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The dot total graph of a commutative ring without the zero element

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ABSTRACT: Let $\mathcal{R}$ be a commutative ring with $1 \neq 0$, $Z(\mathcal{R})$ be the set of zero-divisors of $\mathcal{R}$, and $Reg(\mathcal{R}) = \mathcal{R} \setminus Z(\mathcal{R})$ be the set of regular elements of $\mathcal{R}$. The dot total graph of $\mathcal{R}$ is the simple (undirected) graph $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}))$ with vertices as all elements of $\mathcal{R}$, and two distinct vertices x and y are adjacent if and only if $xy \in Z(\mathcal{R})$. In this paper, we study the (induced) subgraph $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ of $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}))$, with vertices $\mathcal{R}^* = \mathcal{R} \setminus \{0\}$. After that, connectivity, clique number, and girth of $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ have also been studied. Finally, we determine the cases when $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ is Eulerian, Hamiltonian, and $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ contains an Eulerian trail.

Key Words: commutative rings, dot total graph, zero-divisor graph, zero-divisors.

Contents

1. Introduction

Throughout this paper, $\mathcal{R}$ will represent an associative and commutative ring with non-zero unity. The symbols $Z(\mathcal{R})$ and $Reg(\mathcal{R}) = \mathcal{R} \setminus Z(\mathcal{R})$ stands for zero-divisors of $\mathcal{R}$ and regular elements of $\mathcal{R}$, respectively. In 1988, Beck [10] considered $\Gamma(\mathcal{R})$ as a simple graph, whose vertices are the elements of $\mathcal{R}$ and any two different elements x and y are adjacent if and only if $xy = 0$, but he was mainly interested in colorings. In 1993, Anderson and Naseer [6] continued this study by giving a counter example, where $\mathcal{R}$ is a finite local ring. In 1999, Anderson and Livingston [2], associated a (simple) graph $\Gamma(\mathcal{R})$ to $\mathcal{R}$ with vertices $Z(\mathcal{R})^* = Z(\mathcal{R}) \setminus \{0\}$, the set of nonzero zero-divisors of $\mathcal{R}$, and for distinct $x, y \in Z(\mathcal{R})^*$, the vertices x and y are adjacent if and only if $xy = 0$ and they were interested to study the interplay of ring-theoretic properties of $\mathcal{R}$ with graph-theoretic properties of $\Gamma(\mathcal{R})$. In 2008, Anderson and Badawi [3] introduced the total graph of $\mathcal{R}$, denoted by $T(\Gamma(\mathcal{R}))$, as the (undirected) graph with all elements of $\mathcal{R}$ as vertices and for distinct $x, y \in \mathcal{R}$, the vertices x and y are adjacent if and only if $x + y \in Z(\mathcal{R})$. Also, in 2012 Anderson and Badawi [4] studied the two (induced) subgraphs $Z_0(\Gamma(\mathcal{R}))$ and $T_0(\Gamma(\mathcal{R}))$ of $T(\Gamma(\mathcal{R}))$, with vertices $Z(R) \setminus \{0\}$ and $\mathcal{R} \setminus \{0\}$, respectively. Recently, Ashraf *et.al.*, in [8] introduced the dot total graph of $\mathcal{R}$ to be the simple (undirected) graph $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}))$ with vertices all elements of $\mathcal{R}$, and two distinct vertices x and y are adjacent if and only if $xy \in Z(\mathcal{R})$. Also, Ashraf *et.al.*, in [7] introduced an ideal-based dot total graph of $\mathcal{R}$ denoted by $T_I(\Gamma(\mathcal{R}))$. The graph $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}))$ is connected with $diam(T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}))) \leq 2$ since $x - 0 - y$ is a path between any two vertices x and y in $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}))$. In this paper, we consider the (induced) subgraph $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ of $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}))$, with vertices $\mathcal{R}^* = \mathcal{R} \setminus \{0\}$. In addition, some fundamental graphs with zero-divisors can be identified in [1,5,9,11,13,14].

Let $G(V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. If there is a path between every two vertices $x, y \in V(G)$, then G is said to be connected. The distance between x and y denoted by $d(x, y)$ is defined as the shortest path from x to y (if there is no such path, then $d(x, y) = \infty$). The

