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On a phosphorus cycling model with nonlinear boundary conditions

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ABSTRACT: In this paper we study the existence and multiplicity of positive solutions for a phosphorus cycling model with nonlinear boundary conditions, namely

$$\begin{cases} -\Delta u = \lambda\Big(k - u + c\frac{u^4}{1+u^4}\Big) =: \lambda f(u), & x \in \Omega, \\ \mathbf{n}.\nabla u + a(u)u = 0, & x \in \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, Δ is the Laplacian operator, $1/\lambda > 0$ is a the diffusion coefficient, k and c are positive parameters and $a : [0,\infty) \to (0,\infty)$ is nondecreasing C^1 function. This model describes the steady states of phosphorus cycling in stratified lakes. Also, it describes the colonization of barren soils in drylands by vegetation. We prove our results by the method of sub- and supersolutions.

Key Words: Phosphorus cycling model, nonlinear boundary conditions, sub-supersolutions.

Contents

1. Introduction

solutions to the nonlinear boundary value problem

$$\begin{cases} -\Delta u = \lambda\Big(k - u + c\frac{u^4}{1+u^4}\Big) =: \lambda f(u), & x \in \Omega, \\ \mathbf{n}.\nabla u + a(u)u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, Δ is the Laplacian operator, $1/\lambda > 0$ is a the diffusion coefficient, k and c are positive parameters and $a : [0,\infty) \to (0,\infty)$ is nondecreasing C^1 function.

This model describes phosphorus cycling in stratified lakes (see [4]). In particular, it illustrates the decrease in the amount of phosphorus in the epilimnion (upper layer) and the rapid recycling that occurs when the hypolimnion (lower layer) is depleted of oxygen. Here, u is the mass or concentration of phosphorous (P) in the water column, and k is the rate of P input from the watershed. The rate of recycling of P is given by $cu^4/1 + u^4$, where c is the maximum recycling rate. The assumption here is that the recycling is primarily from the sediments. The same equation has also been used to describe plant colonization of barren soils in drylands (see [10]). In this case, u is the amount of barren soil, and $cu^4/1 + u^4$ represents erosion by wind and runoff.

The motivation for this study cames from the work in [3] where the authors established the existence and multiplicity of positive solutions for the nonlinear boundary value problem

$$\begin{cases} -\Delta u = \lambda\Big(k - u + c\frac{u^4}{1+u^4}\Big) =: \lambda f(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

Here we extend this study to the nonlinear boundary condition of the form (1). One can refer to [5,7,8,9] for some recent existence and uniqueness results of elliptic problems with nonlinear boundary conditions.

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2. Main results

In this section, we give our main results concerning existence and multiplicity of positive solutions for the problem (1). We establish our results by the method of sub and super-solutions. By a sub-solution of (1) we mean a function $\psi : \bar{\Omega} \to \mathbb{R}$ satisfying:

$$\begin{cases} -\Delta\psi \leq \lambda f(\psi), & x \in \Omega, \\ \mathbf{n}.\nabla\psi + a(\psi)\psi \leq 0, & x \in \partial\Omega, \end{cases} \tag{2.1}$$

and by a super-solution of (1) we mean a function $Z : \bar{\Omega} \to \mathbb{R}$ satisfying:

$$\begin{cases} -\Delta Z \geq \lambda f(Z), & x \in \Omega, \\ \mathbf{n}.\nabla Z + a(Z)Z \geq 0, & x \in \partial\Omega. \end{cases} \tag{2.2}$$

By strict sub and super-solutions we understand functions ψ and Z for which strict inequalities (2) and (3) hold.

It is well known that if there exist sub and supersolutions ψ and Z respectively of (1) such that $\psi \leq Z$. Then (1) has a solution u such that $u \in [\psi, Z]$ (see [2,6]).

To establish our multiplicity result we use the following very useful result discussed in [11]. Note here that by $\psi_1 < \psi_2$ we mean that $\psi_1 \leq \psi_2$ and $\psi_1 \neq \psi_2$.

