivatives in each coordinate direction separately, with potentially different orders of regularity for each direction.
An important result in this area is the work of Boccardo et al. [17] who have studied some anisotropic equations with measures data on the right-hand side, of the type

$$\begin{cases} -\text{div}\,(j(Du)) = f & \text{in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

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where $j(\xi)$ is the vector field whose components are $|\xi_i|^{p_i-2}\xi_i \quad (i=1,\dots,N; \quad p_i>1)$. They have proved the existence of solutions to this equation in the anisotropic Sobolev space. In [23], Feng-Quan has studied the existence of solution for the following anisotropic elliptic equation:

$$\begin{cases} -\text{div } (a(x,u,Du)) = f & \text{in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.2}$$

in the sense of distributions, where $f \in L^m(\Omega)$ for $1 < m < \overline{m} = \dfrac{N\overline{p}}{N\overline{p} - N + \overline{p}}$.
Antontsev et al. have studied in [4] the uniqueness of weak solutions for elliptic equations of the form:

$$-\sum_{i=1}^{N} -\partial_{x_i}(a_i(x,u)|\partial_{x_i}u|^{p_i-2}\partial_{x_i}u) + b(x,u) = f \quad \text{ in } \Omega, \tag{1.3}$$

in a bounded Lipschitz domain of $I\!R^N$ with mixed boundary conditions. Moreover, they have established a similar result for the parabolic case.
Bendahmane et al. have applied Hedberg-type approximation in [13] to prove the existence of solutions in the sense of distributions for some nonlinear anisotropic elliptic equations of the type

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial_{x_i}}(|\frac{\partial u}{\partial x_i}|^{p_i-2}\frac{\partial u}{\partial x_i}) + g(x,u) = f & \text{in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\Omega$ is an open subset of $I\!R^N$ and the exponents $p_i > 1$ for $i = 1,\dots,N$. The nonlinear function $g : \Omega \times I\!R \mapsto I\!R$ is assumed to be Carathéodory function that verifying some growth and sign condition.
In [21], Di Nardo et al. have studied the existence and uniqueness of weak solutions for some classes of anisotropic elliptic equations with homogeneous Dirichlet boundary conditions, giving by

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \partial_{x_i}(a_i(x,u)(\varepsilon + |\partial_{x_i}u|^2)^{(p_i-2)/2}\partial_{x_i}u) = f - \partial_{x_i}g_i \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.5}$$

where $f \in L^{p'_\infty}(\Omega)$, and $g_i \in L^{p'}(\Omega)$ for $i = 1,\dots,N$.
Recently, Azroul et al. have studied in [6] the existence of entropy solutions for the anisotropic quasilinear elliptic problems

$$\begin{cases} -\text{div } (a(x,u,\nabla u)) + |u|^{s-1}u = f + \rho\dfrac{|u|^{p_0-2}u}{|x|^{p_0}} \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.6}$$

where $-\text{div } (a(x,u,\nabla u))$ is a Leray-Lions operator acted from $W_0^{1,\vec{p}}(\Omega,\omega)$ into its dual, $\Omega$ is an open bounded subset of $I\!R^N$ containing the origin, the datum $f$ is assumed to be in $L^1(\Omega)$ and merely integrable and $\rho$ is a positive constant, we refer the reader also to [2], [8] and [9] for more details.

The present paper extends the study of this class of problems, we apply the variational method and some a priori estimates to establish the existence of at least one renormalized solution to the following nonlinear and non-coercive elliptic problem:

$$\begin{cases} Au + g(x,u,\nabla u) + \alpha(x)|u|^{r-1}u = f & \text{in } \Omega, \\ \displaystyle\sum_{i=1}^{N} a_i(x,u,\nabla u).n_i + \lambda u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.7}$$

where $\Omega$ is a regular bounded domain of $I\!R^N$, the data $f$ belong to $L^1(\Omega)$, and $Au = -\displaystyle\sum_{i=1}^{N} a_i(x,u,\nabla u)$ is the Leray-Lions operator acting from $W^{1,\vec{p}}(\Omega)$ into its dual. The nonlinear lower-order term $g(x,s,\xi)$ : $\Omega \times I\!R \times I\!R^N \to I\!R^N$ is a Carathéodory function that satisfying the sign and some growth conditions.

This paper is organized as follows. In the section 2 we introduce some assumptions for which our problem has at least one solution. The section 3 is devoted to study the existence of a solution in the sense of distributions for the equation with right-hand side $F(x) \in L^\infty(\Omega)$. In the last section, we establish the existence of renormalized solutions for the non-coercive elliptic problem (1.7) with $f(x) \in L^1(\Omega)$.

## 2. Preliminaries

Let $\Omega$ be an open bounded domain in $I\!R^N$ $(N \geq 2)$, with smooth boundary $\partial\Omega$.
Let $p_1, \ldots, p_N$ be $N$ real constants numbers, with $1 < p_i < \infty$ for $i = 1, \ldots, N$.
We denote

$$\vec{p} = (1, p_1, \ldots, p_N), \qquad D^0 u = u \qquad \text{and} \qquad D^i u = \frac{\partial u}{\partial x_i} \qquad \text{for} \qquad i = 1, \ldots, N.$$

We set

$$\underline{p} = \min\{p_1, p_2, \ldots, p_N\} \qquad \text{and} \qquad p_m = \max\{p_1, p_2, \ldots, p_N\}.$$

We define the anisotropic Sobolev space $W^{1,\vec{p}}(\Omega)$ as follows :

$$W^{1,\vec{p}}(\Omega) = \left\{ u \in W^{1,1}(\Omega) \quad \text{such that} \quad D^i u \in L^{p_i}(\Omega) \quad \text{for} \quad i = 1, 2, \ldots, N \right\},$$

endowed with the norm

$$\|u\|_{1,\vec{p}} = \|u\|_{1,1} + \sum_{i=1}^{N} \|D^i u\|_{L^{p_i}(\Omega)}. \tag{2.1}$$

The space $\left(W^{1,\vec{p}}(\Omega), \|\cdot\|_{1,\vec{p}}\right)$ is a separable and reflexive Banach space (cf [29]).
Let us recall the Poincaré and Sobolev type inequalities in the anisotropic Sobolev space.

**Proposition 2.1** *(cf [25], [34])*
*Let $u \in W^{1,\vec{p}}(\Omega)$, we have*
***(i)*** *Poincaré Wirtinger inequality: there exists a constant $C_p > 0$, such that*

$$\|u - m(u)\|_{L^{p_i}(\Omega)} \leq C_p \sum_{i=1}^{N} \|D^i u\|_{L^{p_i}(\Omega)},$$

*where*

$$m(u) = \frac{1}{|\Omega|} \int_\Omega |u(x)|\, dx,$$

*is the mean-value of $u$.*
***(ii)*** *Sobolev inequality : there exists an other constant $C_s > 0$ such that*

$$\|u - m(u)\|_q \leq \frac{C_s}{N} \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{p_i},$$

*where*

$$\frac{1}{\overline{p}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i} \qquad \text{and} \qquad \begin{cases} q = \overline{p}^* = \dfrac{N\overline{p}}{N - \overline{p}} & \text{if} \quad \overline{p} < N, \\ q \in [1, +\infty[ & \text{if} \quad \overline{p} \geq N. \end{cases}$$

**Lemma 2.1** *Let $\Omega$ be a bounded open subset in $I\!R^N$ $(N \geq 2)$, we set*

$$s = \max(q,\ p_m),$$

*then, we have the following embedding :*

- *if $\overline{p} < N$ then the embedding $W^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$ is compact for any $r \in [1, s[$,*

- *if* $\overline{p} = N$ *then the embedding* $W^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$ *is compact for any* $r \in [1, +\infty[$,
- *if* $\overline{p} > N$ *then the embedding* $W^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^\infty(\Omega) \cap C^0(\overline{\Omega})$ *is compact.*

The proof of this lemma follows from the Proposition 2.1.

**Definition 2.1** Let $k > 0$, we consider the truncation function $T_k(\cdot) : I\!R \longmapsto I\!R$ given by

$$T_k(s) = \begin{cases} s & \text{if} \quad |s| \leq k, \\ k\dfrac{s}{|s|} & \text{if} \quad |s| > k, \end{cases}$$

and we define

$$T^{1,\vec{p}}(\Omega) := \{u : \Omega \mapsto I\!R \text{ measurable, such that } T_k(u) \in W^{1,\vec{p}}(\Omega) \text{ for any } k > 0\}.$$

**Proposition 2.2** *Let* $u \in T^{1,\vec{p}}(\Omega)$. *For any* $i \in \{1, \ldots, N\}$, *there exists a unique measurable function* $v_i : \Omega \mapsto I\!R$ *such that*

$$\forall k > 0 \qquad D^i T_k(u) = v_i.\chi_{\{|u|<k\}} \qquad a.e. \quad x \in \Omega,$$

*where* $\chi_A$ *denotes the characteristic function of a measurable set* $A$. *The functions* $v_i$ *are called the weak partial derivatives of* $u$ *and are still denoted* $D^i u$. *Moreover, if* $u$ *belongs to* $W^{1,1}(\Omega)$, *then* $v_i$ *coincides with the standard distributional derivative of* $u$, *that is,* $v_i = D^i u$.

The proof of the Proposition 2.2 follows the usual techniques developed in [14] for the case of Sobolev spaces. For more details concerning the anisotropic Sobolev spaces, we refer the reader to [13,21,22,4].

