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Summation formulas for the function $R_1[\mu,\delta,\delta';\gamma;\nu,\tau,z_1,z_2]$

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ABSTRACT: In this paper, we obtain finite and infinite summation formulas for Appell-type extension of ${}_pR_q(\nu,\tau;z)$ function, denoted as $R_1[\mu,\delta,\delta';\gamma;\nu,\tau,z_1,z_2]$ and confluent functions $R\Phi_1[\mu,\delta;\gamma;\nu,\tau,z_1,z_2]$, $R\Phi_2[\delta,\delta';\gamma;\nu,\tau,z_1,z_2]$ and $R\Phi_3[\delta;\gamma;\nu,\tau,z_1,z_2]$.

Key Words: Gamma function, Beta function, Appell hypergeometric function, Mittag-Leffler function.

Contents

1. Introduction and preliminaries

The ${}_pR_q(\nu,\tau;z)$ function with p numerator and q denominator parameters, introduced by Desai and Shukla [2,3] is defined as

$$\begin{aligned}{}_pR_q(\nu,\tau;z) &= {}_pR_q\left(\begin{array}{c} a_1,a_2,...,a_p \\ b_1,b_2,...,b_q\end{array}\middle|\,\nu,\tau;z\right)\\ &= \sum_{k=0}^{\infty}\frac{1}{\Gamma(\nu k+\tau)}\frac{(a_1)_k(a_2)_k...(a_p)_k}{(b_1)_k(b_2)_k...(b_q)_k}\frac{z^k}{k!}\end{aligned} \tag{1.1}$$

where $\nu,\tau\in\mathbf{C},\Re(\nu)>0,\Re(\tau)>0,\Re(a_i)>0,\Re(b_j)>0$ for all $i=1,2,...,p$ and $j=1,2,...,q$ and $(a)_k$ is a Pochhammer symbol [5] defined as

$$(a)_k := \frac{\Gamma(a+k)}{\Gamma(a)} = \begin{cases}1 & (k=0;a\in\mathbf{C}\setminus\{0\})\\ a(a+1)\dots(a+k-1) & (k\in\mathbf{N};a\in\mathbf{C}).\end{cases}$$

and Γ is a Gamma function [5] defined as

$$\Gamma(z)=\int_0^{\infty}u^{z-1}e^{-z}dz, \qquad \Re(z)>0.$$

Here, we modify the restriction on parameter ν as $\Re(\nu)\geq 0$ instead of $\Re(\nu)>0$, which satisfy the convergence condition of the ${}_pR_q(\nu,\tau;z)$ function, as mentioned in [3].
For $\nu=0$ and $\tau=1$, (1.1) reduces to generalized hypergeometric function [4,5].

$${}_pR_q\left(\begin{array}{c} a_1,a_2,...,a_p \\ b_1,b_2,...,b_q\end{array}\middle|\,0,1;z\right) = {}_pF_q\left(\begin{array}{c} a_1,a_2,...,a_p \\ b_1,b_2,...,b_q\end{array}\middle|\,z\right). \tag{1.2}$$

Recently, authors defined and introduced [7,8,9] Appell-type extension of the ${}_pR_q(\nu,\tau;z)$ function denoted as $R_1[\mu,\delta,\delta';\gamma;\nu,\tau,z_1,z_2]$ and its confluent functions $R\Phi_1[\mu,\delta;\gamma;\nu,\tau,z_1,z_2]$,

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$R\Phi_2\left[\delta,\delta';\gamma;\nu,\tau,z_1,z_2\right]$ and $R\Phi_3\left[\delta;\gamma;\nu,\tau,z_1,z_2\right]$ which are listed below.

These functions are defined for $\mu,\delta,\delta',\gamma,\nu,\tau,z_1,z_2\in\mathbf{C},\gamma\notin\mathbf{Z}^-\cup\{0\},\Re(\nu)\geq 0,\Re(\tau)>0,|z_1|,|z_2|<\infty$.

$$R_1\left[\mu,\delta,\delta';\gamma;\nu,\tau,z_1,z_2\right]=\sum_{p\geq 0}\sum_{q\geq 0}\frac{1}{\Gamma\left(\nu\left(p+q\right)+\tau\right)}\frac{(\mu)_{p+q}(\delta)_p(\delta')_q}{(\gamma)_{p+q}}\frac{{z_1}^p}{p!}\frac{{z_2}^q}{q!} \tag{1.3}$$

$$R\Phi_1\left[\mu,\delta;\gamma;\nu,\tau,z_1,z_2\right]=\sum_{p\geq 0}\sum_{q\geq 0}\frac{1}{\Gamma\left(\nu\left(p+q\right)+\tau\right)}\frac{(\mu)_{p+q}(\delta)_p}{(\gamma)_{p+q}}\frac{{z_1}^p}{p!}\frac{{z_2}^q}{q!} \tag{1.4}$$

$$R\Phi_2\left[\delta,\delta';\gamma;\nu,\tau,z_1,z_2\right]=\sum_{p\geq 0}\sum_{q\geq 0}\frac{1}{\Gamma\left(\nu\left(p+q\right)+\tau\right)}\frac{(\delta)_p(\delta')_q}{(\gamma)_{p+q}}\frac{{z_1}^p}{p!}\frac{{z_2}^q}{q!} \tag{1.5}$$

$$R\Phi_3\left[\delta;\gamma;\nu,\tau,z_1,z_2\right]=\sum_{p\geq 0}\sum_{q\geq 0}\frac{1}{\Gamma\left(\nu\left(p+q\right)+\tau\right)}\frac{(\delta)_p}{(\gamma)_{p+q}}\frac{{z_1}^p}{p!}\frac{{z_2}^q}{q!} \tag{1.6}$$