Lemma 2.1 *Suppose there exist a subsolution ψ_1, a strict supersolution Z_1, a strict subsolution ψ_2 and a supersolution Z_2 for (1) such that $\psi_1 < Z_1 < Z_2$, $\psi_1 < \psi_2 < Z_2$, and $\psi_2 \nleq Z_1$. Then (1) has at least three distinct solutions u_1, u_2 and u_3 such that $\psi_1 < u_1 < u_2 < u_3 \leq Z_2$.*

To precisely state our existence result we consider the unique classical solution e of the following linear elliptic problem

$$\begin{cases} -\Delta e = 1, & x \in \Omega, \\ \mathbf{n}.\nabla e + a_0 e = 0, & x \in \partial\Omega, \end{cases} \tag{2.3}$$

where $a_0 = a(0)$.

Instead of working with the particular reaction term in (1), we will prove our results for a class of functions f which satisfy the following hypothesis:

(A1) $f \in C^2([0, \infty))$, $f(u > 0$ on $[0, r_0)$ and $f(u) < 0$ for $u > r_0$.

Then we establish:

Theorem 2.1 *Let (A1) holds. Then (1) has a positive solution u for all $\lambda > 0$.*

Proof: First observe that $\psi = 0$ is a subsolution of (1). Let us now show there is a positive super-solution Z for problem (1). We can choose C_λ large enough so that $C_\lambda > \lambda \max_{[0,r_0)} f(t)$. Setting $Z = C_\lambda e$, where e is defined in (4), and substituting in (3) we have

$$-\Delta Z = C_\lambda > \lambda \max_{[0,r_0)} f(t) \geq \lambda f(C_\lambda e) \quad = \lambda f(Z) \ \text{ in}\, \Omega,$$

and

$$\begin{aligned} \mathbf{n}.\nabla Z + a(Z)Z &\geq C_\lambda \mathbf{n}.\nabla e + C_\lambda e a_0 \\ &= C_\lambda(\mathbf{n}.\nabla e + e a_0) \\ &= 0 \ \text{ on}\, \partial\Omega. \end{aligned}$$

which implies that Z is indeed a positive supersolution of (1). Therefore (1) has a positive solution for all $\lambda > 0$. □

Our second result concerns with multiplicity of solution for the problem (1) and gives an estimate on the parameter λ when such a situation occurs. For positive constants $0 < a < b$, define

$$Q(a,b,\Omega) = \frac{\frac{b}{f(b)}(\frac{N+1}{N})^{N+1}\frac{N^2}{R^2}}{\min\{\frac{a}{\|e\|_\infty \hat{f}(a)}, \frac{2NM}{f(b)R^2}\}},$$

where $B_R = B(0,R)$ be the largest ball of radius R inscribed in Ω, e is defined in (4) and $\hat{f}(s) = \max_{[0,s)} f(t)$. We establish:

Theorem 2.2 *Let $m, M \in (0, r_0)$ be such that f is non-decreasing in (m, M). Assume there exist $b \in [m, M]$ and $a \in (0, b)$ such that $Q(a, b, \Omega) < 1$. Then (1) has three positive solutions for $\lambda \in (\lambda_*, \lambda^*)$ where*

$$\lambda_* = \frac{b}{f(b)}(\frac{N+1}{N})^{N+1}\frac{N^2}{R^2}, \;\; \lambda^* = \min\{\frac{a}{\|e\|_\infty \hat{f}(a)}, \frac{2NM}{f(b)R^2}\}.$$

Proof: To establish the multiplicity result we have to construct a subsolution ψ_1, a strict supersolution Z_1, a strict subsolution ψ_2 and a supersolution Z_2 for (1) such that $\psi_1 < Z_1 < Z_2$, $\psi_1 < \psi_2 < Z_2$, and $\psi_2 \not\leq Z_1$. We already know that (1) has at least one positive solution for all $\lambda > 0$. We let $\psi_1 = \psi$ and $Z_2 = Z$ be as in the proof of Theorem 2.2 We need to construct a strict subsolution and a strict supersolution. First we construct a strict supersolution Z_1 of (1). Assume $\lambda < \lambda^*$ and choose $\epsilon > 0$ so small that $\lambda \hat{f}(a) < \frac{a}{\|e\|_\infty + \epsilon}$. We now claim that a function $Z_1 = a\frac{e_a+\epsilon}{\|e\|_\infty+\epsilon}$ is a strict super-solution of (1) for $\lambda \in (0, \lambda^*)$. Indeed, substituting Z_1 into (3) and using (4) we have