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diameter of a graph G is the largest distance between any two vertices of G and is denoted by $diam(G)$. The girth of a graph G is defined as the length of the smallest cycle in G and is denoted by $gr(G)$. if G contains no cycle, then $gr(G) = \infty$. Note that if G contains a cycle, then $gr(G) \leq 2\ diam(G) + 1$. The number of edges incident with a vertex v in a graph G is its degree and is denoted by $deg(v)$. A vertex v is said to be a cut-vertex in a connected graph G if $G \setminus \{v\}$ is disconnected. Also, the subset U of a vertex set $V(G)$ is said to be a vertex-cut if removed together with any incident edges, causing the graph to become disconnected. A graph G has connectivity $\kappa(G) = k$ if k is the cardinality of the smallest subset of the vertex set whose deletion causes the graph to become disconnected. We have the same notions in the edges as well. An edge e is said to be a bridge in a connected graph G if $G \setminus \{e\}$ is disconnected. Also, the subset X of an edge set $E(G)$ is said to be an edge-cut if removed, causing the graph to become disconnected. A graph G has edge-connectivity $\lambda(G) = l$ if l is the cardinality of the smallest subset of the edge set whose deletion causes the graph to become disconnected. A clique is a subset of vertices in a graph G where each pair of different vertices is adjacent. The clique number of a graph G is represented by $\omega(G)$ and is defined as the highest feasible size of a clique in the graph. An Eulerian graph is a graph G that contains an Eulerian circuit, which is a circuit that includes all the edges of G. Also, the Eulerian trail is an open trail that includes all edges of G. A Hamiltonian graph is a graph G that contains a Hamiltonian cycle, which is a cycle that includes all the vertices of G. In addition, the Hamiltonian path is a path that includes all vertices of G. More information about the graphs can be identified in [12]. The following is the structure of the present article:
In Section 2, we study the dot total graph of $\mathcal{R}$ without zero element. We give many examples and prove that if $\mathcal{R}$ is not an integral domain, then $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ is connected, and has a diameter of at most two. We determine whether $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ is a regular graph or a complete graph. Also, we calculate the degree of each vertex in $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$. Further, in Section 3 and 4, we prove certain facts concerning cut-vertices and bridges in $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$. In addition, we compute the $\kappa(T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*)))$, $\omega(T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*)))$ and $gr(T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*)))$. Finally, in Section 5, we demonstrate that $T_{Z(\mathcal{R})}\Gamma(\mathcal{R}^*))$ can be an Eulerian graph and an Eulerian trail. We also determine the graph $T_{Z(\mathcal{R})}\Gamma(\mathcal{R}^*))$ is a Hamiltonian graph.

2. Definition and properties of $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$

In this section, we study the connectedness of $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$. In fact, after removing the zero element from the ring $\mathcal{R}$, $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ is still connected if $\mathcal{R}$ is not an integral domain, and $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ is an empty graph if $\mathcal{R}$ is an integral domain. We find the diameter of $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$, and degree of each vertex of $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$. Recall that $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}))$ is connected (for reference see [8]), and the diameter of $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}))$ is at most two. We now show that the dot total graph of $\mathcal{R}^*$ is not connected if $\mathcal{R}$ is an integral domain.

Theorem 2.1 *Let $\mathcal{R}$ be a commutative ring.*

- (i) *If $\mathcal{R}$ is an integral domain, then $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ is not connected.*
- (ii) *If $\mathcal{R}$ is not an integral domain, then $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ is connected. Moreover, $diam(T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))) \leq 2$.*

Proof:

- (i) Let $\mathcal{R}$ be an integral domain. Then all elements in $\mathcal{R}^* = \mathcal{R} \setminus \{0\}$ are regular elements. Since there is no adjacency between any two elements of $Reg(\mathcal{R})$, implies that $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ is an empty graph. Hence $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ is not connected.
- (ii) Let $\mathcal{R}$ be a commutative ring which is not an integral domain. Then $Z(\mathcal{R})$ has at least two elements. Let x and y be distinct vertices of $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$. Then we have the following cases:

Case(i) If $x, y \in Z(\mathcal{R})$, then $x - y$ is a path in $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$.

Case(ii) If $x, y \in Reg(\mathcal{R})$, then there exists some $0 \neq z \in Z(\mathcal{R})$ such that $xz \in Z(\mathcal{R})$ and $yz \in Z(\mathcal{R})$. Thus $x - z - y$ is a path in $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$.

Case(iii) If $x \in Z(\mathcal{R})$ and $y \in Reg(\mathcal{R})$, then $x - y$ is a path in $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$.

Hence $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ is connected and $diam(T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))) \leq 2$. □

Example 2.1 *We have several rings with the set of zero-divisors $Z(\mathcal{R})$ and the set of regular elements $Reg(\mathcal{R})$ and comparisons $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}))$ and $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$:*

(i) $\mathcal{R} = \mathbb{Z}_4, Z(\mathcal{R}) = \{0, 2\}$ *and* $Reg(\mathcal{R}) = \{1, 3\}$ *(see Fig. 1).*

Figure 1: (a) $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ and (b) $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}))$, when $\mathcal{R} = \mathbb{Z}_4$

(ii) $\mathcal{R} = \mathbb{Z}_9, Z(\mathcal{R}) = \{0, 3, 6\}$ *and* $Reg(\mathcal{R}) = \{1, 2, 4, 5, 7, 8\}$ *(see Fig. 2).*

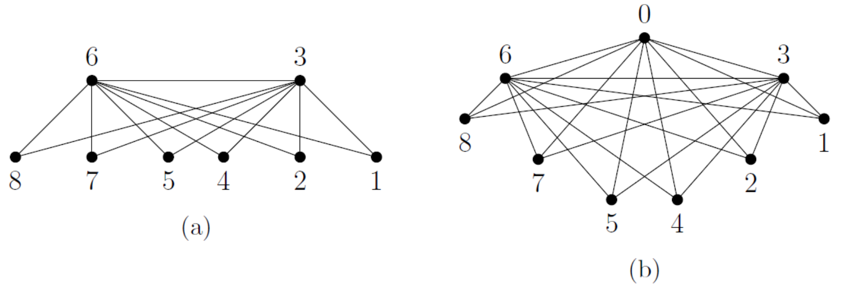

Figure 2: (a) $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ and (b) $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}))$, when $\mathcal{R} = \mathbb{Z}_9$