For $\nu=0$ and $\tau=1$, these functions reduce to the Appell function $F_1(a,b,b';c;w,z)$ and its confluent functions $\phi_1\left[a,b;c;w,z\right],\phi_2\left[b,b';c;w,z\right]$ and $\phi_3\left[b;c;w,z\right]$, which are the part of Horn's list [4] of convergent hypergeometric series of order two.

2. Finite and Infinite Summation Formulas

Inspired by the work of Brychkov and Saad [1] and Wang [10], we establish some finite and infinite summation formulas for the functions (1.3)-(1.6)

Theorem 2.1 *The following finite summation formula for (1.3) holds true*

$$\sum_{k=0}^{n}\binom{n}{k}\frac{(\mu)_k}{(\gamma)_k}{z_1}^kR_1\left[\mu+k,\delta,\delta'+k;\gamma+k;\nu,\nu k+\tau,z_1,z_2\right]=R_1\left[\mu,\delta,\delta'+n;\gamma;\nu,\tau,z_1,z_2\right] \tag{2.1}$$

Proof: Applying Leibnitz formula for differentiation of a product of two functions on the left hand side of the following result [8]

$$D_{z_2}^n\left\{{z_2}^{\delta'+n-1}R_1\left[\mu,\delta,\delta';\gamma;\nu,\tau,z_1,z_2\right]\right\}=(\delta')_n{z_2}^{\delta'-1}R_1\left[\mu,\delta,\delta'+n;\gamma;\nu,\tau,z_1,z_2\right],$$

we get

$$\sum_{k=0}^{n}\binom{n}{k}D_{z_2}^{n-k}\left\{{z_2}^{\delta'+n-1}\right\}D_{z_2}^k\left\{R_1\left[\mu,\delta,\delta';\gamma;\nu,\tau,z_1,z_2\right]\right\}=(\delta')_n{z_2}^{\delta'-1}R_1\left[\mu,\delta,\delta'+n;\gamma;\nu,\tau,z_1,z_2\right]$$

On using $D_{z_2}^n\left\{{z_2}^m\right\}=(m-n+1)_n{z_2}^{m-n}$ and derivative formula [8] for function (1.3) in the left hand side, we get

$$\sum_{k=0}^{n}\binom{n}{k}(\delta'+k)_{n-k}{z_2}^{\delta'+k-1}\frac{(\mu)_k(\delta')_k}{(\gamma)_k}R_1\left[\mu+k,\delta,\delta'+k;\gamma+k;\nu,\nu k+\tau,z_1,z_2\right]$$

$$=(\delta')_n{z_2}^{\delta'-1}R_1\left[\mu,\delta,\delta'+n;\gamma;\nu,\tau,z_1,z_2\right]$$

On applying the simple transformation $(\delta')_k(\delta'+k)_{n-k}=(\delta')_n$, we get desired result. □

Similarly, we can also prove the following finite summation formulas for (1.3)-(1.6)

$$\sum_{k=0}^{n}\binom{n}{k}\frac{(\mu)_k}{(\gamma)_k}{z_1}^k R_1\left[\mu+k,\delta+k,\delta';\gamma+k;\nu,\nu k+\tau,z_1,z_2\right]=R_1\left[\mu,\delta+n,\delta';\gamma;\nu,\tau,z_1,z_2\right] \tag{2.2}$$

$$\sum_{k=0}^{n}\binom{n}{k}\frac{(\mu)_k}{(\gamma)_k}{z_1}^k R\Phi_1\left[\mu+k,\delta+k;\gamma+k;\nu,\nu k+\tau,z_1,z_2\right]=R\Phi_1\left[\mu,\delta+n;\gamma;\nu,\tau,z_1,z_2\right] \tag{2.3}$$

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{(\gamma)_k}{z_1}^k R\Phi_2\left[\delta+k,\delta';\gamma+k;\nu,\nu k+\tau,z_1,z_2\right]=R\Phi_2\left[\delta+n,\delta';\gamma;\nu,\tau,z_1,z_2\right] \tag{2.4}$$

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{(\gamma)_k}{z_1}^k R\Phi_3\left[\delta+k;\gamma+k;\nu,\nu k+\tau,z_1,z_2\right]=R\Phi_3\left[\delta+n;\gamma;\nu,\tau,z_1,z_2\right] \tag{2.5}$$