Moreover, we introduce the set $T^{1,\vec{p}}_{tr}(\Omega)$ as a subset of $T^{1,\vec{p}}(\Omega)$ for which a generalized notion of trace may be defined (see also [3] for the case of constant exponent). More precisely, $T^{1,\vec{p}}_{tr}(\Omega)$ is the set of function $u$ in $T^{1,\vec{p}}(\Omega)$, such that : there exists a sequence $(u_n)_n$ in $W^{1,\vec{p}}(\Omega)$ and a measurable function $v$ on $\partial\Omega$ that verifying

**(a)** $u_n \longrightarrow u$ a.e. in $\Omega$,

**(b)** $D^i T_k(u_n) \longrightarrow D^i T_k(u)$ in $L^1(\Omega)$ for every $k > 0$.

**(c)** $u_n \longrightarrow v$ a.e. on $\partial\Omega$.

The function $v$ is the trace of $u$ in the generalized sense introduced in [3].
Let $u \in W^{1,\vec{p}}(\Omega)$, the trace of $u$ on $\partial\Omega$ will be denoted by $\tau(u)$, and for any $u \in T^{1,\vec{p}}_{tr}(\Omega)$, the trace of $u$ on $\partial\Omega$ will be denoted by $tr(u)$ or $u$, the operator $tr(\cdot)$ satisfied the following properties

**(i)** if $u \in T^{1,\vec{p}}_{tr}(\Omega)$, then $\tau(T_k(u)) = T_k(tr(u))$ for any $k > 0$.

**(ii)** if $\varphi \in W^{1,\vec{p}}(\Omega)$, then, for any $u \in T^{1,\vec{p}}_{tr}(\Omega)$, we have $u - \varphi \in T^{1,\vec{p}}_{tr}(\Omega)$ and $tr(u - \varphi) = tr(u) - \tau(\varphi)$.

In the case where $u \in W^{1,\vec{p}}(\Omega)$, $tr(u)$ coincides with $\tau(u)$. Obviously, we have

$$W^{1,\vec{p}}(\Omega) \subset T^{1,\vec{p}}_{tr}(\Omega) \subset T^{1,\vec{p}}(\Omega).$$

**Lemma 2.2** *(see [24], Theorem 13.47) Let* $(u_n)_n$ *be a sequence in* $L^1(\Omega)$ *and* $u \in L^1(\Omega)$ *such that*

**(i)** $u_n \to u$ *a.e. in* $\Omega$,

**(ii)** $u_n \geq 0$ *and* $u \geq 0$ *a.e. in* $\Omega$,

**(iii)** $\displaystyle\int_\Omega u_n \, dx \to \int_\Omega u \, dx$,

*then* $u_n \to u$ *strongly in* $L^1(\Omega)$.

## 3. Essential assumptions

Let $\Omega$ be a bounded open subset of $I\!R^N$ ($N \geq 2$), with smooth boundary $\partial\Omega$.
We consider the strongly nonlinear anisotropic elliptic problem

$$\begin{cases} Au + g(x,u,\nabla u) + \alpha(x)|u|^{r-1}u = f & \text{in } \Omega, \\ a(x,u,\nabla u).\vec{n} + \lambda u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

with $0 < r \leq \underline{p} - 1$, and $\lambda > 0$. The data $f(\cdot)$ is assumed to be a measurable function in $L^1(\Omega)$ and the positive function $\alpha(\cdot) \in L^\infty(\Omega)$ such that $\alpha(x) \geq \alpha_0 > 0$ a.e. in $\Omega$.
The Leray-Lions operator $A$ acted from $W^{1,\vec{p}}(\Omega)$ into its dual, defined by

$$Au = -\sum_{i=1}^N D^i a_i(x,u,\nabla u),$$

where $a_i : \Omega \times I\!R \times I\!R^N \longmapsto I\!R$ are Carathéodory functions for $i = 1,...,N$ (measurable with respect to $x$ in $\Omega$ for every $(s,\xi)$ in $I\!R \times I\!R^N$, and continuous with respect to $(s,\xi)$ in $I\!R \times I\!R^N$ for almost every $x$ in $\Omega$), which satisfy the following conditions :

$$|a_i(x,s,\xi)| \leq \beta(K_i(x) + |s|^{p_i-1} + |\xi_i|^{p_i-1}) \qquad \text{for} \quad i = 1,\ldots,N, \tag{3.2}$$

where the nonnegative functions $K_i(\cdot)$ are assumed to be in $L^{p_i'}(\Omega)$ for $i = 1,\ldots,N$, with $\beta > 0$.

$$\sum_{i=1}^N (a_i(x,s,\xi) - a_i(x,s,\xi'))(\xi_i - \xi_i') > 0 \qquad \text{for} \quad \xi_i \neq \xi_i', \tag{3.3}$$

for almost every $x \in \Omega$ and all $(s,\xi)$ in $I\!R \times I\!R^N$.

$$a_i(x,s,\xi)\xi_i \geq b(|s|)|\xi_i|^{p_i} \quad \text{with} \quad \frac{b_0}{(1+|s|)^\delta} \leq b(|s|) \quad \text{for any} \quad s \in I\!R, \tag{3.4}$$

such that $b(|\cdot|) : I\!R^+ \mapsto I\!R^+$ is a decreasing function, with $b_0 > 0$ and $0 \leq \delta < \underline{p} - 1$.
The nonlinear term $g(x,s,\xi)$ is a Carathéodory function which satisfies :

$$g(x,s,\xi)s \geq 0, \tag{3.5}$$

$$|g(x,s,\xi)| \leq d(|s|)(c(x) + \sum_{i=1}^N |\xi_i|^{p_i}), \tag{3.6}$$

where $d(\cdot) : I\!R^+ \mapsto I\!R^+$ is a continuous, nondecreasing function, and $c : \Omega \mapsto I\!R^+$ with $c \in L^1(\Omega)$.
As a consequence of (3.4) and the continuity of the function $a_i(x,s,\cdot)$ with respect to $\xi$, we have

$$a_i(x,s,0) = 0.$$

Now, we will recall the following technical Lemma, useful to prove our main results.

**Lemma 3.1** *(see [10]) Let $k > 0$, assuming that (3.2) $-$ (3.6) hold true, and let $(u_n)_{n\in I\!N}$ be a sequence in $W^{1,\vec{p}}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W^{1,\vec{p}}(\Omega)$ and*

$$\begin{aligned} &\int_\Omega (|u_n|^{\underline{p}-2}u_n - |u|^{\underline{p}-2}u)(u_n - u)\,dx \\ &\quad + \sum_{i=1}^N \int_\Omega (a_i(x,T_k(u_n),\nabla u_n) - a_i(x,T_k(u_n),\nabla u))(D^i u_n - D^i u)\,dx \to 0, \end{aligned} \tag{3.7}$$

*then $u_n \to u$ strongly in $W^{1,\vec{p}}(\Omega)$ for a subsequence.*

## 4. Existence of weak solutions for $L^\infty-$ data

We consider the strongly nonlinear elliptic problem

$$\begin{cases} -\sum_{i=1}^{N} D^i a_i(x, T_n(u), \nabla u) + g_n(x, u, \nabla u) + \alpha(x)|u|^{r-1}u = F(x) & \text{in } \Omega, \\ \sum_{i=1}^{N} a_i(x, T_n(u), \nabla u) \cdot n_i + \lambda T_n(u) = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where

$$|F(x)| \leq C_0, \quad \text{for any} \quad x \in \Omega, \tag{4.2}$$

with $C_0$ is a positive constant.

**Definition 4.1** A measurable function $u$ is called a solution in the sense of distributions for the strongly nonlinear anisotropic elliptic equation (4.1), if $u \in W^{1,\vec{p}}(\Omega)$ and $|u|^{r+1} \in L^1(\Omega)$, such that $u$ verifies the following equality

$$\begin{aligned} \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u), \nabla u) D^i v \,\mathrm{d}x + \int_\Omega g_n(x, u, \nabla u) v \,\mathrm{d}x \\ + \int_\Omega \alpha(x)|u|^{r-1} u v \,\mathrm{d}x + \lambda \int_{\partial\Omega} T_n(u) v \,\mathrm{d}\sigma = \int_\Omega F\ v \,\mathrm{d}x \end{aligned} \tag{4.3}$$

for any $v \in W^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$.

**Theorem 4.1** *Assuming that* (3.2) $-$ (3.6) *and* (4.2) *hold true. Then there exists at least one solution in the sense of distribution* $u \in W^{1,\vec{p}}(\Omega)$ *for the strongly nonlinear elliptic equation* (4.1).

**Proof of Theorem 4.1**

***Step 1 : Approximate problem***
We consider the following approximate problem for the strongly nonlinear elliptic equation (4.1), giving by

$$\begin{cases} -\sum_{i=1}^{N} D^i a_i(x, T_n(u_m), \nabla u_m) + g_n(x, u_m, \nabla u_m) + \alpha(x)|T_m(u_m)|^{r-1} T_m(u_m) \\ \qquad\qquad + \frac{1}{m}|u_m|^{\underline{p}-2} u_m = F(x) & \text{in } \Omega, \\ \sum_{i=1}^{N} a_i(x, T_n(u_m), \nabla u_m) \cdot n_i + \lambda T_n(u_m) = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

We define the operator $A_m$ and $R_m$ acted from $W^{1,\vec{p}}(\Omega)$ into its dual $(W^{1,\vec{p}}(\Omega))'$ giving by :

$$\langle A_m u, v\rangle = \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u), \nabla u) D^i v \,\mathrm{d}x + \frac{1}{m} \int_\Omega |u|^{\underline{p}-2} u v \,\mathrm{d}x + \lambda \int_{\partial\Omega} T_n(u) v \,\mathrm{d}\sigma, \tag{4.5}$$

and

$$\langle R_m u, v\rangle = \int_\Omega g_n(x, u, \nabla u) v \,\mathrm{d}x + \int_\Omega \alpha(x)|T_m(u)|^{r-1} T_m(u) v \,\mathrm{d}x, \tag{4.6}$$

for any $u, v \in W^{1,\vec{p}}(\Omega)$.
In view of Hölder's type inequality we have

$$\begin{aligned} |\langle R_m u, v\rangle| &= \int_\Omega \alpha(x)|T_m(u)|^r |v| \,\mathrm{d}x + \int_\Omega |g_n(x, u, \nabla u)||v| \,\mathrm{d}x \\ &\leq \|\alpha(x)\|_{L^\infty(\Omega)} m^r \int_\Omega |v| \,\mathrm{d}x + n \int_\Omega |v| \,\mathrm{d}x \\ &\leq C_1 \|v\|_{1,\vec{p}}. \end{aligned} \tag{4.7}$$