(iii) $\mathcal{R} = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0),(0,1),(1,0),(1,1)\}, Z(\mathcal{R}) = \{(0,0),(0,1),(1,0)\}$ *and* $Reg(\mathcal{R}) = \{(1,1)\}$ *(see Fig. 3).*

Figure 3: (a) $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ and (b) $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}))$, when $\mathcal{R} = \mathbb{Z}_2 \times \mathbb{Z}_2$

(iv) $\mathcal{R} = \mathbb{Z}_7, Z(\mathcal{R}) = \{0\}$ *and* $Reg(\mathcal{R}) = \{1,2,3,4,5,6\}$ *(see Fig. 4).*

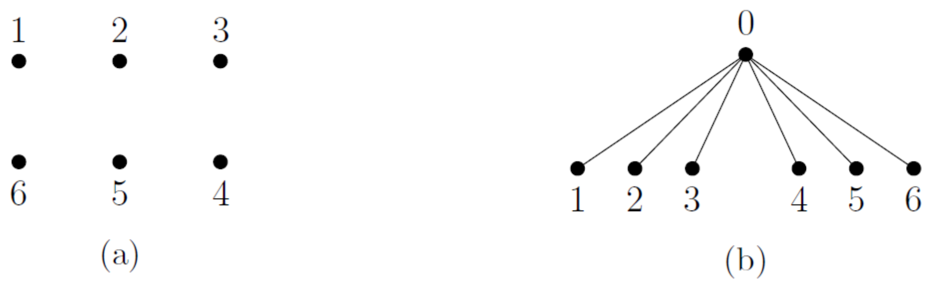

Figure 4: (a) $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ and (b) $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}))$, when $\mathcal{R} = \mathbb{Z}_7$

Notice that $|\mathcal{R}| \leq 3$ if and only if $\mathcal{R}$ is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$. In the next theorem we find the diameter of $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$.

Theorem 2.2 *Let $\mathcal{R}$ be a commutative ring.*

(i) *If $|\mathcal{R}| \leq 3$, then* $$diam(T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))) = \begin{cases} 0 & \text{if } \mathcal{R} \cong \mathbb{Z}_2, \\ \infty & \text{if } \mathcal{R} \cong \mathbb{Z}_3. \end{cases}$$

(ii) *If $|\mathcal{R}| \geq 4$, then* $$diam(T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))) = \begin{cases} \infty & \text{if } |Z(\mathcal{R})| = 1, \\ 1 & \text{if } |Reg(\mathcal{R})| = 1, \\ 2 & \text{if } |Z(\mathcal{R})| \geq 2 \text{ and } |Reg(\mathcal{R})| \geq 2. \end{cases}$$

Proof:

(i) Let $|\mathcal{R}| \leq 3$. Then $\mathcal{R} \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$. If $\mathcal{R} \cong \mathbb{Z}_2$, then $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ is a complete graph of order one. Thus $diam(T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))) = 0$. If $\mathcal{R} \cong \mathbb{Z}_3$, then $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ is a disconnected graph with two vertices and has no edge. Thus $diam(T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))) = \infty$.

(ii) Let $|\mathcal{R}| \geq 4$. Then depending on the cardinality of the zero-divisors and regular elements of $\mathcal{R}$, we have the following cases:

Case(i) If $|Z(\mathcal{R})| = 1$, *i.e.*, $\mathcal{R}$ is an integral domain, then there is no adjacency between any elements in $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$. Hence $diam(T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))) = \infty$.

Case(ii) If $|Reg(\mathcal{R})| = 1$ and $|\mathcal{R}| \geq 4$, then the set of zero-divisors of $\mathcal{R}$ has at least three elements. Therefore, all elements are adjacent. Hence $diam(T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))) = 1$.

Case(iii) If $|Z(\mathcal{R})| \geq 2$ and $|Reg(\mathcal{R})| \geq 2$, then the proof is clear by the same arguments as used in the Theorem 2.1.

This completes the proof. □

Theorem 2.3 *$T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ is a complete graph of order $n = |Z(\mathcal{R})|$ if and only if $Reg(\mathcal{R}) = \{1\}$, where 1 is the unity of $\mathcal{R}$.*

Proof: Suppose that $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ is complete. Then each pair of distinct vertices in $\mathcal{R}^*$ is adjacent. This implies that all vertices of $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ in $Z(\mathcal{R})^*$ except the unity belongs to $Reg(\mathcal{R})$. Otherwise $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ is not complete. Hence $Reg(\mathcal{R}) = \{1\}$.
Conversely, suppose that $Reg(\mathcal{R}) = \{1\}$ and we prove that $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ is complete. Assume, on contrary, that $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ is not complete. Then there are two distinct elements x and y of $\mathcal{R}^*$ such that x is not adjacent to y. This implies that $x, y \in Reg(\mathcal{R})$, which is a contradiction to our assumption. Hence $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ is complete. □

