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Introduction to Gradient h-almost η-Ricci Soliton on Warped Product Spaces *

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ABSTRACT: In this paper, we introduce the new concept of gradient h-almost η-Ricci soliton. We discuss here a steady or expanding gradient h-almost η-Ricci soliton warped product $B^n \times_f F^m$, $m > 1$. We show that the warping function f of this warped product attains minimum as well as maximum and it will be a Riemannian product under certain conditions. We also describe some suitable restrictions to these constructions for the compact base of this warped product. Later, we study h-almost η-Ricci soliton and gradient h-almost η-Ricci soliton on warped product manifolds including a concurrent vector field.

Key Words: Warped product space, Ricci soliton, η-Ricci soliton, h-almost Ricci soliton.

Contents

1. Introduction

Firstly, we discuss the concept of the warped product. A fertile generalization of the concept of the direct or cartesian product is nothing but a concept of the warped product. The notion of the warped product has come in the physical and mathematical literature. The idea of the warped product had been developed due to a surface of revolution. There are two natural elongations of the warped product manifolds which are convolution manifolds and twisted products. There exist so many exact solutions of Einstein field equations and modified field equations. These solutions are warped product. Robertson-walker models and the Schwarzschild solution are the examples of warped product. For the purpose of the study of manifolds of negative curvature Bishop and O'Neill [3] introduced the warped product.

Recently, the perusal of warped product plays a significant role. At first, Bishop and O' Neill [3] had given the idea of warped products and they had constructed examples of complete Riemannian manifolds with negative sectional curvature. Suppose that (B, g_B) and (F, g_F) are two Riemannian manifolds of positive dimensions and $f : B \to (0, \infty)$ is a positive smooth function. Also, suppose that $\pi : B \times F \to B$ and $\eta : B \times F \to F$ are the natural projections on B and F respectively. The warped product $M = B \times_f F$ is the manifold $B \times F$ furnished with the following Riemannian structure such that

$$g = \pi^* g_B + (f \circ \pi)^2 \eta^* g_F, \tag{1.1}$$

where f is known as warping function. π^* and η^* are pull back maps of π and η respectively. B and F are base and fiber of M respectively. Note that if f is a constant function, then M is just a usual Riemannian product.

Though in the Riemannian geometry, the class of warped products with a non-constant warping functions serve a rich class of examples, Kim and Kim [17] showed it there hardly exists a compact Einstein warped product having non-constant warping function under the condition of non-positiveness of scalar curvature. Additionaly, they noticed that one warped product would be an Einstein manifold if its base is a quasi-Einstein metric. It should be focused that some paradigms of expanding quasi - Einstein manifolds with an arbitrary Einstein manifold as a fiber and steady quasi-Einstein manifolds having fiber of non-negative scalar curvature which was developed in Besse [2]. In recent times, Barros, Batista and Ribeiro [1] served few volume estimations of Einstein warped products which are similar to a classical result of Yau [21] and

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Calabi [7] for complete Riemannian manifolds with non-negative Ricci curvature. Their approach is with quasi-Einstein manifold. They also showed a hindrance for the existence of such a class of manifolds. In this regard, we want to mention He, Petersen and Wylie's [15] work on Einstein warped product manifolds. As it is an elongation of Case, Shu and Wei's [9] work and some erstwhile works of Kim and Kim [17], the result of [15] is that the base may have non-void boundary.

Secondly, we discuss the concept of Ricci soliton. Ricci solitons are the generalization of Einstein manifolds. Hamilton [13] developed this idea at the beginning of 80's. A Riemannian manifold M furnished with a metric g is said to be a Ricci soliton if it satisfies the following relation

$$\mathrm{Ric} + \frac{1}{2}\pounds_X g = \lambda g, \tag{1.2}$$

where λ is a scalar quantity and X is a vector field of M. The above equation (1.2) is known as the fundamental equation. Ricci solitons are of three types depending upon the values of λ. If $\lambda > 0$, $\lambda < 0$ and $\lambda = 0$, then a Ricci soliton will be shrinking, expanding and steady respectively. Moreover, If we take $X = \nabla\psi$ in equation (1.2), where ψ being a smooth function on M, then we denote the gradient Ricci soliton as $(M, g, \nabla\psi, \lambda)$. Hence the equation (1.2) becomes

$$\mathrm{Ric} + \nabla^2\psi = \lambda g, \tag{1.3}$$

where $\nabla^2\psi$ is the Hessian of ψ. To know more see [8,13].

J. N. Gomes, Q. Wang, C. Xia introduced a new kind of Ricci soliton, called h-almost Ricci soliton in [12]. They have given the following definition.