Theorem 2.2 *For $t\in\mathbf{C}$ and $|t|<1$, the following infinite summation formula for (1.3) holds true*

$$\sum_{k=0}^{\infty}\frac{(\mu)_k}{k!}t^k R_1\left[\mu+k,\delta,\delta';\gamma;\nu,\tau,z_1,z_2\right]=(1-t)^{-\mu}R_1\left[\mu,\delta,\delta';\gamma;\nu,\tau,\frac{z_1}{1-t},\frac{z_2}{1-t}\right] \tag{2.6}$$

Proof: We have, $\sum\limits_{k=0}^{\infty}\frac{(\mu)_k}{k!}t^k R_1\left[\mu+k,\delta,\delta';\gamma;\nu,\tau,z_1,z_2\right]$

$$=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\frac{(\mu)_k(\mu+k)_{m+p}(\delta)_m(\delta')_p}{k!m!p!(\gamma)_{m+p}\Gamma\left(\nu\left(m+p\right)+\tau\right)}{z_1}^m{z_2}^p t^k$$

On applying the simple transformation $(\mu)_k(\mu+k)_m=(\mu)_m(\mu+m)_k$ and Newton's generalized binomial theorem [5], ${}_1F_0\left[\begin{array}{c}\mu\\ -\end{array}\middle|z\right]=(1-z)^{-\mu}$, we get

$$\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\frac{(\mu)_{m+p}(\delta)_m(\delta')_p}{m!p!(\gamma)_{m+p}\Gamma\left(\nu\left(m+p\right)+\tau\right)}{z_1}^m{z_2}^p(1-t)^{-\mu-m-p}$$

$$=(1-t)^{-\mu}R_1\left[\mu,\delta,\delta';\gamma;\nu,\tau,\frac{z_1}{1-t},\frac{z_2}{1-t}\right]$$

□

Similarly, for $t\in\mathbf{C}$ and $|t|<1$, the following infinite summation formula for (1.4) holds true

$$\sum_{k=0}^{\infty}\frac{(\mu)_k}{k!}t^k R\Phi_1\left[\mu+k,\delta;\gamma;\nu,\tau,z_1,z_2\right]=(1-t)^{-\mu}R\Phi_1\left[\mu,\delta;\gamma;\nu,\tau,\frac{z_1}{1-t},\frac{z_2}{1-t}\right] \tag{2.7}$$

Theorem 2.3 *For $t\in\mathbf{C}$ and $|t|<1$, the following infinite summation formula for (1.3) holds true*

$$\sum_{k=0}^{\infty}\frac{(\delta)_k}{k!}t^k R_1\left[\mu,\delta+k,\delta';\gamma;\nu,\tau,z_1,z_2\right]=(1-t)^{-\delta}R_1\left[\mu,\delta,\delta';\gamma;\nu,\tau,\frac{z_1}{1-t},z_2\right] \tag{2.8}$$

Proof: We have, $\sum\limits_{k=0}^{\infty} \frac{(\delta)_k}{k!} t^k R_1\left[\mu, \delta+k, \delta'; \gamma; \nu, \tau, z_1, z_2\right]$

$$= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \frac{(\mu)_{m+p} (\delta)_k (\delta+k)_m (\delta')_p}{k! m! p! (\gamma)_{m+p} \Gamma\left(\nu\left(m+p\right)+\tau\right)} {z_1}^m {z_2}^p t^k$$

On applying the simple transformation $(\delta)_k (\delta+k)_m = (\delta)_m (\delta+m)_k$ and Newton's generalized binomial theorem [5], ${}_1F_0\left[\begin{array}{c|c} \mu & z \\ - & \end{array}\right] = (1-z)^{-\mu}$, we get

$$\sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \frac{(\mu)_{m+p} (\delta)_m (\delta')_p}{m! p! (\gamma)_{m+p} \Gamma\left(\nu\left(m+p\right)+\tau\right)} {z_1}^m {z_2}^p (1-t)^{-\delta-m}$$

$$= (1-t)^{-\delta} R_1\left[\mu, \delta, \delta'; \gamma; \nu, \tau, \frac{z_1}{1-t}, z_2\right]$$

□

Similarly, for $t \in \mathbf{C}$ and $|t|<1$, the following infinite summation formulas for (1.3)-(1.6) also hold true

$$\sum_{k=0}^{\infty} \frac{(\delta')_k}{k!} t^k R_1\left[\mu, \delta, \delta'+k; \gamma; \nu, \tau, z_1, z_2\right] = (1-t)^{-\delta'} R_1\left[\mu, \delta, \delta'; \gamma; \nu, \tau, z_1, \frac{z_2}{1-t}\right] \tag{2.9}$$

$$\sum_{k=0}^{\infty} \frac{(\delta)_k}{k!} t^k R\Phi_1\left[\mu, \delta+k; \gamma; \nu, \tau, z_1, z_2\right] = (1-t)^{-\delta} R\Phi_1\left[\mu, \delta; \gamma; \nu, \tau, \frac{z_1}{1-t}, z_2\right] \tag{2.10}$$

$$\sum_{k=0}^{\infty} \frac{(\delta)_k}{k!} t^k R\Phi_2\left[\delta+k, \delta'; \gamma; \nu, \tau, z_1, z_2\right] = (1-t)^{-\delta} R\Phi_2\left[\delta, \delta'; \gamma; \nu, \tau, \frac{z_1}{1-t}, z_2\right] \tag{2.11}$$

$$\sum_{k=0}^{\infty} \frac{(\delta)_k}{k!} t^k R\Phi_3\left[\delta+k; \gamma; \nu, \tau, z_1, z_2\right] = (1-t)^{-\delta} R\Phi_3\left[\delta; \gamma; \nu, \tau, \frac{z_1}{1-t}, z_2\right] \tag{2.12}$$