**Lemma 4.1** *The bounded operator $B_m = A_m + R_m$ acting from $W^{1,\vec{p}}(\Omega)$ into $(W^{1,\vec{p}}(\Omega))'$ is a pseudo-monotone operator. Moreover, $B_m$ is coercive in the following sense:*

$$\frac{\langle B_m v, v\rangle}{\|v\|_{1,\vec{p}}} \longrightarrow \infty \qquad \text{as} \quad \|v\|_{1,\vec{p}} \longrightarrow \infty, \tag{4.8}$$

*for any $v \in W^{1,\vec{p}}(\Omega)$.*

The proof of Lemma 4.1 is similar to the arguments in [7] ( see also [8] and [9]) with very few modifications. In view of Lemma 4.1 (cf. [25], Theorem 8.2) there exists at least one weak solution $u_m \in W^{1,\vec{p}}(\Omega)$ for the approximate problem (4.4), i.e.

$$\begin{aligned}\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_m),\nabla u_m)D^i v\,dx + \int_\Omega g_n(x,u_m,\nabla u_m)v\,dx + \int_\Omega \alpha(x)|T_m(u_m)|^{r-1}T_m(u_m)v\,dx\\ +\frac{1}{m}\int_\Omega |u_m|^{\underline{p}-2}u_m v\,dx + \lambda\int_{\partial\Omega} T_n(u_m)v\,d\sigma = \int_\Omega F(x)v\,dx,\end{aligned} \tag{4.9}$$

for any $v \in W^{1,\vec{p}}(\Omega)$.

***Step 2: Weak convergence of the sequence*** $(u_m)_m$

Let $m \geq n \geq 1$, by taking $v = u_m$ as a test function for the approximate problem (4.4), we have

$$\begin{aligned}\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_m),\nabla u_m)D^i u_m\,dx + \int_\Omega g_n(x,u_m,\nabla u_m)u_m\,dx + \int_\Omega \alpha(x)|T_m(u_m)|^{r}\ |u_m|\,dx\\ +\frac{1}{m}\int_\Omega |u_m|^{\underline{p}}\,dx + \lambda\int_{\partial\Omega} T_n(u_m)u_m\,d\sigma = \int_\Omega F(x)u_m\,dx.\end{aligned} \tag{4.10}$$

Thus, in view of (3.4) and (3.6) we obtain

$$\begin{aligned}\sum_{i=1}^{N}\int_\Omega b(|T_n(u_m)|)|D^i u_m|^{p_i(x)}\,dx + \int_\Omega |g_n(x,u_m,\nabla u_m)|\ |u_m|\,dx + \alpha_0\int_\Omega |T_m(u_m)|^{r}\ |u_m|\,dx\\ +\frac{1}{m}\int_\Omega |u_m|^{\underline{p}}\,dx + \lambda\int_{\partial\Omega} T_n(u_m)u_m\,d\sigma \leq C_0\int_\Omega |u_m|\,dx.\end{aligned} \tag{4.11}$$

For the first term on the right-hand side of (4.11), by applying Young's inequality we have

$$\begin{aligned}C_0\int_\Omega |u_m|\,dx &\leq C_0\int_{\{|u_m|\leq C_1\}} |u_m|\,dx + C_0\int_{\{|u_m|>C_1\}} |u_m|\,dx\\ &\leq C_2 + \frac{\alpha_0}{2}\int_{\{|u_m|>C_1\}} |T_m(u_m)|^r\ |u_m|\,dx\\ &\leq C_2 + \frac{\alpha_0}{2}\int_\Omega |T_m(u_m)|^r\ |u_m|\,dx,\end{aligned} \tag{4.12}$$

with $m \geq C_1 = \left(\frac{2}{\alpha_0}C_0\right)^{\frac{1}{r}} + 1$.

By combining (4.11) and (4.12) we conclude that

$$\frac{b_0}{2(1+n)^\delta}\sum_{i=1}^{N}\int_\Omega |D^i u_m|^{p_i}\,dx + \frac{\alpha_0}{2}\int_\Omega |T_m(u_m)|^r\ |u_m|\,dx + \frac{1}{m}\int_\Omega |u_m|^{\underline{p}}\,dx + \lambda\int_{\partial\Omega} |T_n(u_m)||u_m|\,d\sigma \leq C_4. \tag{4.13}$$

Moreover, we deduce that

$$\begin{aligned} \|u_m\|_{1,\vec{p}} &= \|u_m\|_{L^1(\Omega)} + \sum_{i=1}^N \|D^i u_m\|_{L^1(\Omega)} + \sum_{i=1}^N \|D^i u_m\|_{L^{p_i}(\Omega)} \\ &\leq \int_\Omega |u_m|\,dx + 2\sum_{i=1}^N \int_\Omega |D^i u_m|^{p_i}\,dx + N(\text{meas}(\Omega)+1) \\ &\leq \int_\Omega |T_m(u_m)|^r |u_m|\,dx + 2\sum_{i=1}^N \int_\Omega |D^i u_m|^{p_i}\,dx + C_5 \\ &\leq C_6, \end{aligned} \tag{4.14}$$

with $C_6$ is a constant that doesn't depend on $m$. Thus, the sequence $(u_m)_m$ is uniformly bounded in $W^{1,\vec{p}}(\Omega)$, and there exists a subsequence still denoted $(u_m)_m$ such that

$$\begin{cases} u_m \rightharpoonup u & \text{weakly in } W^{1,\vec{p}}(\Omega), \\ u_m \longrightarrow u & \text{strongly in } \; L^{\underline{p}}(\Omega) \quad \text{and} \quad \text{a.e. in} \quad \Omega, \\ u_m \rightharpoonup u & \text{weakly in } \; L^1(\partial\Omega). \end{cases} \tag{4.15}$$

It follows that

$$\frac{1}{m}|u_m|^{\underline{p}-2}u_m \longrightarrow 0 \qquad \text{strongly in} \quad L^{\underline{p}'}(\Omega). \tag{4.16}$$

Moreover, in view of (4.13) we conclude that $(T_m(u_m))_m$ is uniformly bounded in $L^{r+1}(\Omega)$, and since $T_m(u_m) \to u$ almost everywhere in $\Omega$, we get

$$T_m(u_m) \rightharpoonup u \quad \text{weakly in} \quad L^{r+1}(\Omega). \tag{4.17}$$

Furthermore, we have $u_m \to u$ almost everywhere in $\partial\Omega$, it follows that

$$T_n(u_m) \to T_n(u) \qquad \text{a.e in} \quad \partial\Omega,$$

and $|T_n(u_m)| \leq n$, then

$$T_n(u_m) \to T_n(u) \qquad \text{weak} - * \quad \text{in} \quad L^\infty(\partial\Omega). \tag{4.18}$$

***Step 3 : The convergence almost everywhere of the gradient***
By taking $v = u_m - u$ as a test function for the approximate problem (4.1) we obtain

$$\begin{aligned} &\sum_{i=1}^N \int_\Omega a_i(x, T_n(u_m), \nabla u_m)(D^i u_m - D^i u)\,dx + \int_\Omega g_n(x, u_m, \nabla u_m)\,(u_m - u)\,dx \\ &+ \int_\Omega \alpha(x)|T_m(u_m)|^{r-1}T_m(u_m)\,(u_m - u)\,dx + \frac{1}{m}\int_\Omega |u_m|^{\underline{p}-2}u_m(u_m-u)\,dx + \lambda\int_{\partial\Omega} T_n(u_m)(u_m-u)\,d\sigma \\ &= \int_\Omega F(x)\,(u_m - u)\,dx, \end{aligned} \tag{4.19}$$

it follows that

$$
\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega}\left(a_i(x,T_n(u_m),\nabla u_m)-a_i(x,T_n(u_m),\nabla u)\right)(D^i u_m-D^i u)\,dx\\
&+\int_{\Omega}\alpha(x)\left(|T_m(u_m)|^{r-1}T_m(u_m)-|T_m(u)|^{r-1}T_m(u)\right)(u_m-u)\,dx\\
&\quad+\lambda\int_{\partial\Omega}(T_n(u_m)-T_n(u))\,(u_m-u)\,d\sigma\\
&=-\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_n(u_m),\nabla u)(D^i u_m-D^i u)\,dx-\int_{\Omega}g_n(x,u_m,\nabla u_m)(u_m-u)\,dx\\
&\quad-\int_{\Omega}\alpha(x)|T_m(u)|^{r-1}T_m(u)(u_m-u)\,dx-\frac{1}{m}\int_{\Omega}|u_m|^{\underline{p}-2}u_m(u_m-u)\,dx\\
&\quad-\lambda\int_{\partial\Omega}T_n(u)(u_m-u)\,d\sigma+\int_{\Omega}F(x)(u_m-u)\,dx\\
&\le\sum_{i=1}^{N}\int_{\Omega}|a_i(x,T_n(u_m),\nabla u)|\;|D^i u_m-D^i u|\,dx+\int_{\Omega}|g_n(x,u_m,\nabla u_m)||u_m-u|\,dx\\
&\quad+\int_{\Omega}\alpha(x)|T_m(u)|^{r}\;|u_m-u|\,dx+\frac{1}{m}\int_{\Omega}|u_m|^{\underline{p}-1}\;|u_m-u|\,dx\\
&\quad+\lambda\int_{\partial\Omega}|T_n(u)||u_m-u|\,d\sigma+\int_{\Omega}|F(x)|\;|u_m-u|\,dx.
\end{aligned}
\tag{4.20}
$$

For the first term on the right-hand side of (4.20), we have $T_n(u_m)\to T_n(u)$ strongly in $L^{p_i}(\Omega)$ then $|a_i(x,T_n(u_m),\nabla u)|\longrightarrow|a_i(x,T_n(u),\nabla u)|$ strongly in $L^{p_i'}(\Omega)$, and since $D^i u_m\to D^i u$ weakly in $L^{p_i}(\Omega)$, it follows that

$$\sum_{i=1}^{N}\int_{\Omega}|a_i(x,T_n(u_m),\nabla u)|\;|D^i u_m-D^i u|\,dx\longrightarrow 0\qquad\text{as}\quad m\to\infty.\tag{4.21}$$