An h-almost Ricci soliton is a complete Riemannian manifold (M^n, g) which are smooth and satisfy the equation

$$\mathrm{Ric} + \frac{h}{2}\pounds_X g = \lambda g + \mu(\eta \otimes \eta),$$

where $X \in \mathfrak{X}(M)$, $\lambda : M \to R$ is a soliton function and $h : M \to R$ is a function. Then (M^n, g, X, h, λ) is called an h-almost Ricci soliton.

Inspired and motivated by the concept of warped product and Ricci soliton [4,18,20], we introduce a new notion of h-almost η-Ricci soliton as follows.

Definition 1.1. *A complete Riemannian manifold (M^n, g) furnished with a metric g is said to be an h-almost η-Ricci soliton if it satisfies the following relation*

$$\mathrm{Ric} + \frac{h}{2}\pounds_X g = \lambda g + \mu(\eta \otimes \eta), \tag{1.4}$$

where λ is a scalar quantity, X is a vector field on M, $h : M \to R$ is a smooth function and η is a 1-form.

Moreover, if we put $X = \nabla\psi$ in equation (1.4), then we obtain an another definition as follows.

Definition 1.2. *A complete Riemannian manifold (M^n, g) furnished with a metric g is said to be a gradient h-almost η-Ricci soliton if it satisfies the following relation*

$$\mathrm{Ric} + h\nabla^2\psi = \lambda g + \mu(\eta \otimes \eta), \tag{1.5}$$

where ψ is a smooth function on M and $\nabla^2\psi$ is the Hessian of ψ. We denote it as $(M, g, \nabla\psi, h, \eta, \lambda)$ for convenience.

At the beginning of 90's, it was proved that a Ricci soliton which is a compact gradient expanding or steady, is an Einstein manifold [14,16]. Petersen and Wylie [19] gave a theorem in reference to Brinkmann [5] that warped product is nothing but a surface gradient Ricci soliton. Robert Bryant [6,10] constructed an example of a steady Ricci soliton as a warped product $(0, \infty) \times_f S_m$, where m is greater than one, where warping function is a radial function. As the function f is not limited, hence we face two very

simple questions which are given as follows.
(1) When a warped product having a limited warping function would be an h-almost η-Ricci soliton ?
(2) Are there any condition? if yes, what are these conditions ?

In this paper, Theorem 2.7 partly provides an answer to these above questions. The concept of [17] inspires our second theorem. Our first theorem is the natural generalization from Einstein case to Ricci soliton case except the condition of compactness on the product which has been considered in [17]. By the way, one significant fact comes out during the study of h-almost η-Ricci soliton which are felt like a warped product. The base space of them satisfy the following equation

$$\mathrm{Ric} + \nabla^2\phi = \lambda g_B + \frac{m}{f}\nabla^2 f, \tag{1.6}$$

It is the generalization of Einstein metrics containing quasi-Einstein metrics. Theorem 2.8 sets up a criterion of compactness for shrinking gradient h-almost η-Ricci soliton warped product under the assumption that the base is compact. The following two lemmas are very important for further study.

Lemma 1.3. *[3] If Y, $Z \in \Gamma(B)$ and V, $W \in \Gamma(F)$ on M, then the following results hold*

$$\begin{aligned}
&(i) \quad \textit{On } B,\ D_Y Z \textit{ is the lift of } \nabla_Y Z,\\
&(ii) \quad D_Y V = D_V Y = \frac{Y(f)}{f} V,\\
&(iii) \quad H(D_V W) = -\frac{g(V,W)}{f}\nabla f,\\
&(iv) \quad \textit{On } F,\ V(D_V W) \in \Gamma(F) \textit{ is the lift of } F_{\nabla_V W}.
\end{aligned}$$

Particularly,

$$\Delta\tilde{h} = \Delta h + \frac{m}{f}\nabla h(f), \tag{1.7}$$

for all smooth functions h on B.

Lemma 1.4. *[3] If Y, $Z \in \Gamma(B)$ and V, $W \in \Gamma(F)$ on $M = B^n \times_f F^m$, where $m > 1$, then the following results hold*

$$\begin{aligned}
&(i) \quad \mathrm{Ric}(Y,Z) = \mathrm{Ric}_B(Y,Z) - \frac{m}{f}H^f(Y,Z),\\
&(ii) \quad \mathrm{Ric}(Y,V) = 0,\\
&(iii) \quad \mathrm{Ric}(V,W) = \mathrm{Ric}_F(V,W) - \left[\frac{\Delta f}{f} + \frac{|\nabla f|^2}{f^2}(m-1)\right] g(V,W).
\end{aligned}$$