Theorem 2.4 *For $t \in \mathbf{C}$, the following infinite summation formula for (1.3) hold true*

$$\sum_{k=0}^{\infty} \frac{(\mu)_k (\delta)_k}{k! (\gamma)_k} t^k R_1\left[\mu+k, \delta+k, \delta'; \gamma+k; \nu, \nu k+\tau, z_1, z_2\right] = R_1\left[\mu, \delta, \delta'; \gamma; \nu, \tau, t+z_1, z_2\right] \tag{2.13}$$

Proof: We have, $\sum\limits_{k=0}^{\infty} \frac{(\mu)_k (\delta)_k}{k! (\gamma)_k} t^k R_1\left[\mu+k, \delta+k, \delta'; \gamma+k; \nu, \nu k+\tau, z_1, z_2\right]$

$$= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \frac{(\mu)_k (\mu+k)_{m+p} (\delta)_k (\delta+k)_m (\delta')_p}{k! m! p! (\gamma)_k (\gamma+k)_{m+p} \Gamma\left(\nu\left(m+p\right)+\nu k+\tau\right)} {z_1}^m {z_2}^p t^k$$

$$= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \frac{(\mu)_{k+m+p} (\delta)_{k+m} (\delta')_p}{k! m! p! (\gamma)_{k+m+p} \Gamma\left(\nu\left(k+m+p\right)+\tau\right)} {z_1}^m {z_2}^p t^k$$

$$= \sum_{p=0}^{\infty} \frac{(\mu)_p (\delta')_p}{p! (\gamma)_p} {z_2}^p \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\mu+p)_{k+m} (\delta)_{k+m}}{k! m! (\gamma+p)_{k+m} \Gamma\left(\nu\left(k+m\right)+\nu p+\tau\right)} {z_1}^m t^k$$

On using addition formula [6] for the ${}_pR_q(\nu, \tau; z)$ function, we get

$$= \sum_{p=0}^{\infty} \frac{(\mu)_p (\delta')_p}{p! (\gamma)_p} {z_2}^p \cdot {}_2R_1(\mu+p, \delta; \gamma+p; \nu, \nu p+\tau; t+z_1)$$

$$= \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \frac{(\mu)_{m+p} (\delta)_m (\delta')_p}{m! p! (\gamma)_{m+p} \Gamma\left(\nu\left(m+p\right)+\tau\right)} (t+z_1)^m {z_2}^p$$

$$= R_1\left[\mu, \delta, \delta'; \gamma; \nu, \tau, t+z_1, z_2\right]$$

$\square$

Similarly, for $t \in \mathbf{C}$, the following infinite summation formulas for (1.3)-(1.6) also hold true

$$\sum_{k=0}^{\infty} \frac{(\mu)_k (\delta')_k}{k!(\gamma)_k} t^k R_1 \left[\mu+k, \delta, \delta'+k; \gamma+k; \nu, \nu k+\tau, z_1, z_2\right] = R_1 \left[\mu, \delta, \delta'; \gamma; \nu, \tau, z_1, t+z_2\right] \tag{2.14}$$

$$\sum_{k=0}^{\infty} \frac{(\mu)_k (\delta)_k}{k!(\gamma)_k} t^k R\Phi_1 \left[\mu+k, \delta+k; \gamma+k; \nu, \nu k+\tau, z_1, z_2\right] = R\Phi_1 \left[\mu, \delta; \gamma; \nu, \tau, t+z_1, z_2\right] \tag{2.15}$$

$$\sum_{k=0}^{\infty} \frac{(\delta)_k}{k!(\gamma)_k} t^k R\Phi_2 \left[\delta+k, \delta'; \gamma+k; \nu, \nu k+\tau, z_1, z_2\right] = R\Phi_2 \left[\delta, \delta'; \gamma; \nu, \tau, t+z_1, z_2\right] \tag{2.16}$$

$$\sum_{k=0}^{\infty} \frac{(\delta)_k}{k!(\gamma)_k} t^k R\Phi_3 \left[\delta+k; \gamma+k; \nu, \nu k+\tau, z_1, z_2\right] = R\Phi_3 \left[\delta; \gamma; \nu, \tau, t+z_1, z_2\right] \tag{2.17}$$

Theorem 2.5 *For $r \in \mathbf{C}$, the following infinite summation formula for (1.3) holds true*

$$\sum_{k=0}^{\infty} \frac{(\mu)_k (r)_k}{k!(\gamma)_k} {z_1}^k R_1 \left[\mu+k, \delta, \delta'; \gamma+k; \nu, \nu k+\tau, z_1, z_2\right] = R_1 \left[\mu, \delta+r, \delta'; \gamma; \nu, \tau, z_1, z_2\right] \tag{2.18}$$

