Derivation Alternator Rings with $S(a, b, c) = 0$

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ABSTRACT: In this paper, we discuss about the derivation alternator rings which are nonassociative but not (-1,1) rings. By assuming some additional conditions, we prove that derivation alternator rings are (-1,1) rings. Here we validate a semiprime derivation alternator ring with commutators in the left nucleus satisfies the identity $S(a, b, c) = 0$. By using this we show that a semiprime derivation alternator ring with commutators in the left nucleus is a (-1,1) ring.

Key Words: (-1,1) rings, derivation alternator rings, nonassociative rings, semiprime rings.

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1. Introduction

Hentzel [3] studied about derivation alternator rings. Thedy [6] proved that the rings $R$ satisfying the identifies $((R, R), R, R) = 0, ((a, b, a), b) = 0$ for all $a, b$ in $R$, and also $(R, (R, R), R) = 0$ or $(R, R, (R, R)) = 0$. The main solution is that prime rings following these identities are also commutative or associative. Kleinfeld [4] proved the same result by considering the rings with $(a, b, a)$ and commutators in the left nucleus. In [1] the rings with $((R, R), R, R) = 0$ and $(a, b, a), R = 0$ have been studied. Here we validate that semiprime derivation alternator ring with commutators in the left nucleus is a (-1,1) ring.

2. Preliminaries

A 2-divisible nonassociative ring is said to be a derivation alternator ring if it satisfies the following properties [3]

\[(a, a, a) = 0\] (2.1)

\[(bc, a, a) = b(c, a, a) + (b, a, a)c\] (2.2)

and

\[(a, a, bc) = b(a, a, c) + (a, b, a)c\] (2.3)

A (-1,1) ring is a nonassociative ring in which the right alternative law [2] $(a, b, b) = 0$, i.e., $(a, b, c) + (a, c, b) = 0$ and $(a, b, c) + (b, c, a) + (c, a, b) = 0$ hold. The left nucleus $N_l$ of a ring $R$ is defined as $N_l = n \in R/(n, R, R) = 0$. Throughout this paper, $R$ denotes a derivation alternator ring with commutators in the left nucleus. That is,

\[((R, R), R, R) = 0\] (2.4)

Using (2),( 3) and linearized (1) that such rings also satisfy

\[(a, bc, a) = b(a, c, a) + (a, b, a)c\] (2.5)

By using Teichmuller identity [2]

\[(da, b, c)\overline{(d, ab, c)} + (d, a, bc) = d(a, b, c) + (d, a, b)c\] (2.6)
As in [3] the identity (6) along with (5), (3) and (1) gives
\[(a^2, b, a) = (a, a, a) - (a, a, a) + a(a, b, a) + (a, a, b) a \]
a(a, b, a) - (a, a, b) a + a(a, b, a) + (a, a, b) a 
= 2a(a, b, a).
Thus we have proved
\[ (a^2, b, a) = 2a(a, b, a) \] (2.7)
Also we know that (7) becomes
\[ (a, b, a^2) = 2(a, b, a) a \] (2.8)
In [5] the properties of flexile derivation alternator rings have been studied. It is shown that a derivation alternator ring follows the flexible law,
\[ (a, b, a) = 0 \] (2.9)
The following properties hold in an arbitrary ring: [2]
\[ (ab, c) - (a, b, c) (b, c) = (a, c, b) \]
\[ (ab, c) + (bc, a) + (ca, b) = S(a, b, c) \] (2.10)
and
\[ ((a, b, c) + (b, c, a) + (c, a, b) = S(a, b, c) - S(a, c, b) \] (2.11)
where \( S(a, b, c) = (a, b, c) + (b, c, a) + (c, a, b) \).
Using \( (R, R) \in N_l \) in the identity (11), we get \( S(a, b, c) \in N_l \) (2.13)
Moreover, in all ring we have the identity
\[ (a, b, c) = a(b, c) + (a, c) b + S(a, b, c) = (a, c, b) \] (2.14)
A linearization of (9) implies \( (a, b, c) + (c, b, a) = 0 \). Then connecting this with (14), (13) and (4) we obtain
\[ a(b, c) + (a, c) b \subseteq N_l \] (2.15)
Assume that \( n \in N_l \). Then with \( d = n \) in (6), we obtain \( (na, b, c) = n(a, b, c) \). Combining this with (4) leads
\[ (na, b, c) = n(a, b, c) = (an, b, c) \] (2.16)
A combination of (15) and (16) yields
\[ (b, c)(a, r, s) = -(a, c)(b, r, s) \] (2.17)
If we substitute a commutator \( v \) for \( b \) and using (8), we get
\[ (v, c)(a, r, s) = 0 \] (2.18)
If we linearize \( (a, a, a) = 0 \), we get
\[ S(a, b, a) + S(a, c, b) = 0. \] (2.19)
By multiplying (12) and (19) with \( (p, q, r) \) and using (8) we obtain,
\[ S(a, b, c)(p, q, r) S(a, c, b)(p, q, r) = 0 \]
\[ S(a, b, c)(p, q, r) + S(a, c, b)(p, q, r) = 0 \]
By adding the above two equations, we have \( 2S(a, b, c)(p, q, r) = 0 \). Since \( R \) is 2-divisible,
\[ S(a, b, c)(p, q, r) = 0 \] (2.20)
3. Main Results