Besides these two lemmas, the following two identities will help us to prove Proposition 2.3.

$$\mathrm{div}(\nabla^2\phi) = \mathrm{Ric}(\nabla\phi, .) + \mathrm{d}(\Delta\phi), \tag{1.8}$$

$$\frac{1}{2}\mathrm{d}(|\nabla\phi|^2) = (\nabla^2\phi)(\nabla\phi, .). \tag{1.9}$$

Now, by taking trace of the equation (1.5), we gain

$$\mathrm{R} + h\Delta\psi = k\lambda + \mu.$$

Hamilton [14] proved the following result

$$2\lambda\psi - |\nabla\psi|^2 + \Delta\psi = c, \tag{1.10}$$

where c is some constant. In this way, we have derived similar equation to (1.10) for gradient h-almost η- Ricci soliton warped product's base, cf. equation (2.4). It is the first outcome of our next section.

2. The conditions for existence of h-almost η-Ricci soliton on warped product spaces

Now a Riemannian manifold (B^n, g_B) has been taken as possible base of a gradient h-almost η-Ricci soliton warped product $(M = B^n \times_f F^m, g, \nabla\psi, h, \eta, \lambda)$. We cosider that ψ is the potential function and ψ being the lift of ϕ, which is a smooth function defined on B^n, that is, the crucial information of M will be carried by base. Keeping in mind with these considerations, we set up some conditions on the functions which parametrize a gradient h-almost η-Ricci soliton by the almost η-Ricci soliton warped product. Hamilton's equation (1.10) for B^n is the first condition.

Proposition 2.1. *Let $M = B^n \times_f F^m$ be a warped product and ϕ defined on B is a smooth function such that $(M, g, \nabla\tilde{\phi}, h, \eta, \lambda)$ is a gradient h-almost η-Ricci soliton. Then we obtain*

$$2\lambda\phi - \mid \nabla\phi \mid^2 + \Delta\phi + \frac{m}{f}\nabla\phi(f) = c,$$

where c is a constant.

Proof. Hamilton's equation (1.10) on manifold M is given by

$$2\lambda\tilde{\phi} - \mid \nabla\tilde{\phi} \mid^2 + \Delta\tilde{\phi} = c, \tag{2.1}$$

where c is some constant. Note that

$$\nabla\tilde{\phi} = \tilde{\nabla\phi}, \tag{2.2}$$

$$\Delta\tilde{\phi} = \Delta\phi + \frac{m}{f}\nabla\phi(f). \tag{2.3}$$

Using equations (2.2) and (2.3) in equation (2.1), we gain

$$2\lambda\phi - \mid \nabla\phi \mid^2 + \Delta\phi + \frac{m}{f}\nabla\phi(f) = c. \tag{2.4}$$

This completes the proof. □

Proposition 2.2. *Let $M = B^n \times_f F^m$ be a warped product and ϕ defined on B is a smooth function such that $(M, g, \nabla\tilde{\phi}, h, \eta, \lambda)$ is a gradient h-almost η-Ricci soliton, where $m > 1$. Then we gain*

$$\mathrm{Ric}_B + hH^\phi = \lambda g_B + \frac{m}{f}H^f + \mu(\eta \otimes \eta),$$
$$\mathrm{Ric}_F = [\lambda f^2 + f\Delta f + (m-1) \mid \nabla f \mid^2 - hf\nabla\phi(f)]g_F + \mu(\eta \otimes \eta).$$

Proof. From Lemma 1.4, it is clear that

$$\mathrm{Ric}(Y, Z) = \mathrm{Ric}_B(Y, Z) - \frac{m}{f}H^f(Y, Z), \ \forall \ Y, Z \in \Gamma(B). \tag{2.5}$$

The gradient h-almost η-Ricci soliton is

$$\mathrm{Ric} + h\nabla^2\tilde{\phi} = \lambda g + \mu(\eta \otimes \eta).$$
$$i.e., \ \mathrm{Ric}(Y, Z) = \lambda g_B(Y, Z) + \mu(\eta \otimes \eta)(Y, Z) - hH^\phi(Y, Z). \tag{2.6}$$

From the equations (2.5) and (2.6), it follows that

$$\mathrm{Ric}_B + hH^\phi = \lambda g_B + \frac{m}{f}H^f + \mu(\eta \otimes \eta). \tag{2.7}$$

Hence, this completes the proof of the first assertion of Proposition 2.2.