Proof: We have, $\sum\limits_{k=0}^{\infty} \frac{(\mu)_k (r)_k}{k!(\gamma)_k} {z_1}^k R_1 \left[\mu+k, \delta, \delta'; \gamma+k; \nu, \nu k+\tau, z_1, z_2\right]$

$$\begin{aligned}
&= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \frac{(\mu)_k (\mu+k)_{m+p} (r)_k (\delta)_m (\delta')_p}{k!m!p!(\gamma)_k (\gamma+k)_{m+p} \Gamma\left(\nu\left(m+p\right)+\nu k+\tau\right)} {z_1}^{k+m} {z_2}^p \\
&= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \frac{(\mu)_{k+m+p} (r)_k (\delta)_m (\delta')_p}{k!m!p!(\gamma)_{k+m+p} \Gamma\left(\nu\left(k+m+p\right)+\tau\right)} {z_1}^{k+m} {z_2}^p \\
&= \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(\mu)_{n+p} (\delta)_n (\delta')_p}{n!p!(\gamma)_{n+p} \Gamma\left(\nu\left(n+p\right)+\tau\right)} {z_1}^n {z_2}^p \cdot {}_2F_1 \left[\begin{array}{c} -n, r \\ 1-\delta-n \end{array} \middle| 1\right] \\
&= \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(\mu)_{n+p} (\delta+r)_n (\delta')_p}{n!p!(\gamma)_{n+p} \Gamma\left(\nu\left(n+p\right)+\tau\right)} {z_1}^n {z_2}^p \\
&= R_1 \left[\mu, \delta+r, \delta'; \gamma; \nu, \tau, z_1, z_2\right]
\end{aligned}$$

$\square$

Similarly, for $r \in \mathbf{C}$, the following infinite summation formula for (1.3)-(1.6) also hold true

$$\sum_{k=0}^{\infty} \frac{(\mu)_k (r)_k}{k!(\gamma)_k} {z_2}^k R_1 \left[\mu+k, \delta, \delta'; \gamma+k; \nu, \nu k+\tau, z_1, z_2\right] = R_1 \left[\mu, \delta, \delta'+r; \gamma; \nu, \tau, z_1, z_2\right] \tag{2.19}$$

$$\sum_{k=0}^{\infty} \frac{(\mu)_k (r)_k}{k!(\gamma)_k} {z_1}^k R\Phi_1 \left[\mu+k, \delta; \gamma+k; \nu, \nu k+\tau, z_1, z_2\right] = R\Phi_1 \left[\mu, \delta+r; \gamma; \nu, \tau, z_1, z_2\right] \tag{2.20}$$

$$\sum_{k=0}^{\infty} \frac{(r)_k}{k!(\gamma)_k} {z_1}^k R\Phi_2 \left[\delta, \delta'; \gamma+k; \nu, \nu k+\tau, z_1, z_2\right] = R\Phi_2 \left[\delta+r, \delta'; \gamma; \nu, \tau, z_1, z_2\right] \tag{2.21}$$

$$\sum_{k=0}^{\infty} \frac{(r)_k}{k!(\gamma)_k} {z_1}^k R\Phi_3 \left[\delta; \gamma+k; \nu, \nu k+\tau, z_1, z_2\right] = R\Phi_3 \left[\delta+r; \gamma; \nu, \tau, z_1, z_2\right] \tag{2.22}$$

For special values of parameters, one can easily obtain the following finite summation formulas

$$\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)\frac{(\mu)_k}{(\gamma)_k}(-z_1)^k R_1\left[\mu+k,\delta,\delta';\gamma+k;\nu,\nu k+\tau,z_1,z_2\right]=R_1\left[\mu,\delta-n,\delta';\gamma;\nu,\tau,z_1,z_2\right] \tag{2.23}$$

$$\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)\frac{(\mu)_k}{(\gamma)_k}(-z_2)^k R_1\left[\mu+k,\delta,\delta';\gamma+k;\nu,\nu k+\tau,z_1,z_2\right]=R_1\left[\mu,\delta,\delta'-n;\gamma;\nu,\tau,z_1,z_2\right] \tag{2.24}$$

$$\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)\frac{(\mu)_k}{(\gamma)_k}(-z_1)^k R\Phi_1\left[\mu+k,\delta;\gamma+k;\nu,\nu k+\tau,z_1,z_2\right]=R\Phi_1\left[\mu,\delta-n;\gamma;\nu,\tau,z_1,z_2\right] \tag{2.25}$$

$$\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)\frac{(-z_1)^k}{(\gamma)_k}R\Phi_2\left[\delta,\delta';\gamma+k;\nu,\nu k+\tau,z_1,z_2\right]=R\Phi_2\left[\delta-n,\delta';\gamma;\nu,\tau,z_1,z_2\right] \tag{2.26}$$

$$\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)\frac{(-z_1)^k}{(\gamma)_k}R\Phi_3\left[\delta;\gamma+k;\nu,\nu k+\tau,z_1,z_2\right]=R\Phi_3\left[\delta-n;\gamma;\nu,\tau,z_1,z_2\right] \tag{2.27}$$