Remark 2.1 *$T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}))$ and $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ have the same condition to be a complete graph. In case they are complete, then $|V(T_{Z(\mathcal{R})}(\Gamma(\mathcal{R})))| = |V(T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*)))| + 1$.*

Corollary 2.1 *$T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ is not complete if and only if $|Reg(\mathcal{R})| \geqslant 2$.*

Corollary 2.2 *Let $\mathcal{R}$ be an integral domain, Then $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*)) \cong \overline{K}_n$, where $n = |Reg(\mathcal{R})|$.*

Remark 2.2 *For any integral domain $\mathcal{R}$, we know that $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}))$ is a star graph. But the graph $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ is an empty graph (see Example 2.1, Figure 4).*

In the next theorem, we find the degree of each vertex of $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$.

Theorem 2.4 *$deg(x) = |\mathcal{R}| - 2$ or $|Z(\mathcal{R})| - 1$, where $x \in V(T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*)))$.*

Proof: Since each vertex in $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ is either nonzero zero-divisor or regular element, we have two cases as follows:

(*i*) If $0 \neq x \in Z(\mathcal{R})$, then x is adjacent to all vertices in $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ except x, that is, x is adjacent to $(|\mathcal{R}|-2)$ vertices and hence $deg(x) = (|\mathcal{R}|-2)$.

(*ii*) If $x \in Reg(\mathcal{R})$, then x is adjacent to all vertices in $Z(\mathcal{R})^*$, that is, x is adjacent to $(|Z(\mathcal{R})|-1)$ vertices and hence $deg(x) = (|Z(\mathcal{R})|-1)$.

Hence the degree of each vertex of $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ is either $(|\mathcal{R}|-2)$ or $(|Z(\mathcal{R})|-1)$. □

Corollary 2.3 *$T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ is a regular graph if and only if $|Reg(\mathcal{R})| = 1$ (i.e., $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ is a complete graph).*

Remark 2.3 *The minimum degree of $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ is $\delta(T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))) = |Z(\mathcal{R})|-1$, and the maximum degree of $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ is $\Delta(T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))) = |\mathcal{R}|-2$.*

3. Connectivity of $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$

In this section, we investigate the conditions that must be met to obtain a cut-vertex or bridge in the graph $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$. Also, we find the connectivity of $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$.

Theorem 3.1 *$T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ has a cut-vertex if and only if $|\mathcal{R}| \geq 4$ and $|Z(\mathcal{R})| = 2$.*

Proof: Suppose that the graph $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ has a cut-vertex (say z). Then there exist at least two vertices $u, w \in T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ such that z lies on all paths from u to w. If u is adjacent to w, then we get a contradiction. So we assume that u is not adjacent to w. Then $u, w \in Reg(\mathcal{R})$ and each element of $Reg(\mathcal{R})$ is adjacent to the element of $Z(\mathcal{R})$. Thus $z \in Z(\mathcal{R})^*$. Now, if $|\mathcal{R}| \leq 3$, then $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ is without edges. Therefore, we assume that $|\mathcal{R}| \geq 4$ and we have the following two cases:

(*i*) If $|Z(\mathcal{R})| > 2$, then $Z(\mathcal{R})$ has at least one non-zero element (say z_0). Now let $z_0, z \in Z(\mathcal{R})^*$. This implies that u, w are adjacent to each element of $Z(\mathcal{R})$. In particular adjacent to z_0 and z, that is, there is at least one path from u to w and z does not lie on it, which is a contradiction.

(*ii*) If $|Z(\mathcal{R})| = 2$, then $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*)) \cong K_{1,m}$, where $m = |Reg(\mathcal{R})|$. Now, we find that if z is the cut-vertex of $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$, then $|Z(\mathcal{R})| = 2$ and $|\mathcal{R}| \geq 4$.

Conversely, assume that $|\mathcal{R}| \geq 4$ and $|Z(\mathcal{R})| = 2$. Then it is clear that $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ has a cut-vertex. □

Theorem 3.2 *$T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ has a bridge if and only if $|Z(\mathcal{R})| = 2$ (i.e., $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ is a star graph).*

Proof: Assume that $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ has a bridge. We know that if $|\mathcal{R}| = 2$ or 3, then $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*)) \cong K_1$ or $\overline{K}_2$. Therefore, $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ has no bridge in these cases. Now let $|\mathcal{R}| \geq 4$. According to cardinality of zero-divisor of $\mathcal{R}$, we have the following cases:

(*i*) If $|Z(\mathcal{R})| = 1$, then $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*)) \cong \overline{K}_m$, where $m = |Reg(\mathcal{R})|$ which has no bridges.

(*ii*) If $|Z(\mathcal{R})| = 2$, then $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*)) \cong K_{1,m}$, where $m = |Reg(\mathcal{R})|$. Thus all edges are bridge.

(*iii*) If $|Z(\mathcal{R})| > 2$ and $|Reg(\mathcal{R})| = 1$, then $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*)) \cong K_m$, where $m = |Z(\mathcal{R})|$. Thus $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ is a complete graph of order at least 3. Hence $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ has no bridge.