**Lemma 3.1.** Let $T = t \in N_l/t(R, R, R) = 0$. Then $T$ is an ideal of $R$ and $T(R, R, R) = 0$.

*Proof.* By substituting $t$ for $n$ in (16), we get 
\[(ta, b, c) = t(a, b, c) = (at, b, c) = 0.\]
Thus $tR \in N_l$ and $Rt \in N_l$. First note that $td. (a, b, c) = t.d. (a, b, c)$. But (6) multiplied on left by $t$ yields 
\[t.d. (a, b, c) = −t.(d, a, b)c = −t(d, a, b)c = 0.\] Thus $tw. (a, b, c) = 0$. However (17) yields $wt. (a, b, c) = 0$. Thus, $T$ is an ideal of $R$ and obviously $T(R, R, R) = 0$.

Let $A$ be the associator ideal of $R$. We know that $A$ is the set of all finite sums of associators and right multiples of associators. We know that $R$ is semiprime, if the only ideal of $R$ which squares to zero is the zero ideal. □

**Lemma 3.2.** In a semiprime derivation alternator ring $R$ with commutators in the left nucleus $S(a, b, c) = 0$.

*Proof.* By using lemma (1) and equation (6) we establish readily that $T.A = 0$.

But then $T \cap A$ is an ideal of $R$ which squares to zero. Since $R$ is semiprime, then $T \cap A = 0$.

From (13) and (20), we obtain $S(a, b, c) \in T$. Also 
\[S(a, b, c) \text{ is in } A.\] Thus $S(a, b, c) = 0$. Let $I$ be the ideal generated by \{(b, a, a)/a, b ∈ R\} and we know that the linear span $I$ of the alternators is an ideal. □

**Theorem 3.3.** A semiprime derivation alternator ring $R$ with commutators in the left nucleus is a (-1,1) ring.

*Proof.* Since $(a, b, a) = 0$, from (7) we have $(aa^2, b, a) = 0$.

This means that by definition a flexible derivation alternator ring is non-commutative Jordan, and non-commutative Jordan rings satisfy 
\[(b, a^2, c) = ao(b, a, c).\] (3.2)

Next from the identity (6) 
\[(a^2, b, c)\sim (a, ab, c) + (a, a, bc) = a(a, b, c) + (a, a, b)c.\] (3.3)

Likewise (6) implies 
\[(cb, a, a)\sim (c, ba, a) + (c, b, a^2) = c(b, a, a) + (c, b, a)a.\] (3.4)

By taking this into our previous equation leads to $2(a^2, b, c)\sim ao(a, b, c) = 0$. Since $R$ is 2-divisible, we have 
\[(a^2, b, c) = ao(a, b, c).\] (3.5)

Flexibility and (25) imply 
\[(b, c, a^2) = ao(b, c, a).\] (3.6)

Linearizing (1) and using (9), we get $(b, a, a) + (a, a, b) = 0$.

This implies that $(b, a, a) = −(a, a, b)$.

Then $(b, a, a, a, a) = −((a, a, b), a, a) = ((a, a, a), a, b) = 0$.

Applying (25), we write 
\[0 = (b^2, a, a), a, a) = (bo(b, a, a), a, a) = bo((b, a, a), a, a) + (b, a, a)ao(b, a, a) = 2(b, a, a)^2.\]

Since $R$ is 2-divisible, we get that $(b, a, a)^2 = 0$. Since $I$ is an ideal and $R$ is semiprime, we have $(b, a, a) = 0$.

From lemma (2) and $(b, a, a) = 0$, $R$ is a (-1,1) ring. □
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