Concerning the second term on the right-hand side of (4.20), we have $u_m\to u$ strongly in $L^1(\Omega)$, then

$$\int_{\Omega}|g_n(x,u_m,\nabla u_m)||u_m-u|\,dx\le n\int_{\Omega}|u_m-u|\,dx\longrightarrow 0\qquad\text{as}\quad m\to\infty.\tag{4.22}$$

For the third term on the right-hand side of (4.20), we have $|T_m(u)|^r\in L^{\frac{r+1}{r}}(\Omega)$ and since $u_m\rightharpoonup u$ weakly in $L^{r+1}(\Omega)$, it follows that

$$\int_{\Omega}\alpha(x)|T_m(u)|^r|u_m-u|\,dx\longrightarrow 0\qquad\text{as}\quad m\to\infty.\tag{4.23}$$

Similarly, in view of (4.16) and (3.6), we deduce that

$$\frac{1}{m}\int_{\Omega}|u_m|^{\underline{p}-1}\;|u_m-u|\,dx\longrightarrow 0\qquad\text{as}\quad m\to\infty,\tag{4.24}$$

and

$$\int_{\Omega}|F(x)|\;|u_m-u|\,dx\longrightarrow 0\qquad\text{as}\quad m\to\infty.\tag{4.25}$$

Furthermore, we have $T_n(u)$ belongs to $L^\infty(\partial\Omega)$, and since $u_m\rightharpoonup u$ weakly in $L^1(\partial\Omega)$ it follows that

$$\lambda\int_{\partial\Omega}|T_n(u)|\;|u_m-u|\,d\sigma\longrightarrow 0\qquad\text{as}\quad m\to\infty.\tag{4.26}$$

We have $u_m\to u$ strongly in $L^{\underline{p}}(\Omega)$, and y combining (4.20) and (4.21) – (4.26) we conclude that

$$
\begin{aligned}
\lim_{m\to\infty}\Big(&\sum_{i=1}^{N}\int_{\Omega}\left(a_i(x,T_n(u_m),\nabla u_m)-a_i(x,T_n(u_m),\nabla u)\right)(D^i u_m-D^i u)\,dx\\
&+\int_{\Omega}\left(|u_m|^{\underline{p}-2}u_m-|u|^{\underline{p}-2}u\right)(u_m-u)\,dx\Big)=0.
\end{aligned}
\tag{4.27}
$$

In view of Lemma 3.1, we conclude that

$$\begin{cases} u_m \to u \quad \text{strongly in} \quad W^{1,\vec{p}}(\Omega), \\ D^i u_m \to D^i u \qquad \text{a.e. in} \quad \Omega \quad \text{for} \quad i = 1, ..., N. \end{cases} \tag{4.28}$$

Thus, $a_i(x, T_n(u_m), \nabla u_m) \to a_i(x, T_n(u), \nabla u)$ and $g_n(x, u_m, \nabla u_m) \to g_n(x, u, \nabla u)$ almost everywhere in $\Omega$, and since $(a_i(x, T_n(u_m), \nabla u_m))_m$ is uniformly bounded in $L^{p_i'}(\Omega)$, it follows that

$$a_i(x, T_n(u_m), \nabla u_m) \rightharpoonup a_i(x, T_n(u), \nabla u) \qquad \text{weakly in} \quad L^{p_i'}(\Omega), \tag{4.29}$$

for $i = 1, \ldots, N$. Moreover, in view of Lebesgue dominated convergence theorem, we obtain

$$g_n(x, u_m, \nabla u_m) \to g_n(x, u, \nabla u) \qquad \text{strongly in} \quad L^{\underline{p}'}(\Omega). \tag{4.30}$$

***Step 4 : Passage to the limit***
By taking $v \in W^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$ as a test function for the approximate problem (4.1) we have

$$\begin{aligned} \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_m), \nabla u_m) D^i v\, dx + \int_\Omega g_n(x, u_m, \nabla u_m) v\, dx + \int_\Omega \alpha(x) |T_m(u_m)|^{r-1} T_m(u_m) v\, dx \\ + \frac{1}{m} \int_\Omega |u_m|^{\underline{p}-1} u_m v\, dx + \lambda \int_{\partial\Omega} T_n(u_m) v\, d\sigma = \int_\Omega F(x)\ v\, dx. \end{aligned} \tag{4.31}$$

In view of (4.16) $-$ (4.18), (4.29) and (4.30), then letting $m$ tends to infinity we conclude that

$$\sum_{i=1}^N \int_\Omega a_i(x, T_n(u), \nabla u) D^i v dx + \int_\Omega g_n(x, u, \nabla u) v dx + \int_\Omega \alpha(x) |u|^{r-1} u v dx + \lambda \int_\Omega T_n(u) v d\sigma e = \int_\Omega F(x)\ v dx. \tag{4.32}$$

Thus, the proof of the theorem 4.1 is concluded.

## 5. Main result

Let $f(x) \in L^1(\Omega)$, we begin by introducing the definition of renormalized solution for the non-coercive elliptic equation (3.1).

**Definition 5.1** A measurable function $u$ is called a renormalized solution for the strongly nonlinear and non-coercive anisotropic elliptic equation (3.1), if $u \in T_{tr}^{1,\vec{p}}(\Omega)$, with $g(x, u, \nabla u) \in L^1(\Omega)$ and $|u|^{r-1} u \in L^1(\Omega)$, such that

$$\lim_{h\to\infty} \frac{1}{h} \sum_{i=1}^N \int_{\{|u|\le h\}} a_i(x, u, \nabla u) D^i u\, dx = 0, \tag{5.1}$$

and $u$ verifies the following equality

$$\begin{aligned} \sum_{i=1}^N \int_\Omega a_i(x, u, \nabla u)(S'(u)\varphi D^i u + S(u) D^i \varphi)\, dx + \int_\Omega g(x, u, \nabla u) S(u) \varphi\, dx \\ + \int_\Omega \alpha(x) |u|^{r-1} u S(u) \varphi\, dx + \lambda \int_{\partial\Omega} u S(u) \varphi\, d\sigma = \int_\Omega f\ S(u) \varphi\, dx, \end{aligned} \tag{5.2}$$

for every $\varphi \in W^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$ and any smooth function $S(\cdot) \in W^{1,\infty}(I\!R)$ with a compact support.

**Theorem 5.1** *Assuming that (3.2) $-$ (3.5) hold true and $f \in L^1(\Omega)$, then there exists at least one renormalized solution $u$ for the strongly nonlinear and non-coercive anisotropic elliptic Neumann problem (3.1).*

**Remark 5.1** In the case of $g(x, s, \xi) \equiv 0$ and $a_i(x, s, \xi) = a_i(x, s) |\xi_i|^{p_i-2} \xi_i$, the uniqueness of renormalized solution for the problem (3.1) can be obtained, we refer the reader to [1], [15], [21] and [22] for more details.

## 6. Proof of Theorem 5.1

**Step 1: Approximate problems**

We set $f_n(\cdot) = T_n(f(\cdot))$, then $f_n(\cdot)$ is a bounded sequence in $L^\infty(\Omega) \cap L^1(\Omega)$, such that :

$$f_n \longrightarrow f \qquad \text{strongly in} \quad L^1(\Omega).$$

We consider the approximate problem :

$$\begin{cases} -\sum_{i=1}^{N} D^i a_i(x, T_n(u_n), \nabla u_n) + g_n(x, u_n, \nabla u_n) + \alpha(x)|u_n|^{r-1}u_n = f_n(x) & \text{in } \Omega, \\ \sum_{i=1}^{N} a_i(x, T_n(u_n), \nabla u_n).n_i + \lambda T_n(u_n) = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.1}$$

where $g_n(x, s, \xi) = T_n(g(x, s, \xi))$.
In view of theorem 4.1, there exists at least one solution in the sense of distributions $u_n \in W^{1,\vec{p}}(\Omega)$ for the strongly nonlinear elliptic problem (6.1), i.e. :

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i v \, dx + \int_\Omega g_n(x, u_n, \nabla u_n) v \, dx \\ &+ \int_\Omega \alpha(x)|u_n|^{r-1}u_n v \, dx + \lambda \int_{\partial\Omega} T_n(u_n) v \, \mathrm{d}\sigma = \int_\Omega f_n v \, dx \qquad \text{for any} \quad v \in W^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega). \end{aligned} \tag{6.2}$$

**Step 2: Weak convergence of truncations.**

By taking $v = T_k(u_n) \in W^{1,\vec{p}}(\Omega)$ as a test function for the approximate problem (6.2), we have

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i T_k(u_n) \, dx + \int_\Omega g_n(x, u_n, \nabla u_n) T_k(u_n) \, dx \\ &\quad + \int_\Omega \alpha(x)|u_n|^{r-1}u_n T_k(u_n) \, dx + \lambda \int_{\partial\Omega} T_n(u_n) T_k(u_n) \, \mathrm{d}\sigma = \int_\Omega f_n T_k(u_n) \, dx. \end{aligned} \tag{6.3}$$

In view of (3.4) and (3.5), we conclude that

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega b(|T_n(u_n)|)|D^i T_k(u_n)|^{p_i} \, dx + \int_\Omega \alpha(x)|u_n|^{r-1}u_n T_k(u_n) \, dx \\ &\quad + \lambda \int_{\partial\Omega} |T_n(u_n)| \; |T_k(u_n)| \, \mathrm{d}\sigma \le \int_\Omega |f(x)||T_k(u_n)| \, dx. \end{aligned}$$