Theorem 2.6 *For $r\in\mathbf{C}$, the following infinite summation formula of (1.3) holds true*

$$\sum_{k=0}^{\infty}\frac{(\mu)_k(r)_k}{k!(\gamma)_k}(-z_1)^k R_1\left[\mu+k,\delta+k,\delta';\gamma+k;\nu,\nu k+\tau,z_1,z_2\right]=R_1\left[\mu,\delta-r,\delta';\gamma;\nu,\tau,z_1,z_2\right] \tag{2.28}$$

Proof: We have, $\sum\limits_{k=0}^{\infty}\frac{(\mu)_k(r)_k}{k!(\gamma)_k}(-z_1)^k R_1\left[\mu+k,\delta+k,\delta';\gamma+k;\nu,\nu k+\tau,z_1,z_2\right]$

$$\begin{aligned}
&=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\frac{(\mu)_k(\mu+k)_{m+p}(r)_k(\delta+k)_m(\delta')_p}{k!m!p!(\gamma)_k(\gamma+k)_{m+p}\Gamma\left(\nu\left(m+p\right)+\nu k+\tau\right)}(-1)^k {z_1}^{k+m}{z_2}^p\\
&=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\frac{(\mu)_{k+m+p}(r)_k(\delta+k)_m(\delta')_p}{k!m!p!(\gamma)_{k+m+p}\Gamma\left(\nu\left(k+m+p\right)+\tau\right)}(-1)^k {z_1}^{k+m}{z_2}^p\\
&=\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(\mu)_{n+p}(\delta)_n(\delta')_p}{n!p!(\gamma)_{n+p}\Gamma\left(\nu\left(n+p\right)+\tau\right)}{z_1}^n{z_2}^p\cdot {}_2F_1\left[\begin{array}{c} -n,r \\ \delta \end{array}\middle| 1\right]\\
&=\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(\mu)_{n+p}(\delta-r)_n(\delta')_p}{n!p!(\gamma)_{n+p}\Gamma\left(\nu\left(n+p\right)+\tau\right)}{z_1}^n{z_2}^p\\
&=R_1\left[\mu,\delta-r,\delta';\gamma;\nu,\tau,z_1,z_2\right]
\end{aligned}$$

□

Similarly, for $r\in\mathbf{C}$, the following infinite summation formulas for (1.3)-(1.6) also hold true

$$\sum_{k=0}^{\infty}\frac{(\mu)_k(r)_k}{k!(\gamma)_k}(-z_2)^k R_1\left[\mu+k,\delta,\delta'+k;\gamma+k;\nu,\nu k+\tau,z_1,z_2\right]=R_1\left[\mu,\delta,\delta'-r;\gamma;\nu,\tau,z_1,z_2\right] \tag{2.29}$$

$$\sum_{k=0}^{\infty}\frac{(\mu)_k(r)_k}{k!(\gamma)_k}(-z_1)^k R\Phi_1\left[\mu+k,\delta+k;\gamma+k;\nu,\nu k+\tau,z_1,z_2\right]=R\Phi_1\left[\mu,\delta-r;\gamma;\nu,\tau,z_1,z_2\right] \tag{2.30}$$

$$\sum_{k=0}^{\infty}\frac{(r)_k}{k!(\gamma)_k}(-z_1)^k R\Phi_2\left[\delta+k,\delta';\gamma+k;\nu,\nu k+\tau,z_1,z_2\right]=R\Phi_2\left[\delta-r,\delta';\gamma;\nu,\tau,z_1,z_2\right] \tag{2.31}$$

$$\sum_{k=0}^{\infty}\frac{(r)_k}{k!(\gamma)_k}(-z_1)^k R\Phi_3\left[\delta+k;\gamma+k;\nu,\nu k+\tau,z_1,z_2\right]=R\Phi_3\left[\delta-r;\gamma;\nu,\tau,z_1,z_2\right] \tag{2.32}$$

Theorem 2.7 *For $t \in \mathbf{C}$ and $|t| < 1$, the following infinite summation formula of (1.3) holds true*

$$\sum_{k=0}^{\infty} \frac{(\mu)_k}{k!} t^k R_1\left[-k, \delta, \delta'; \gamma; \nu, \tau, z_1, z_2\right] = (1-t)^{-\mu} R_1\left[\mu, \delta, \delta'; \gamma; \nu, \tau, \frac{tz_1}{t-1}, \frac{tz_2}{t-1}\right] \tag{2.33}$$