(*iv*) If $|Z(\mathcal{R})| > 2$ and $|Reg(\mathcal{R})| \geq 2$, then there exist at least two non-zero elements in $Z(\mathcal{R})$. Thus each edge lies on cycle of $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$. Hence $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ has no bridge.

From all the above, we find that if graph $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ has a bridge, then $|Z(\mathcal{R})| = 2$.

Converse of the proof is trivial. □

Corollary 3.1 *Let $|\mathcal{R}| \leq 5$. Then $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ has a bridge if and only if $\mathcal{R} \cong \mathbb{Z}_4$ or $\mathcal{R}_1$, where* $\mathcal{R}_1 = \left\{ \begin{pmatrix} a & 0 \\ b & a \end{pmatrix} \mid a, b \in \mathbb{Z}_2 \right\}$.

Theorem 3.3 $\kappa(T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))) = |Z(\mathcal{R})| - 1$.

Proof: We know that, for any graph G, $\kappa(G) \leqslant \lambda(G) \leqslant \delta(G)$ and by Remark 2.3, $\delta(T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))) = (|Z(\mathcal{R})| - 1)$. Therefore,

$$\kappa(T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))) \leqslant (|Z(\mathcal{R})| - 1).$$

Let $w \in V(T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*)))$ such that w is adjacent to each vertex $y \in \mathcal{R}^*$ implies that $w \in Z(\mathcal{R})^*$. Hence $Z(\mathcal{R})^*$ is the minimum vertex-cut of $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$, otherwise, $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ is connected. Hence

$$\kappa(T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))) = (|Z(\mathcal{R})| - 1).$$

□

Remark 3.1 *The set of non zero zero-divisors of $\mathcal{R}$, (i.e., $Z(\mathcal{R})^*$) is the minimum vertex-cut of $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$.*

4. Clique number of $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$

In this section, we discuss the clique number of $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$. Note that if $|\mathcal{R}| = 2$, then $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*)) \cong K_1$. Thus $\omega(T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))) = 1$. Also, if $|\mathcal{R}| = 3$, then $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*)) \cong \overline{K}_2$. Hence $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ has no clique if and only if $|\mathcal{R}| = 3$.

Theorem 4.1 *Let $\mathcal{R}$ be a commutative ring with non-zero unity such that $|\mathcal{R}| \geq 4$, and $|Z(\mathcal{R})| > 2$. Then $\omega(T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))) = |Z(\mathcal{R})|$.*

Proof: We know that $Z(\mathcal{R})$ is closed under multiplication, so that each pair of elements in $Z(\mathcal{R})$ are adjacent. In general, they are adjacent to all elements of $\mathcal{R}$. Thus each element of $Z(\mathcal{R})$ is adjacent at least to the unity. Since $\mathcal{R}^*$ is the vertex set of $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$, we find that $|Z(\mathcal{R})|$ elements are adjacent. This completes the proof. □

Theorem 4.2 *Let $\mathcal{R}$ be a commutative ring with $1 \neq 0$, $Z(\mathcal{R})$ be the set of zero-divisors of $\mathcal{R}$. If $|Z(\mathcal{R})| = n > 2$, then $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ has a cycle. Moreover, $gr(T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))) = 3$.*

Proof: Using the same reasoning as in the previous theorem, for $|Z(\mathcal{R})| = n > 2$, we find that at least two non-zero elements are in $Z(\mathcal{R})$. Let $u, v \in Z(\mathcal{R})^*$. Also, $\mathcal{R}$ has unity $1 \in Reg(\mathcal{R})$. Then $u - 1 - v - u$ is a cycle of length 3. Thus $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ contains a cycle of length 3. Hence $gr(T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))) = 3$. □

Corollary 4.1 *Let $Z(\mathcal{R})$ be the set of zero-divisors of $\mathcal{R}$. Then*

$$gr(T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))) = \begin{cases} 3 & |Z(\mathcal{R})| > 2, \\ \infty & otherwise. \end{cases}$$

5. Traversability of $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$

In this section, we show that $T_{Z(\mathcal{R})}\Gamma(\mathcal{R}^*))$ can be an Eulerian graph and contains an Eulerian trail. Also, we find out when the graph $T_{Z(\mathcal{R})}\Gamma(\mathcal{R}^*))$ is a Hamiltonian graph.

Theorem 5.1 *Let $\mathcal{R}$ be a commutative ring with non-zero unity such that $|\mathcal{R}| \geq 4$, and $|Z(\mathcal{R})| \geq 3$. Then $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ is Eulerian if and only if $|\mathcal{R}|$ is even and $|Z(\mathcal{R})|$ is odd. Moreover, $|Reg(\mathcal{R})|$ is odd.*

Proof: Assume that $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ is Eulerian. Then each vertex in the graph $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ has even degree. According to Theorem 2.4, the degree of each vertex of $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ is either $(|\mathcal{R}|-2)$ or $(|Z(\mathcal{R})|-1)$. Now we have the following two cases:

(*i*) If $x \in Z(\mathcal{R})^*$, then $deg(x) = |\mathcal{R}|-2$, which is even. Thus $|\mathcal{R}|$ is even.