Thus, we conclude that

$$b_0 \sum_{i=1}^{N} \int_\Omega \frac{|D^i T_k(u_n)|^{p_i}}{(1 + |u_n|)^\delta} \, dx + \alpha_0 \int_\Omega |u_n|^r |T_k(u_n)| \, dx + \lambda \int_{\partial\Omega} |T_n(u_n)| \; |T_k(u_n)| \, \mathrm{d}\sigma \le k\|f\|_{L^1(\Omega)}. \tag{6.4}$$

It follows that

$$\frac{b_0}{(1+k)^\delta} \sum_{i=1}^{N} \int_\Omega |D^i T_k(u_n)|^{p_i} \, dx + \alpha_0 \int_\Omega |T_k(u_n)|^{r+1} \, dx + \lambda \int_{\partial\Omega} |T_k(u_n)|^2 \, \mathrm{d}\sigma \le k\|f\|_{L^1(\Omega)}. \tag{6.5}$$

Therefore, we conclude that

$$\begin{aligned}\|T_k(u_n)\|_{1,\vec{p}} &= \|T_k(u_n)\|_{1,1} + \sum_{i=1}^{N} \|D^i T_k(u_n)\|_{p_i} \\ &= \int_\Omega |T_k(u_n)|\,dx + \sum_{i=1}^{N}\int_\Omega |D^i T_k(u_n)|\,dx + \sum_{i=1}^{N}\Big(\int_\Omega |D^i T_k(u_n)|^{p_i}\,dx\Big)^{\frac{1}{p_i}} \\ &\le k\cdot \mathrm{meas}(\Omega) + 2\sum_{i=1}^{N}\int_\Omega |D^i T_k(u_n)|^{p_i}\,dx + N + N\cdot|\Omega| \\ &\le C_1 k^{1+\delta} \qquad \text{for any} \quad k\ge 1,\end{aligned} \tag{6.6}$$

where $C_1$ is a positive constant that does not depend on $k$ and $n$. Thus, the sequence $(T_k(u_n))_n$ is uniformly bounded in $W^{1,\vec{p}}(\Omega)$, and there exists a subsequence still denoted $(T_k(u_n))_n$ and a measurable function $v_k \in W^{1,\vec{p}}(\Omega)$ such that

$$\begin{cases} T_k(u_n) \rightharpoonup v_k & \text{weakly in} \quad W^{1,\vec{p}}(\Omega), \\ T_k(u_n) \to v_k & \text{strongly in} \quad L^1(\Omega) \quad \text{and} \quad \text{a.e in} \quad \Omega. \end{cases} \tag{6.7}$$

Let $k\ge 1$, thanks to (6.4) we have

$$\int_{\partial\Omega} |T_n(u_n)|\mathrm{d}\sigma \le \mathrm{meas}_\Gamma(\partial\Omega) + \int_{\{|u_n|>1\}} |T_n(u_n)|d\sigma \le \mathrm{meas}_\Gamma(\partial\Omega) + \int_{\partial\Omega} |T_n(u_n)|\,|T_k(u_n)|d\sigma \le C_2, \tag{6.8}$$

and

$$\int_\Omega |u_n|^r\,dx \le \mathrm{meas}(\Omega) + \int_{\{|u_n|>1\}} |u_n|^r\,dx \le \mathrm{meas}(\Omega) + \int_\Omega |u_n|^r |T_k(u_n)|\,dx \le C_3. \tag{6.9}$$

with $C_2$ and $C_3$ are two positive constants that doesn't depend on $k$ and $n$.
Thus, for any $h>0$ we obtain

$$h^r\cdot \mathrm{meas}(\{h<|u_n|\}) \le \int_{\{|u_n|>h\}} |u_n|^r\,dx \le \int_\Omega |u_n|^r\,dx \le C_3,$$

it follows that

$$\limsup_{n\to\infty}\ \mathrm{meas}(\{h<|u_n|\}) \le \frac{C_3}{h^r} \longrightarrow 0 \quad \text{as} \quad h\to\infty. \tag{6.10}$$

Now, we will show that $(u_n)_n$ is a Cauchy sequence in measure.
For all $\varrho>0$, we have

$$\mathrm{meas}\{|u_n-u_m|>\varrho\} \quad \le \mathrm{meas}\{|u_n|>k\} + \ \mathrm{meas}\{|u_m|>k\} + \ \mathrm{meas}\{|T_k(u_n)-T_k(u_m)|>\varrho\}.$$

Let $\varepsilon>0$, using (6.10) we may choose $k=k(\varepsilon)$ large enough such that

$$\mathrm{meas}\{|u_n|>k\} \le \frac{\varepsilon}{3} \qquad \text{and} \qquad \mathrm{meas}\{|u_m|>k\} \le \frac{\varepsilon}{3}. \tag{6.11}$$

On the other hand, thanks to (6.7) we have $T_k(u_n)\to v_k$ strongly in $L^1(\Omega)$ and a.e. in $\Omega$. Thus, we have $(T_k(u_n))_n$ is a Cauchy sequence in measure, it follows that : for all $k>0$ and $\varepsilon,\varrho>0$, there exists $n_0=n_0(k,\varepsilon,\varrho)$ such that

$$\mathrm{meas}\{|T_k(u_n)-T_k(u_m)|>\varrho\} \le \frac{\varepsilon}{3} \qquad \text{for all } m,n\ge n_0(k,\varepsilon,\varrho). \tag{6.12}$$

By combining (6.11) – (6.12), we conclude that

$$\forall \varepsilon,\varrho>0 \quad \text{there exists} \quad n_0=n_0(\varepsilon,\varrho) \quad \text{such that} \quad \mathrm{meas}\{|u_n-u_m|>\varrho\}\le\varepsilon \quad \text{for any} \quad n,m\ge n_0(\varepsilon,\varrho).$$

We conclude that $(u_n)_n$ is a Cauchy sequence in measure, then converges almost everywhere, for a subsequence, to some measurable function $u$. Consequently, we have

$$\begin{cases} T_k(u_n) \rightharpoonup T_k(u) & \text{weakly in } W^{1,\vec{p}}(\Omega), \\ T_k(u_n) \to T_k(u) & \text{strongly in } L^1(\Omega) \text{ and a.e in } \Omega, \\ T_k(u_n) \to T_k(u) & \text{weakly in } L^1(\partial\Omega) \text{ and a.e in } \Omega. \end{cases} \tag{6.13}$$

In view of Lebesgue's dominated convergence theorem, we obtain

$$T_k(u_n) \to T_k(u) \quad \text{strongly in} \quad L^{p_i}(\Omega) \quad \text{and} \quad \text{a.e. in} \quad \Omega \quad \text{for} \quad i = 1, \dots, N. \tag{6.14}$$

Moreover, thanks to (6.5) it's clear that : for any $i = 1, \dots, N$

$$\int_\Omega \frac{|D^i T_k(u_n)|^{p_i}}{k^{p_i}}\, dx \;\leq\; \frac{\|f\|_{L^1(\Omega)} k(1+k)^\delta}{b_0 k^{p_i}} \longrightarrow 0 \qquad \text{as} \quad k \to \infty,$$

and in view of (6.10) we have $\Big\|\dfrac{T_k(u_n)}{k}\Big\|_{L^1(\Omega)} \longrightarrow 0$ as $k$ tends to infinity, then

$$\begin{aligned} \lim_{k\to\infty} \Big\|\frac{T_k(u_n)}{k}\Big\|_{L^1(\partial\Omega)} &\leq \lim_{k\to\infty} C \Big\|\frac{T_k(u_n)}{k}\Big\|_{W^{1,1}(\Omega)} \\ &\leq C \lim_{k\to\infty} \Big\|\frac{T_k(u_n)}{k}\Big\|_{L^1(\Omega)} + C \lim_{k\to\infty} \sum_{i=1}^N \Big\|\frac{D^i T_k(u_n)}{k}\Big\|_{L^1(\Omega)} \\ &\leq C \lim_{k\to\infty} \Big\|\frac{T_k(u_n)}{k}\Big\|_{L^1(\Omega)} + C' \lim_{k\to\infty} \sum_{i=1}^N \Big\|\frac{D^i T_k(u_n)}{k}\Big\|_{L^{p_i}(\Omega)} \\ &= 0. \end{aligned}$$

We conclude that

$$\frac{T_k(u_n)}{k} \longrightarrow 0 \quad \text{weak} - * \ \text{ in } L^\infty(\partial\Omega). \tag{6.15}$$

**Step 3 : Some a priori estimates.**

In this section, we will show that :

$$\lim_{h\to\infty} \limsup_{n\to\infty} \sum_{i=1}^N \frac{1}{h} \int_{\{|u_n|\leq h\}} a_i(x, T_n(u_n), \nabla u_n) D^i u_n \, dx = 0.$$

By taking $v = \dfrac{T_h(u_n)}{h}$ as a test function for the approximate problem (6.2), we obtain

$$\begin{aligned} &\frac{1}{h} \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i T_h(u_n)\, dx + \int_\Omega g_n(x, u_n, \nabla u_n) \frac{T_h(u_n)}{h}\, dx \\ &+ \int_\Omega \alpha(x)|u_n|^{r-1} u_n \frac{T_h(u_n)}{h}\, dx + \lambda \int_{\partial\Omega} T_n(u_n) \frac{T_h(u_n)}{h}\, d\sigma = \int_\Omega f_n \frac{T_h(u_n)}{h}\, dx, \end{aligned} \tag{6.16}$$

using (3.4) and (3.5), we conclude that

$$\begin{aligned} &\frac{1}{h} \sum_{i=1}^N \int_{\{|u_n|\leq h\}} a_i(x, T_n(u_n), \nabla u_n) D^i T_h(u_n)\, dx + \int_\Omega |g_n(x, u_n, \nabla u_n)| \ \frac{|T_h(u_n)|}{h}\, dx \\ &+ \int_\Omega \alpha(x)|u_n|^r \frac{|T_h(u_n)|}{h}\, dx + \lambda \int_{\partial\Omega} |T_n(u_n)| \frac{|T_h(u_n)|}{h}\, d\sigma \\ &\leq \int_\Omega |f_n| \frac{|T_h(u_n)|}{h}\, dx. \end{aligned} \tag{6.17}$$