Proof: We have, $\sum_{k=0}^{\infty} \frac{(\mu)_k}{k!} t^k R_1\left[-k, \delta, \delta'; \gamma; \nu, \tau, z_1, z_2\right]$

$$\begin{aligned}
&= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty} \frac{(\mu)_k(-k)_{m+p}(\delta)_m(\delta')_p}{k!m!p!(\gamma)_{m+p}\Gamma\left(\nu\left(m+p\right)+\tau\right)} {z_1}^m {z_2}^p t^k \\
&= \sum_{m=0}^{\infty}\sum_{p=0}^{\infty} \frac{(\delta)_m(\delta')_p}{m!p!(\gamma)_{m+p}\Gamma\left(\nu\left(m+p\right)+\tau\right)} {z_1}^m {z_2}^p \sum_{k=m+p}^{\infty} \frac{(-1)^{m+p}(\mu)_k}{(k-m-p)!} t^k \\
&= \sum_{m=0}^{\infty}\sum_{p=0}^{\infty} \frac{(\mu)_{m+p}(\delta)_m(\delta')_p}{m!p!(\gamma)_{m+p}\Gamma\left(\nu\left(m+p\right)+\tau\right)} {z_1}^m {z_2}^p \sum_{l=0}^{\infty} \frac{(-1)^{m+p}(\mu+m+p)_l}{l!} t^{l+m+p} \\
&= \sum_{m=0}^{\infty}\sum_{p=0}^{\infty} \frac{(\mu)_{m+p}(\delta)_m(\delta')_p}{m!p!(\gamma)_{m+p}\Gamma\left(\nu\left(m+p\right)+\tau\right)} {z_1}^m {z_2}^p (-t)^{m+p}(1-t)^{-(\mu+m+p)} \\
&= (1-t)^{-\mu} R_1\left[\mu, \delta, \delta'; \gamma; \nu, \tau, \frac{tz_1}{t-1}, \frac{tz_2}{t-1}\right]
\end{aligned}$$

□

Similarly, for $t \in \mathbf{C}$, the following infinite summation formula for (1.3), (1.4) also hold true

$$\sum_{k=0}^{\infty} \frac{(\mu)_k}{k!} t^k R\Phi_1\left[-k, \delta; \gamma; \nu, \tau, z_1, z_2\right] = (1-t)^{-\mu} R\Phi_1\left[\mu, \delta; \gamma; \nu, \tau, \frac{tz_1}{t-1}, \frac{tz_2}{t-1}\right] \tag{2.34}$$

$$\sum_{k=0}^{\infty} \frac{(\mu)_k}{k!} (-t)^k R_1\left[-k, \delta, \delta'; \gamma; \nu, \tau, z_1, z_2\right] = (1+t)^{-\mu} R_1\left[\mu, \delta, \delta'; \gamma; \nu, \tau, \frac{tz_1}{t+1}, \frac{tz_2}{t+1}\right] \tag{2.35}$$

$$\sum_{k=0}^{\infty} \frac{(\mu)_k}{k!} (-t)^k R\Phi_1\left[-k, \delta; \gamma; \nu, \tau, z_1, z_2\right] = (1+t)^{-\mu} R\Phi_1\left[\mu, \delta; \gamma; \nu, \tau, \frac{tz_1}{t+1}, \frac{tz_2}{t+1}\right] \tag{2.36}$$

For special values of parameters, one can easily obtain the following finite sums

$$\sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} t^k R_1\left[-k, \delta, \delta'; \gamma; \nu, \tau, z_1, z_2\right] = (1+t)^{n} R_1\left[-n, \delta, \delta'; \gamma; \nu, \tau, \frac{tz_1}{t+1}, \frac{tz_2}{t+1}\right] \tag{2.37}$$

$$\sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} t^k R\Phi_1\left[-k, \delta; \gamma; \nu, \tau, z_1, z_2\right] = (1+t)^{n} R\Phi_1\left[-n, \delta; \gamma; \nu, \tau, \frac{tz_1}{t+1}, \frac{tz_2}{t+1}\right] \tag{2.38}$$

Theorem 2.8 *For $t \in \mathbf{C}$, the following infinite summation formula for (1.3) holds true*

$$\sum_{k=0}^{\infty} \frac{(\delta)_k}{k!} (-t)^k R_1\left[\mu, -k, \delta'; \gamma; \nu, \tau, \frac{1+t}{t} z_1, z_2\right] = (1+t)^{-\delta} R_1\left[\mu, \delta, \delta'; \gamma; \nu, \tau, z_1, z_2\right] \tag{2.39}$$

Proof: We have, $\sum\limits_{k=0}^{\infty} \frac{(\delta)_k}{k!}(-t)^k R_1\left[\mu,-k,\delta';\gamma;\nu,\tau,\frac{1+t}{t}z_1,z_2\right]$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty} \frac{(\mu)_{m+p}(\delta)_k(\delta')_p(-k)_m(-1)^k(1+t)^m t^{k-m} {z_1}^m {z_2}^p}{k!m!p!(\gamma)_{m+p}\Gamma\left(\nu\left(m+p\right)+\tau\right)} \\
&= \sum_{m=0}^{\infty}\sum_{p=0}^{\infty} \frac{(\mu)_{m+p}(\delta)_m(\delta')_p}{m!p!(\gamma)_{m+p}\Gamma\left(\nu\left(m+p\right)+\tau\right)}(1+t)^m {z_1}^m {z_2}^p \sum_{n=0}^{\infty}\frac{(\delta+m)_n}{n!}(-t)^n \\
&= \sum_{m=0}^{\infty}\sum_{p=0}^{\infty} \frac{(\mu)_{m+p}(\delta)_m(\delta')_p}{m!p!(\gamma)_{m+p}\Gamma\left(\nu\left(m+p\right)+\tau\right)}(1+t)^{-\delta} {z_1}^m {z_2}^p \\
&= (1+t)^{-\delta} R_1\left[\mu,\delta,\delta';\gamma;\nu,\tau,z_1,z_2\right]
\end{aligned}
$$