$$\begin{aligned} -\Delta Z_1 = \frac{a}{\epsilon + \|e\|_\infty} &> \lambda \hat{f}(a) \\ &\geq \lambda f\Big(a\frac{e+\epsilon}{\|e\|_\infty + \epsilon}\Big) \\ &= \lambda f(Z_1) \;\; \text{in}\, \Omega, \end{aligned}$$

and

$$\begin{aligned} \mathbf{n}.\nabla Z_1 + a(Z_1)Z_1 &\geq \frac{a}{\epsilon + \|e\|_\infty}\Big(\mathbf{n}.\nabla e + (e+\epsilon)a_0\Big) \\ &= \frac{a}{\epsilon + \|e\|_\infty}(\mathbf{n}.\nabla e + a_0 e + a_0\epsilon) \\ &= \frac{a a_0 \epsilon}{\epsilon + \|e\|_\infty} \\ &> 0 \;\; \text{on}\, \partial\Omega. \end{aligned}$$

Next let us construct a strict sub-solution ψ_2 of (1). First note that a problem

$$\begin{cases} -\Delta u_D = \lambda f(u_D), & x \in \Omega, \\ u_D = 0, & x \in \partial\Omega, \end{cases}$$

admits a strict sub-solution ψ_D with $\|\psi_D\|_\infty \geq b$ provided $\lambda < \lambda^*$ (see [3]). By the Hopf's lemma we have that $\mathbf{n}.\nabla\psi_D < 0$. Therefore, setting $\psi_2 = \psi_D$ we obtain a strict sub-solution for (1) for $\lambda > \lambda_*$.

Thus for $\lambda \in (\lambda_*, \lambda^*)$ we have strict positive sub and super-solutions ψ_2 and Z_1 of the problem (1) such that $\psi_2 \not\leq Z_1$. The latter is guaranteed by the fact that $\|Z_1\|_\infty = a$ and $\|\psi_2\|_\infty \geq b > a$. In a view that e, the solution of (4) is bounded away from zero we conclude that all the conditions of 2.1 are satisfied for $\lambda \in (\lambda_*, \lambda^*)$ and thus solutions of the problem (1) are multiple in this range of λ's. □

References

1. C. M. Dafermos, C. M., *An abstract Volterra equation with application to linear viscoelasticity*. J. Differential Equations 7, 554-589, (1970).

2. H. Amann, *Fixed point equations and nonlinear eigenvalue problems in ordered banach spaces*. SIAM Rev. 18 , 620-709, (1976).

3. D. Butler, S. Sasi, and R. Shivaji, *Existence of Alternate Steady States in a Phosphorous Cycling Model*. ISRN Mathematical Analysis, 2012, Article ID 869147, 11 pages, (2012).

4. S.R. Carpenter, D. Ludwig, and W.A. Brock, *Management of eutrophication for lakes subject to potentially irreversible change*. Ecological Applications, 9 (3), 751-771, (1999).

5. N. Fonseka, R. Shivaji, B. Son, K. Spetzer, *Classes of reaction diffusion equations where a parameter influences the equation as well as the boundary condition*. J. Math. Anal. Appl, 476, 480-494, (2019).

6. F. Inkmann, *Existence and multiplicity theorems for semilinear elliptic equations with nonlinear boundary conditions*. Indiana Univ. Math. J. 31 (2), 213-221, (1982).

7. M. Mallick, S. Sasi, R. Shivaji, S. Sundar, *Bifurcation, uniqueness and multiplicity results for classes of reaction diffusion equations arising in ecology with nonlinear boundary conditions*. Commu. Pure. Appl. Anal, 21 (2), 705-726, (2022).
   Math. J. 31 (2) (1982) 213-221.

8. S.H. Rasouli, *On a mathematical model of thermal explosion with multiple parameters and nonlinear boundary conditions*. Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis, 28, 145-152, (2021)

9. S.H. Rasouli, *On a population model with nonlinear boundary conditions arising in ecosystems*. Bulletin of the Belgian Mathematical Society-Simon Stevin, 26, 63-69, (2019).

10. M. Scheffer, W. Brock, and F. Westley, *Socioeconomic mechanisms preventing optimum use of ecosystem services: an interdisciplinary theoretical analysis*. Ecosystems, 3(5), 451-471, (2000).

11. R. Shivaji, *A remark on the existence of three solutions via sub-super solutions*. Nonlinear Analysis and Application, Lecture notes in pure and applied mathematics. 109, 561-566, (1987).

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