(*ii*) If $x \in Reg(\mathcal{R})$, then $deg(x) = |Z(\mathcal{R})|-1$, which is even. Thus $|Z(\mathcal{R})|$ is odd.

Hence $|\mathcal{R}|$ is even and $|Z(\mathcal{R})|$ is odd. Moreover, $|Reg(\mathcal{R})|$ is odd.

Conversely, assume that $|\mathcal{R}|$ is even and $|Z(\mathcal{R})|$ is odd. Then $(|\mathcal{R}|-2)$ is even and $|Z(\mathcal{R})^*|$ is also even. Using Theorem 2.4, we conclude that all the vertices of $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ have even degree. Hence $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ is Eulerian. □

Remark 5.1 *For any commutative ring $\mathcal{R}$ with $1 \neq 0$, we know that $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}))$ can not be an Eulerian graph (for reference see [8]). But, in view of previous theorem, $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ can be an Eulerian graph.*

Example 5.1 *The following two examples represent the Eulerian graph.*

(*i*) *Let $\mathcal{R} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Then $Z(\mathcal{R}) = \{(0,0,0),(0,0,1),(0,1,0),(0,1,1),(1,0,0),(1,0,1),(1,1,0)\}$ and $Reg(\mathcal{R}) = \{(1,1,1)\}$ (see Fig. 5).*

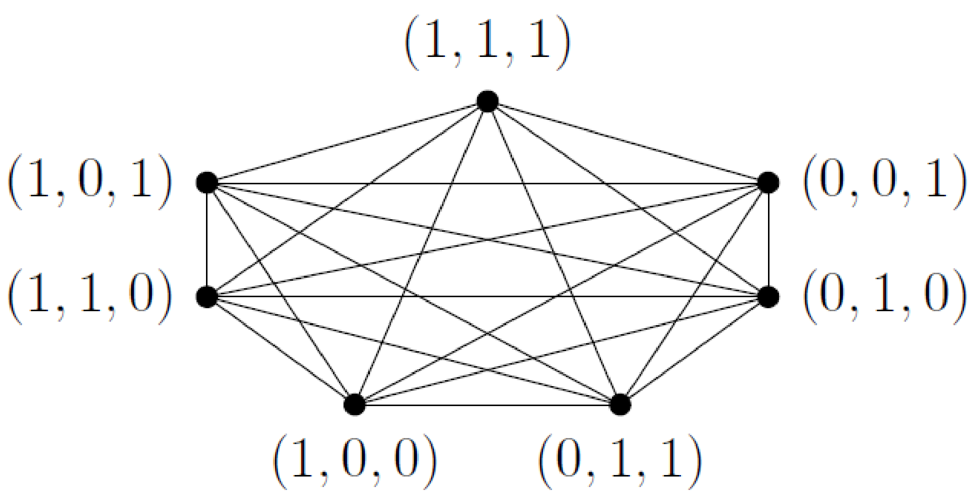

Figure 5: $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$, where $\mathcal{R} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

(*ii*) *Let $\mathcal{R} = \mathbb{Z}_2 \times \mathbb{F}_4$, where $\mathbb{F}_4 \cong \frac{\mathbb{Z}_2[x]}{\langle 1+x+x^2 \rangle}$. Then $Reg(\mathcal{R}) = \{(1,1),(1,x),(1,1+x)\}$ and $Z(\mathcal{R}) = \{(0,0),(0,1),(0,x),(0,1+x),(1,0)\}$ (see Fig. 6).*

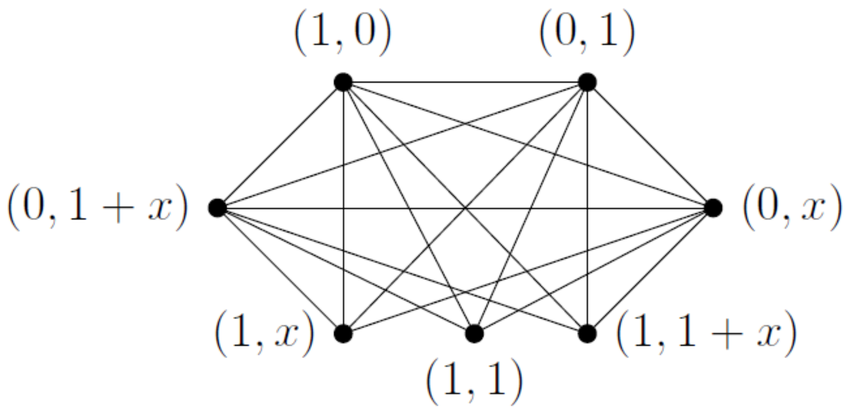

Figure 6: $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$, where $\mathcal{R} = \mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{\langle 1+x+x^2 \rangle}$

Theorem 5.2 *Let $\mathcal{R}$ be a commutative ring with non-zero unity such that $|\mathcal{R}| \geq 4$, and $|Z(\mathcal{R})| \geq 3$. Then $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ contains an Eulerian trail if and only if either $|Z(\mathcal{R})| = 3$ and $|Reg(\mathcal{R})|$ is even or $|Reg(\mathcal{R})| = 2$ and $|Z(\mathcal{R})|$ is even.*