Thanks to (6.10) we have : meas $\{|u_n| > h\} \to 0$ as $h$ tends to infinity, thus $\dfrac{|T_h(u_n)|}{h} \rightharpoonup 0$ weak$-*$ in $L^\infty(\Omega)$. Thanks to Lebesgue's dominated convergence theorem, we get

$$\lim_{h\to\infty}\limsup_{n\to\infty}\int_\Omega |f|\frac{|T_h(u_n)|}{h}\,dx = 0. \tag{6.18}$$

Thus, by letting $h$ tends to infinity in (6.17) we obtain

$$\lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\sum_{i=1}^N\int_{\{|u_n|\le h\}} a_i(x,T_n(u_n),\nabla u_n)D^iu_n\,dx = 0. \tag{6.19}$$

Moreover, we conclude that

$$\lim_{h\to\infty}\limsup_{n\to\infty}\sum_{i=1}^N\int_{\{|u_n|>h\}} |g_n(x,u_n,\nabla u_n)|\,dx = 0, \tag{6.20}$$

$$\lim_{h\to\infty}\limsup_{n\to\infty}\int_{\{|u_n|>h\}} \alpha(x)|u_n|^r\,dx = 0, \tag{6.21}$$

and

$$\lim_{h\to\infty}\limsup_{n\to\infty}\int_{\{|u_n|>h\}\cap\partial\Omega} |T_n(u_n)|\,d\sigma = 0. \tag{6.22}$$

**Step 4: Strong convergence of truncations.**

In the sequel, we denote by $\varepsilon_i(n)$, $i = 1, 2, \dots$, some various real-valued functions of real variables that converges to 0 as $n$ tends to infinity. Similarly, we define $\varepsilon_i(h)$, and $\varepsilon_i(n,h)$.
In this step, we will show the convergence of the sequence $(D^iu_n)_n$ to $D^iu$ almost everywhere in $\Omega$ for any $i = 1, \dots, N$.
We set

$$S_h(\tau) = 1 - \frac{|T_{2h}(\tau) - T_h(\tau)|}{h} \qquad \text{and} \qquad \varphi(s) = s.\exp(\frac{\gamma^2 s^2}{2}),$$

where $\gamma = \Big\|\dfrac{d(|T_k(s)|)}{b(|T_k(s)|)}\Big\|_{L^\infty(I\!R)}$, note that $\varphi'(s) - \gamma|\varphi(s)| \ge \dfrac{1}{2}$ for any $s \in I\!R$.
By taking $v = \varphi(T_k(u_n) - T_k(u))S_h(u_n)$ as a test function for the approximate problem (6.2), we have

$$\begin{aligned}
&\sum_{i=1}^N\int_\Omega a_i(x,T_n(u_n),\nabla u_n)(D^iT_k(u_n) - D^iT_k(u))\varphi'(T_k(u_n) - T_k(u))S_h(u_n)\,dx\\
&-\sum_{i=1}^N\frac{1}{h}\int_{\{h<|u_n|\le 2h\}} a_i(x,T_n(u_n),\nabla u_n)D^iu_n|\varphi(T_k(u_n) - T_k(u))|\,dx\\
&+\int_\Omega g_n(x,u_n,\nabla u_n)\varphi(T_k(u_n) - T_k(u))S_h(u_n)\,dx\\
&+\int_\Omega \alpha(x)|u_n|^{r-1}u_n\varphi(T_k(u_n) - T_k(u))S_h(u_n)\,dx + \lambda\int_{\partial\Omega} T_n(u_n)\varphi(T_k(u_n) - T_k(u))S_h(u_n)\,d\sigma\\
&= \int_\Omega f_n\varphi(T_k(u_n) - T_k(u))S_h(u_n)\,dx.
\end{aligned}$$

We have $a_i(x,r,0)=0$, and $S_h(u_n)=1$ on the set $\{|u_n|\le h\}$. Moreover, $\varphi(T_k(u_n)-T_k(u))$ have the same sign as $u_n$ on the set $\{|u_n|>k\}$. By using (3.4) and (3.5) we obtain

$$\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u_n))(D^iT_k(u_n)-D^iT_k(u))\varphi'(T_k(u_n)-T_k(u))\,dx\\
&-\sum_{i=1}^{N}\int_{\{k<|u_n|\le 2h\}} a_i(x,T_n(u_n),\nabla u_n)D^iT_k(u)\varphi'(T_k(u_n)-T_k(u))S_h(u_n)\,dx\\
&+\int_{\{|u_n|\le k\}}\alpha(x)|u_n|^{r-1}u_n\varphi(T_k(u_n)-T_k(u))\,dx+\int_{\{k<|u_n|\le 2h\}}\alpha(x)|u_n|^r|\varphi(T_k(u_n)-T_k(u))|S_h(u_n)\,dx\\
&-\int_{\{|u_n|\le k\}}|g_n(x,u_n,\nabla u_n)|\;|\varphi(T_k(u_n)-T_k(u))|\,dx\\
&+\int_{\{k<|u_n|\le 2h\}}|g_n(x,u_n,\nabla u_n)|\;|\varphi(T_k(u_n)-T_k(u))|S_h(u_n)\,dx\\
&+\lambda\int_{\{|u_n|\le k\}\cap\partial\Omega}T_n(u_n)\varphi(T_k(u_n)-T_k(u))\,d\sigma\\
&+\lambda\int_{\{k<|u_n|\le 2h\}\cap\partial\Omega}|T_n(u_n)|\;|\varphi(T_k(u_n)-T_k(u))|S_h(u_n)\,d\sigma\\
&\le\int_\Omega|f_n||\varphi(T_k(u_n)-T_k(u))|\,dx+\frac{\varphi(2k)}{h}\sum_{i=1}^{N}\int_{\{h<|u_n|\le 2h\}}a_i(x,T_n(u_n),\nabla u_n)D^iu_n\,dx.
\end{aligned}$$

In view of (3.6) we conclude that

$$\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u_n))(D^iT_k(u_n)-D^iT_k(u))\varphi'(T_k(u_n)-T_k(u))\,dx\\
&+\int_{\{|u_n|\le k\}}\alpha(x)|u_n|^{r-1}u_n\varphi(T_k(u_n)-T_k(u))\,dx\\
&+\lambda\int_{\{|u_n|\le k\}\cap\partial\Omega}T_n(u_n)\varphi(T_k(u_n)-T_k(u))\,d\sigma\\
&\le\int_\Omega|f_n||\varphi(T_k(u_n)-T_k(u))|\,dx+\frac{\varphi(2k)}{h}\sum_{i=1}^{N}\int_{\{h<|u_n|\le 2h\}}a_i(x,T_n(u_n),\nabla u_n)D^iu_n\,dx\\
&+\int_{\{|u_n|\le k\}}d(|T_k(u_n)|)\Big(c(x)+\sum_{i=1}^{N}|D^iT_k(u_n)|^{p_i}\Big)\;|\varphi(T_k(u_n)-T_k(u))|\,dx\\
&+\varphi'(2k)\sum_{i=1}^{N}\int_{\{k<|u_n|\le 2h\}}|a_i(x,T_n(u_n),\nabla u_n)||D^iT_k(u)|\,dx.
\end{aligned}$$

it follows that

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u_n))(D^iT_k(u_n)-D^iT_k(u))\varphi'(T_k(u_n)-T_k(u))\,dx\\
&\quad-\sum_{i=1}^{N}\int_{\Omega} d(|T_k(u_n)|)|D^iT_k(u_n)|^{p_i}\ |\varphi(T_k(u_n)-T_k(u))|\,dx\\
&\quad+\int_{\{|u_n|\le k\}}\alpha(x)(|T_k(u_n)|^{r-1}T_k(u_n)-|T_k(u)|^{r-1}T_k(u))\varphi(T_k(u_n)-T_k(u))\,dx\\
&\quad+\lambda\int_{\{|u_n|\le k\}\cap\partial\Omega}(T_n(u_n)-T_n(u))\varphi(T_k(u_n)-T_k(u))\,d\sigma\\
&\le\int_{\Omega}(|f(x)|+d(k)c(x))|\varphi(T_k(u_n)-T_k(u))|\,dx+\frac{\varphi(2k)}{h}\sum_{i=1}^{N}\int_{\{h<|u_n|\le 2h\}}a_i(x,T_n(u_n),\nabla u_n)D^iu_n\,dx\\
&\quad+\varphi'(2k)\sum_{i=1}^{N}\int_{\{k<|u_n|\le 2h\}}|a_i(x,T_n(u_n),\nabla u_n)||D^iT_k(u)|\,dx\\
&\quad+\int_{\{|u_n|\le k\}}\alpha(x)|T_k(u)|^r|\varphi(T_k(u_n)-T_k(u))|\,dx+\lambda\int_{\{|u_n|\le k\}\cap\partial\Omega}|T_k(u)|\ |\varphi(T_k(u_n)-T_k(u))|\,d\sigma.
\end{aligned}\tag{6.23}$$

Using the fact that $\varphi(s)$ is an increasing function, we conclude that

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u_n))(D^iT_k(u_n)-D^iT_k(u))\varphi'(T_k(u_n)-T_k(u))\,dx\\
&\quad-\Big\|\frac{d(|T_k(s)|)}{b(|T_k(s)|)}\Big\|_{L^\infty(I\!R)}\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u_n))D^iT_k(u_n)\ |\varphi(T_k(u_n)-T_k(u))|\,dx\\
&\le\int_{\Omega}(|f(x)|+d(k)c(x))|\varphi(T_k(u_n)-T_k(u))|\,dx+\frac{\varphi(2k)}{h}\sum_{i=1}^{N}\int_{\{h<|u_n|\le 2h\}}a_i(x,T_n(u_n),\nabla u_n)D^iu_n\,dx\\
&\quad+\varphi'(2k)\sum_{i=1}^{N}\int_{\{k<|u_n|\le 2h\}}|a_i(x,T_{2h}(u_n),\nabla T_{2h}(u_n))||D^iT_k(u)|\,dx\\
&\quad+\|\alpha(x)\|_{L^\infty(\Omega)}\int_{\{|u_n|\le k\}}|T_k(u)|^r|\varphi(T_k(u_n)-T_k(u))|\,dx\\
&+\lambda\int_{\{|u_n|\le k\}\cap\partial\Omega}|T_k(u)|\ |\varphi(T_k(u_n)-T_k(u))|\,d\sigma.
\end{aligned}\tag{6.24}$$