□

Similarly, for $t \in \mathbf{C}$, the following infinite summation formulas for (1.3)-(1.6) also hold true

$$\sum_{k=0}^{\infty}\frac{(\delta')_k}{k!}(-t)^k R_1\left[\mu,\delta,-k;\gamma;\nu,\tau,z_1,\frac{1+t}{t}z_2\right] = (1+t)^{-\delta'} R_1\left[\mu,\delta,\delta';\gamma;\nu,\tau,z_1,z_2\right] \tag{2.40}$$

$$\sum_{k=0}^{\infty}\frac{(\delta)_k}{k!}(-t)^k R\Phi_1\left[\mu,-k;\gamma;\nu,\tau,\frac{1+t}{t}z_1,z_2\right] = (1+t)^{-\delta} R\Phi_1\left[\mu,\delta;\gamma;\nu,\tau,z_1,z_2\right] \tag{2.41}$$

$$\sum_{k=0}^{\infty}\frac{(\delta)_k}{k!}(-t)^k R\Phi_2\left[-k,\delta';\gamma;\nu,\tau,\frac{1+t}{t}z_1,z_2\right] = (1+t)^{-\delta} R\Phi_2\left[\delta,\delta';\gamma;\nu,\tau,z_1,z_2\right] \tag{2.42}$$

$$\sum_{k=0}^{\infty}\frac{(\delta)_k}{k!}(-t)^k R\Phi_3\left[-k;\gamma;\nu,\tau,\frac{1+t}{t}z_1,z_2\right] = (1+t)^{-\delta} R\Phi_3\left[\delta;\gamma;\nu,\tau,z_1,z_2\right] \tag{2.43}$$

$$\sum_{k=0}^{\infty}\frac{(\delta)_k}{k!}\left(\frac{t+z_1}{z_1-1}\right)^k R_1\left[\mu,-k,\delta';\gamma;\nu,\tau,\frac{1+t}{t+z_1}z_1,z_2\right] = \left(\frac{1-z_1}{1+t}\right)^{\delta} R_1\left[\mu,\delta,\delta';\gamma;\nu,\tau,z_1,z_2\right] \tag{2.44}$$

$$\sum_{k=0}^{\infty}\frac{(\delta')_k}{k!}\left(\frac{t+z_2}{z_2-1}\right)^k R_1\left[\mu,\delta,-k;\gamma;\nu,\tau,z_1,\frac{1+t}{t+z_2}z_2\right] = \left(\frac{1-z_2}{1+t}\right)^{\delta'} R_1\left[\mu,\delta,\delta';\gamma;\nu,\tau,z_1,z_2\right] \tag{2.45}$$

$$\sum_{k=0}^{\infty}\frac{(\delta)_k}{k!}\left(\frac{t+z_1}{z_1-1}\right)^k R\Phi_1\left[\mu,-k;\gamma;\nu,\tau,\frac{1+t}{t+z_1}z_1,z_2\right] = \left(\frac{1-z_1}{1+t}\right)^{\delta} R\Phi_1\left[\mu,\delta;\gamma;\nu,\tau,z_1,z_2\right] \tag{2.46}$$

$$\sum_{k=0}^{\infty}\frac{(\delta)_k}{k!}\left(\frac{t+z_1}{z_1-1}\right)^k R\Phi_2\left[-k,\delta';\gamma;\nu,\tau,\frac{1+t}{t+z_1}z_1,z_2\right] = \left(\frac{1-z_1}{1+t}\right)^{\delta} R\Phi_2\left[\delta,\delta';\gamma;\nu,\tau,z_1,z_2\right] \tag{2.47}$$

$$\sum_{k=0}^{\infty}\frac{(\delta)_k}{k!}\left(\frac{t+z_1}{z_1-1}\right)^k R\Phi_3\left[-k;\gamma;\nu,\tau,\frac{1+t}{t+z_1}z_1,z_2\right] = \left(\frac{1-z_1}{1+t}\right)^{\delta} R\Phi_3\left[\delta;\gamma;\nu,\tau,z_1,z_2\right] \tag{2.48}$$

3. Concluding Remarks

In present paper, we obtain some finite and infinite summation formulas for the functions (1.3)-(1.6). By selecting values of parameters $\nu = 0$ and $\tau = 1$, we can deduce the results given by Brychkov and Saad [1] and Wang [10] for the Appell function $F_1(a,b,b';c;w,z)$ and its confluent functions $\phi_1\left[a,b;c;w,z\right], \phi_2\left[b,b';c;w,z\right]$ and $\phi_3\left[b;c;w,z\right]$. These formulas may be very useful in the theory of classical analysis and special functions.

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