Proof: Assume that $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ contains an Eulerian trail. Then the graph $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ has exactly two vertices with odd degree. Suppose that the two vertices w and z have odd degree and the vertices $x_1, x_2, \ldots, x_n$ have even degrees. Then we have the following cases:

(*i*) If $w, z \in Z(\mathcal{R})^*$ and $x_i \in Reg(\mathcal{R})$ for all $1 \leq i \leq n$, then $deg(w) = deg(z)$ is odd and $deg(x_i)$ for all $1 \leq i \leq n$ is even, therefore $(|\mathcal{R}| - 2)$ is odd and $|Z(\mathcal{R})^*| = 2$ is even, thus $|\mathcal{R}|$ is odd and $|Z(\mathcal{R})| = 3$. Hence $|Z(\mathcal{R})| = 3$ and $|Reg(\mathcal{R})|$ is even.

(*ii*) If $w, z \in Z(\mathcal{R})^*$ and there exists at least one vertex $x_j \in Z(\mathcal{R})^*$, then $deg(w) = deg(z) = deg(x_j)$ is odd. Hence each vertex of $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ has odd degree, which contradicts our assumption.

(*iii*) If $w, z \in Reg(\mathcal{R})$ and $x_i \in Z(\mathcal{R})^*$ for all $1 \leq i \leq n$, then $deg(w) = deg(z)$ is odd and $deg(x_i)$ for all $1 \leq i \leq n$ is even. Note that $|Z(\mathcal{R})^*|$ is odd and $|\mathcal{R}| - 2$ is even. So $|Z(\mathcal{R})|$ is even and $|\mathcal{R}|$ is even. Since $w, z \in Reg(\mathcal{R})$ only, we have $|Reg(\mathcal{R})| = 2$. Hence $|Reg(\mathcal{R})| = 2$ and $|Z(\mathcal{R})|$ is even.

(*iv*) If $w, z \in Reg(\mathcal{R})$ and there exists at least one vertex $x_j \in Reg(\mathcal{R})$, then $deg(w) = deg(z) = deg(x_j)$ is odd. Hence each vertex of $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ has odd degree, which contradicts our assumption.

(*v*) If $w \in Z(\mathcal{R})^*$ and $z \in Reg(\mathcal{R})$, then $deg(w) = deg(z) = deg(x_i)$ for all $1 \leq i \leq n$ is odd. Hence each vertex of $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ has odd degree, which contradicts our assumption.

As a result, we have either $|Z(\mathcal{R})| = 3$ and $|Reg(\mathcal{R})|$ is even or $|Reg(\mathcal{R})| = 2$ and $|Z(\mathcal{R})|$ is even.

Conversely, assume that either $|Z(\mathcal{R})| = 3$ and $|Reg(\mathcal{R})|$ is even or $|Reg(\mathcal{R})| = 2$ and $|Z(\mathcal{R})|$ is even. We first suppose that $|Z(\mathcal{R})| = 3$ and $|Reg(\mathcal{R})|$ is even. For any vertex x of $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$, we have the following two cases:

(*i*) If $x \in Z(\mathcal{R})^*$, then $deg(x) = (|\mathcal{R}| - 2)$. Therefore, $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ has two non-zero elements in $Z(\mathcal{R})$ with odd degree.

(*ii*) If $x \in Reg(\mathcal{R})$, then $deg(x) = |Z(\mathcal{R})^*| = 2$. Therefore, $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ has $|Reg(\mathcal{R})|$ elements in $Reg(\mathcal{R})$ with even degree.

Thus, $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ has only two vertices in $Z(\mathcal{R})^*$ with odd degree and $|Reg(\mathcal{R})|$ vertices in $Reg(\mathcal{R})$ with even degree. Hence $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ contains an Eulerian trail.

Now, we suppose that $|Reg(\mathcal{R})| = 2$ and $|Z(\mathcal{R})|$ is even. Then $|\mathcal{R}|$ is even. For any vertex x of $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$, we have the following two cases:

(*i*) If $x \in Z(\mathcal{R})^*$, then $deg(x) = (|\mathcal{R}| - 2)$. Therefore, $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ has $|Z(\mathcal{R})^*|$ elements in $Z(\mathcal{R})$ with even degree.

(*ii*) If $x \in Reg(\mathcal{R})$, then $deg(x) = (|Z(\mathcal{R})| - 1)$. Therefore, $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ has two elements in $Reg(\mathcal{R})$ with odd degree.