For the first term on the right-hand side of (6.25), we have $|f(x)|$ and $d(k)c(x)$ belongs to $L^1(\Omega)$, and since $\varphi(T_k(u_n)-T_k(u))\rightharpoonup 0$ weak$-\star$ in $L^\infty(\Omega)$, it follows that

$$\varepsilon_1(n)=\int_{\Omega}(|f(x)|+d(k)c(x))|\varphi(T_k(u_n)-T_k(u))|\,dx\longrightarrow 0\qquad\text{as}\quad n\to\infty.\tag{6.25}$$

Moreover, in view of (6.19) we have

$$\varepsilon_2(h)=\frac{\varphi(2k)}{h}\sum_{i=1}^{N}\int_{\{h<|u_n|\le 2h\}}a_i(x,T_n(u_n),\nabla u_n)D^iu_n\,dx\longrightarrow 0\ \text{ as }\ h\to\infty.\tag{6.26}$$

Concerning the third term on the right-hand side of (6.25), thanks to (3.4) we have $(a_i(x,T_{2h}(u_n),\nabla T_{2h}(u_n)))_n$ is a uniformly bounded sequence in $L^{p_i'}(\Omega)$, then there exists a measurable function $\vartheta_{i,h}\in L^{p_i'}(\Omega)$ such that $a_i(x,T_{2h}(u_n),\nabla T_{2h}(u_n))\rightharpoonup\vartheta_{i,h}$ weakly in $L^{p_i'}(\Omega)$ for any $i=1,\dots,N$, we conclude that

$$\begin{aligned}\varepsilon_3(n) &= \sum_{i=1}^{N}\int_{\{k<|u_n|\le 2h\}} |a_i(x,T_{2h}(u_n),\nabla T_{2h}(u_n))|\ |D^iT_k(u)|\,dx \\ &\longrightarrow \sum_{i=1}^{N}\int_{\{k<|u|\le 2h\}} |\vartheta_{i,h}|\ |D^iT_k(u)|\,dx = 0 \qquad \text{as} \quad n\to\infty.\end{aligned} \tag{6.27}$$

For the two last terms on the right-hand side of (6.25), in view of Lebesgue dominated convergence theorem, we have $|T_k(u_n)|^r \to |T_k(u)|^r$ strongly in $L^1(\Omega)$, and since $\varphi(T_k(u_n)-T_k(u)) \rightharpoonup 0$ weak$-\star$ in $L^\infty(\Omega)$, it follows that

$$\varepsilon_4(n) = \int_{\{|u_n|\le k\}} |T_k(u_n)|^r|\varphi(T_k(u_n)-T_k(u))|dx \longrightarrow 0 \qquad \text{as} \quad n\to\infty. \tag{6.28}$$

Similarly, we have $T_k(u)\in L^1(\partial\Omega)$, and since $\varphi(T_k(u_n)-T_k(u)) \rightharpoonup 0$ weak$-\star$ in $L^\infty(\partial\Omega)$, then

$$\varepsilon_5(n) = \int_{\partial\Omega} |T_k(u_n)|\ |\varphi(T_k(u_n)-T_k(u))|\,d\sigma \longrightarrow 0 \quad \text{as} \quad n\to\infty. \tag{6.29}$$

By combining (6.24) and (6.26) – (6.29), we conclude that

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u_n))(D^iT_k(u_n)-D^iT_k(u))\varphi'(T_k(u_n)-T_k(u))\,dx \\ &\quad -\Big\|\frac{d(|T_k(s)|)}{b(|T_k(s)|)}\Big\|_{L^\infty(I\!R)}\sum_{i=1}^{N}\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u_n))D^iT_k(u_n)\ |\varphi(T_k(u_n)-T_k(u))|\,dx \\ &\le \varepsilon_6(n,h).\end{aligned} \tag{6.30}$$

We have $\gamma = \Big\|\frac{d(|T_k(s)|)}{b(|T_k(s)|)}\Big\|_{L^\infty(I\!R)}$, it follows that

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega (a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u)))(D^iT_k(u_n)-D^iT_k(u)) \\ &\qquad\qquad \times\Big(\varphi'(T_k(u_n)-T_k(u))-\gamma|\varphi(T_k(u_n)-T_k(u))|\Big)\,dx \\ &\le \varepsilon_6(n,h)+\Big(\varphi'(2k)+\gamma\varphi(2k)\Big)\sum_{i=1}^{N}\int_\Omega |a_i(x,T_k(u_n),\nabla T_k(u))|\ |D^iT_k(u_n)-D^iT_k(u)|\,dx \\ &\quad +\gamma\varphi(2k)\sum_{i=1}^{N}\int_\Omega |a_i(x,T_k(u_n),\nabla T_k(u_n))|\ |D^iT_k(u)|\,dx.\end{aligned} \tag{6.31}$$

For the second term on the left-hand side of (6.31), in view of (6.14) we have $T_k(u_n)\to T_k(u)$ strongly in $L^{p_i}(\Omega)$, then

$$a_i(x,T_k(u_n),\nabla T_k(u)) \to a_i(x,T_k(u),\nabla T_k(u)) \qquad \text{strongly in} \quad L^{p_i'}(\Omega),$$

and since $D^iT_k(u_n)$ tends to $D^iT_k(u)$ weakly in $L^{p_i}(\Omega)$, we obtain

$$\varepsilon_7(n) \le \sum_{i=1}^{N}\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u))(D^iT_k(u_n)-D^iT_k(u))\,dx \to 0 \quad \text{as} \quad n\to\infty. \tag{6.32}$$

Concerning the last term on the left-hand side of (6.31), we have $(|a_i(x,T_k(u_n),\ \nabla T_k(u_n))|)_n$ is bounded in $L^{p_i'}(\Omega)$, then there exists a measurable function $\vartheta_{i,k}\in L^{p_i'}(\Omega)$ such that $|a_i(x,T_k(u_n),\nabla T_k(u_n))| \rightharpoonup$

$\vartheta_{i,k}$ weakly in $L^{p'_i}(\Omega)$, and since $|D^iT_k(u)|\ |\varphi(T_k(u_n)-T_k(u))|$ tends strongly to 0 in $L^{p_i}(\Omega)$ for any $i=1,\dots,N$, it follows that

$$\varepsilon_8(n)=\sum_{i=1}^N\int_\Omega |a_i(x,T_k(u_n),\nabla T_k(u_n))||D^iT_k(u)|\ |\varphi(T_k(u_n)-T_k(u))|dx\to 0 \qquad \text{as} \quad n\to\infty. \tag{6.33}$$

By combining (6.31) and (6.32) – (6.33), we conclude that

$$\begin{aligned}
0\ &\leq \frac{1}{2}\sum_{i=1}^N\int_\Omega \Big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\Big)(D^iT_k(u_n)-D^iT_k(u))\,dx\\
&\leq \sum_{i=1}^N\int_\Omega \Big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\Big)(D^iT_k(u_n)-D^iT_k(u))\\
&\qquad\qquad\qquad\times\Big(\varphi'(T_k(u_n)-T_k(u))-\gamma|\varphi(T_k(u_n)-T_k(u))|\Big)\,dx\\
&\leq \varepsilon_9(n,h)\longrightarrow 0 \qquad \text{as} \quad n,h\to 0.
\end{aligned} \tag{6.34}$$

Having in mind $T_k(u_n)\to T_k(u)$ strongly in $L^{\underline{p}}(\Omega)$, we conclude that

$$\begin{aligned}
&\sum_{i=1}^N\int_\Omega \Big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\Big)(D^iT_k(u_n)-D^iT_k(u))\,dx\\
&+\int_\Omega(|T_k(u_n)|^{r-1}T_k(u_n)-|T_k(u)|^{r-1}T_k(u))(T_k(u_n)-T_k(u))\,dx\to 0 \quad \text{as} \quad n\to\infty.
\end{aligned} \tag{6.35}$$

Thanks to Lemma 3.1, we obtain

$$\begin{cases} T_k(u_n)\to T_k(u) & \text{strongly in} \quad W^{1,\vec{p}}(\Omega),\\ D^iu_n\to D^iu & \text{a.e. in} \quad \Omega \quad \text{for} \quad i=1,\dots,N,\\ T_k(u_n)\to T_k(u) & \text{strongly in} \quad L^1(\partial\Omega) \quad \text{and} \quad \text{a.e. on} \quad \partial\Omega.\end{cases} \tag{6.36}$$

Moreover, we have $a_i(x,T_n(u_n),\nabla u_n)D^iu_n$ tends to $a_i(x,u,\nabla u)D^iu$ almost everywhere in $\Omega$, and in view of Fatou's lemma and (6.19) we conclude that

$$\begin{aligned}
&\lim_{h\to\infty}\frac{1}{h}\sum_{i=1}^N\int_\Omega a_i(x,T_h(u),\nabla T_h(u))D^iT_h(u)\,dx\\
&\leq \lim_{h\to\infty}\liminf_{n\to\infty}\frac{1}{h}\sum_{i=1}^N\int_\Omega a_i(x,T_h(u_n),\nabla T_h(u_n))D^iT_h(u_n)\,dx\\
&\leq \lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\sum_{i=1}^N\int_\Omega a_i(x,T_h(u_n),\nabla T_h(u_n))D^iT_h(u_n)\,dx=0.
\end{aligned} \tag{6.37}$$