Thus, $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ has only two vertices in $Reg(\mathcal{R})$ with odd degree and $|Z(\mathcal{R})^*|$ vertices in $Z(\mathcal{R})$ with even degree. Hence $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ contains an Eulerian trail. □

Remark 5.2 *In the above theorem, if $|Z(\mathcal{R})| = 3$, then the graph $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ has an Eulerian trail begins at one of these two non zero elements of $Z(\mathcal{R})$ and ends at other. Also, if $|Reg(\mathcal{R})| = 2$, then the graph $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ has an Eulerian trail begins at one of these two elements of $Reg(\mathcal{R})$ and ends at other.*

Example 5.2 *The following two examples have the Eulerian trail.*

(*i*) *Let $\mathcal{R} = \mathbb{Z}_2 \times \mathbb{Z}_4$. Then $Z(\mathcal{R}) = \{(0,0),(0,1),(0,2),(0,3),(1,0),(1,2)\}$ and $Reg(\mathcal{R}) = \{(1,1),(1,3)\}$ (see Fig.7).*

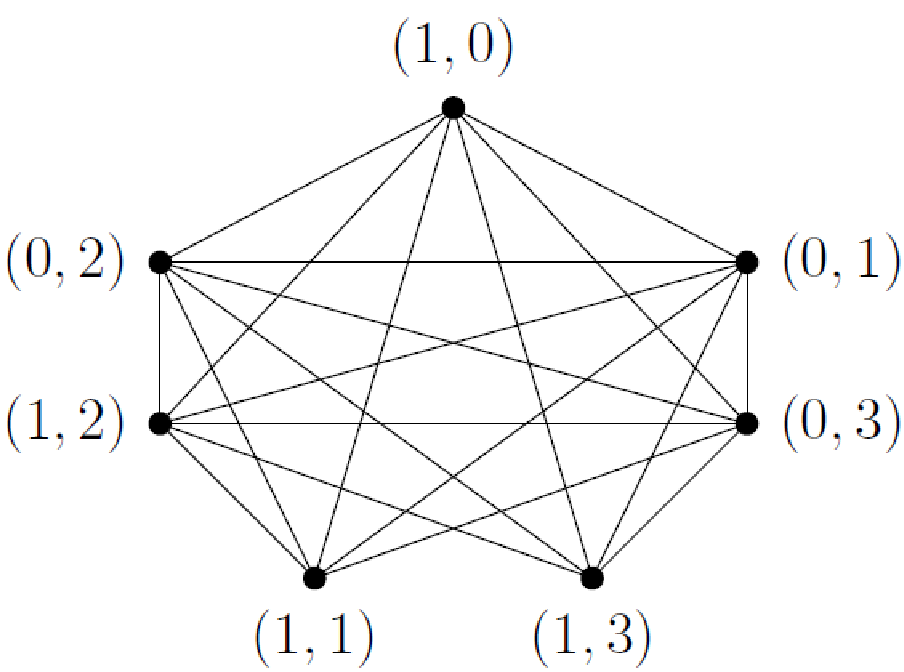

Figure 7: $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$, where $\mathcal{R} = \mathbb{Z}_2 \times \mathbb{Z}_4$

(*ii*) *Let $\mathcal{R} = \dfrac{\mathbb{Z}_3[x]}{\langle x^2 \rangle}$. Then $Z(\mathcal{R}) = \{0, x, 2x\}$ and $Reg(\mathcal{R}) = \{1, 2, 1+x, 2+x, 1+2x, 2+2x\}$ (see Fig.8).*

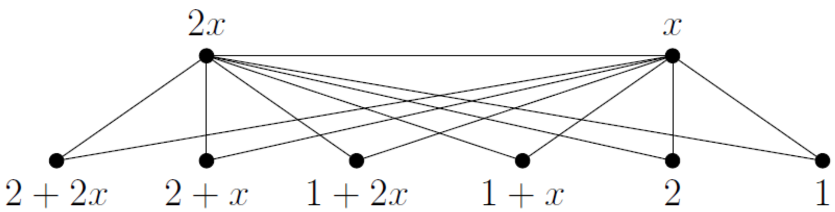

Figure 8: $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$, where $\mathcal{R} = \frac{\mathbb{Z}_3[x]}{\langle x^2 \rangle}$

Corollary 5.1 *Let $\mathcal{R}$ be a commutative ring with non-zero unity such that $|\mathcal{R}| \leq 3$. Then $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ can not be an Eulerian graph and has no Eulerian trail.*

Theorem 5.3 *Let $\mathcal{R}$ be a finite commutative ring with $1 \neq 0$, such that $|\mathcal{R}| = n \geq 4$. If $|Z(\mathcal{R})| \geq \frac{n}{2} + 1$, then $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ is Hamiltonian.*

Proof: Assume that w and z are any two vertices of $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$. Then we have the following two cases:

(i) If w and z are adjacent for all $w, z \in T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$, then $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ is complete. Therefore, $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ is Hamiltonian.

(ii) If w and z are nonadjacent for some $w, z \in T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$, then $w, z \in Reg(\mathcal{R})$. Therefore, $deg(w) = deg(z) = |Z(\mathcal{R})| - 1$. Thus

$$deg(w) + deg(z) = |Z(\mathcal{R})| - 1 + |Z(\mathcal{R})| - 1 \geq \frac{n}{2} + \frac{n}{2} = n.$$

Hence $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ is Hamiltonian. □

Corollary 5.2 *Let $\mathcal{R}$ be a finite commutative ring with $1 \neq 0$, such that $|\mathcal{R}| = n \geq 4$. If $|Z(\mathcal{R})| \geq \frac{n}{2} + 1$ for each pair w, z of $Reg(\mathcal{R})$, then $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*)) + wz$ is Hamiltonian if and only if $T_{Z(\mathcal{R})}(\Gamma(\mathcal{R}^*))$ is Hamiltonian.*

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