***Step 4 : The equi-integrability of the sequences*** $(g_n(x,u_n,\nabla u_n))_n$.
In order to pass to the limit in the approximate problem (6.2), we shall show that

$$g_n(x,u_n,\nabla u_n)\longrightarrow g(x,u,\nabla u) \ \text{ and } \ \alpha(x)|u_n|^{r-1}u_n\longrightarrow \alpha(x)|u|^{r-1}u \ \text{ strongly in } \ L^1(\Omega). \tag{6.38}$$

and

$$T_n(u_n)\longrightarrow u \qquad \text{strongly in} \ \ L^1(\partial\Omega). \tag{6.39}$$

Thanks to (6.36) we have

$$g_n(x,u_n,\nabla u_n)\longrightarrow g(x,u,\nabla u) \qquad \text{and} \qquad \alpha(x)|u_n|^{r-1}u_n\longrightarrow \alpha(x)|u|^{r-1}u \qquad \text{a.e. in} \quad \Omega,$$

and

$$T_n(u_n)\longrightarrow u \qquad \text{a.e. on} \quad \partial\Omega.$$

Then, in view of Vitali's theorem, it suffices to prove that the sequences $(g_n(x,u_n,\nabla u_n))_n$, $(\alpha(x)|u_n|^{r-1}u_n)_n$ and $(T_n(u_n))_n$ are uniformly equi-integrable.
On the one hand, for any measurable subset $E\subseteq\Omega$ we have

$$\begin{aligned}
&\sum_{i=1}^N\int_E|g_n(x,u_n,\nabla u_n)|\,dx+\int_E\alpha(x)|u_n|^r\,dx\\
&\leq\sum_{i=1}^N\int_E|g_n(x,T_{h(\eta)}(u_n),\nabla T_{h(\eta)}(u_n))|\,dx+\int_E\alpha(x)|T_{h(\eta)}(u_n)|^r\,dx\\
&\quad+\sum_{i=1}^N\int_{\{h(\eta)<|u_n|\}}|g_n(x,u_n,\nabla u_n)|\,dx+\int_{\{h(\eta)<|u_n|\}}\alpha(x)|u_n|^r\,dx.
\end{aligned}\tag{6.40}$$

Thanks to (6.36), there exists $\beta(\eta)>0$ such that

$$\sum_{i=1}^N\int_E|g_n(x,T_{h(\eta)}(u_n),\nabla T_{h(\eta)}(u_n))|\,dx+\int_E\alpha(x)|T_{h(\eta)}(u_n)|^r\,dx\leq\frac{\eta}{2},\tag{6.41}$$

for any $E\subset\Omega$ with $\text{meas}(E)\leq\beta(\eta)$.
Moreover, in view of (6.20) and (6.21) we have : for all $\eta>0$, there exists $h(\eta)>0$ such that

$$\sum_{i=1}^N\int_{\{h(\eta)<|u_n|\}}d(|u_n|)|D^iu_n|^{p_i}\,dx+\int_{\{h(\eta)<|u_n|\}}\alpha(x)|u_n|^r\,dx\leq\frac{\eta}{2}\quad\text{for all}\quad h\geq h(\eta).\tag{6.42}$$

Thanks to (3.5), and in view of (6.40), (6.41) and (6.42), one easily has

$$\int_E|g_n(x,u_n,\nabla u_n)|\,dx+\int_E\alpha(x)|u_n|^r\,dx\leq\eta\qquad\text{for all } E \text{ such that } \text{meas}(E)\leq\beta(\eta).\tag{6.43}$$

Then, the sequences $\big(\alpha(x)|u_n|^{r-1}u_n\big)_n$ and $(g_n(x,u_n,\nabla u_n))_n$ are uniformly equi-integrable. In view of Vitali's theorem, the convergence (6.38) is concluded.
On the other hand, in view of (6.22) we have : For any $\eta>0$, there exists $h(\eta)>0$ such that

$$\int_{\{h(\eta)<|u_n|\}\cap\partial\Omega}|T_n(u_n)|\,d\sigma\leq\frac{\eta}{2}\quad\text{for all}\quad h\geq h(\eta).\tag{6.44}$$

Moreover, there exists $\beta(\eta)>0$ such that

$$\int_U|T_{h(\eta)}(u_n)|\,d\sigma\leq\frac{\eta}{2}\qquad\text{for any subset}\quad U\subset\partial\Omega\quad\text{with}\quad\text{meas}_\Gamma(U)\leq\beta(\eta).\tag{6.45}$$

Using (6.44) and (6.45) we conclude that : for any $\eta>0$, there exists $\beta(\eta)>0$ such that

$$\int_U|T_n(u_n)|d\sigma\leq\int_U|T_{h(\eta)}(u_n)|d\sigma+\int_{\{h(\eta)<|u_n|\}\cap\partial\Omega}|T_n(u_n)|d\sigma\leq\eta\quad\forall U\subset\partial\Omega\quad\text{with}\quad\text{meas}_\Gamma(U)\leq\beta(\eta).\tag{6.46}$$

Thus, we conclude that $(T_n(u_n))_n$ is a sequences uniformly equi-integrable in $L^1(\partial\Omega)$. In view of Vitali's theorem, the convergence (6.39) is deduced.

***Step 5 : Passage to the limit***
Let $\varphi\in W^{1,\vec{p}}(\Omega)\cap L^\infty(\Omega)$, and choosing $S(\cdot)$ a smooth function in $W^{1,\infty}(I\!R)$ such that supp $(S(\cdot))\subseteq[-M,M]$ for some $M\geq0$.
By choosing $S(u_n)\varphi\in W^{1,\vec{p}}(\Omega)\cap L^\infty(\Omega)$ as a test function in the approximate problem (6.2), we obtain

$$\begin{aligned}
&\sum_{i=1}^N\int_\Omega a_i(x,T_n(u_n),\nabla u_n)(S'(u_n)\varphi D^iu_n+S(u_n)D^i\varphi)\,dx+\int_\Omega g_n(x,u_n,\nabla u_n)S(u_n)\varphi\,dx\\
&\quad+\int_\Omega\alpha(x)|u_n|^{s-1}u_nS(u_n)\varphi\,dx+\lambda\int_{\partial\Omega}T_n(u_n)S(u_n)\varphi\,d\sigma=\int_\Omega f_nS(u_n)\varphi\,dx.
\end{aligned}\tag{6.47}$$

In view of (6.36), we have $(a_i(x, T_M(u_n), \nabla T_M(u_n)))_n$ is bounded in $L^{p'_i}(\Omega)$, and since $a_i(x, T_M(u_n), \nabla T_M(u_n))$ tends to $a_i(x, T_M(u), \nabla T_M(u))$ almost everywhere in $\Omega$, it follows that

$$a_i(x, T_M(u_n), \nabla T_M(u_n)) \rightharpoonup a_i(x, T_M(u), \nabla T_M(u)) \quad \text{weakly in} \quad L^{p'_i}(\Omega),$$

and since $S'(u_n)\varphi D^i T_M(u_n) + S(T_M(u_n))D^i\varphi$ tends strongly to $S'(u)\varphi D^i T_M(u) + S(T_M(u))D^i\varphi$ in $L^{p_i}(\Omega)$, we deduce that

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n)(S'(u_n)\varphi D^i u_n + S(u_n)D^i\varphi)\, dx \\
&= \lim_{n\to\infty} \sum_{i=1}^{N} \int_\Omega a_i(x, T_M(u_n), \nabla T_M(u_n)) \left(S'(u_n)\varphi D^i T_M(u_n) + S(T_M(u_n))D^i\varphi\right)\, dx \\
&= \sum_{i=1}^{N} \int_\Omega a_i(x, T_M(u), \nabla T_M(u)) \left(S'(u)\varphi D^i T_M(u) + S(T_M(u))D^i\varphi\right)\, dx \\
&= \sum_{i=1}^{N} \int_\Omega a_i(x, u, \nabla u) \left(S'(u)\varphi D^i u + S(u)D^i\varphi\right)\, dx.
\end{aligned} \tag{6.48}$$

Concerning the second and third terms on the left-hand side of (6.47), we have $S(u_n)\varphi \rightharpoonup S(u)\varphi$ weak$-*$ in $L^\infty(\Omega)$, and thanks to (6.38) we deduce that

$$\lim_{n\to\infty} \int_\Omega g_n(x, u_n, \nabla u_n)S(u_n)\varphi\, dx = \int_\Omega g(x, u, \nabla u)S(u)\varphi\, dx, \tag{6.49}$$

and

$$\lim_{n\to\infty} \int_\Omega \alpha(x)|u_n|^{r-1}u_n S(u_n)\varphi\, dx = \int_\Omega \alpha(x)|u|^{r-1}uS(u)\varphi\, dx. \tag{6.50}$$

Moreover, we have

$$\lim_{n\to\infty} \int_\Omega f_n S(u_n)\varphi\, dx = \int_\Omega f S(u)\varphi\, dx. \tag{6.51}$$

Similarly, we have $S(u_n)\varphi \rightharpoonup S(u)\varphi$ weak$-*$ in $L^\infty(\partial\Omega)$, and thanks to (6.39) we get

$$\lim_{n\to\infty} \lambda \int_{\partial\Omega} T_n(u_n)S(u_n)\varphi\, d\sigma = \lambda \int_{\partial\Omega} uS(u)\varphi\, d\sigma. \tag{6.52}$$

Hence, putting all the terms (6.47) and (6.48) − (6.52) together, we obtain

$$\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega a_i(x, u, \nabla u) \left(S'(u)\varphi D^i u + S(u)D^i\varphi\right)\, dx + \int_\Omega g(x, u, \nabla u)S(u)\varphi\, dx \\
&\qquad + \int_\Omega \alpha(x)|u|^{r-1}uS(u)\varphi\, dx + \lambda \int_{\partial\Omega} uS(u)\varphi\, d\sigma = \int_\Omega fS(u)\varphi\, dx,
\end{aligned} \tag{6.53}$$

which conclude the proof of Theorem 5.1.

**Remark 6.1** Note that the existence of renormalized solutions for our equation (3.1) in the parabolic case can be proved using similar arguments as in the Theorem 5.1.

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