For $V, W \in \Gamma(F)$ Lemma 1.4 gives

$$\mathrm{Ric}(V, W) = \mathrm{Ric}_F(V, W) - \left[\frac{\Delta f}{f} + (m-1)\frac{\mid \nabla f \mid^2}{f^2}\right] g(V, W), \ \forall \ V, W \in \Gamma(F). \tag{2.8}$$

Also, from the equation (1.5), we obtain

$$\mathrm{Ric}(V,W) = \lambda f^2 g_F(V,W) - h\nabla^2\tilde{\phi}(V,W) + \mu(\eta\otimes\eta)(V,W). \tag{2.9}$$

In view of the equations (2.8) and (2.9), we have

$$\begin{aligned}\mathrm{Ric}_F(V,W) =&\lambda f^2 g_F(V,W) - h\nabla^2\tilde{\phi}(V,W) + \mu(\eta\otimes\eta)(V,W)\\ &+ f\left[\Delta f + \frac{(m-1)\mid\nabla f\mid^2}{f}\right] g_F(V,W).\end{aligned} \tag{2.10}$$

Since, $\nabla\tilde{\phi}\in\Gamma(B)$ and using the equation (1.7), we obtain

$$\nabla^2\tilde{\phi}(V,W) = g(D_V\nabla\tilde{\phi},W) = g\left(\frac{\nabla\tilde{\phi}(f)}{f}V,W\right) = f\nabla\phi(f)g_F(V,W). \tag{2.11}$$

In view of the equation (2.11), the equation (2.10) implies that

$$\begin{aligned}\mathrm{Ric}_F(V,W) =&[\lambda f^2 + f\Delta f + (m-1)\mid\nabla f\mid^2\\ &- hf\nabla\phi(f)]g_F(V,W) + \mu(\eta\otimes\eta)(V,W).\end{aligned} \tag{2.12}$$

Hence, this completes the proof of the second assertion of Proposition 2.2 and consequently, the proof of Proposition 2.2 has been completed. □

Proposition 2.3. *Let (B^n, g) be a Riemannian manifold having two smooth functions ϕ and $f(>0)$ which satisfy the following equations*

$$\mathrm{Ric} + h\nabla^2\phi = \lambda g + \frac{m}{f}\nabla^2 f + \mu(\eta\otimes\eta), \tag{2.13}$$

$$2\lambda\phi - \mid\nabla\phi\mid^2 + \Delta\phi + \frac{m}{f}\nabla\phi(f) = c, \tag{2.14}$$

for some constants $m(\neq 0)$, c, λ and $\mu\in\mathbb{R}$. Then f and ϕ will satisfy the following equation

$$\lambda f^2 + f\Delta f + (m-1)\mid\nabla f\mid^2 - hf\nabla\phi(f) = \beta, \tag{2.15}$$

where $\beta\in\mathbb{R}$ is a constant, if the following condition is satisfied

$$\begin{aligned}0 =& - hf\mathrm{d}(\nabla\phi(f)) + \frac{hf^2}{m}\mathrm{d}(h\mid\nabla\phi\mid^2) - \frac{hf^2}{m}\mathrm{d}(\mid\nabla\phi\mid^2)\\ &+ 2f\mu(\eta\otimes\eta)(\nabla f,.) + \frac{f^2}{m}\Delta\phi\mathrm{d}h - \frac{2h\mu f^2}{m}(\eta\otimes\eta)(\nabla\phi,.)\\ &- \frac{2f^2}{m}(\nabla^2\phi)(\nabla h,.) + \mathrm{d}hf(\nabla\phi(f)).\end{aligned} \tag{2.16}$$

Proof. By taking trace on both sides of the equation (2.13), we have

$$\mathrm{S} = n\lambda + \frac{m}{f}\Delta f + \mu - h\Delta\phi, \tag{2.17}$$

where scalar curvature of B is S. Hence,

$$\mathrm{dS} = -\frac{m}{f^2}\Delta f\mathrm{d}f + \frac{m}{f}\mathrm{d}(\Delta f) - \Delta\phi\mathrm{d}h - h\mathrm{d}(\Delta\phi). \tag{2.18}$$

Now, we use the second contracted Bianchi identity, which is

$$-\frac{1}{2}\mathrm{dS} + \mathrm{div}(\mathrm{Ric}) = 0. \tag{2.19}$$

We obtain by computation from the equation (2.13),

$$\begin{aligned}\operatorname{div}(\mathrm{Ric}) =&\frac{m}{f}\mathrm{Ric}(\nabla f, .) + \frac{m}{f}\mathrm{d}(\Delta f) - \frac{m}{2f^2}\mathrm{d}(\mid \nabla f \mid^2)\\ &- h\mathrm{Ric}(\nabla \phi, .) - h\mathrm{d}(\Delta \phi) - (\nabla^2\phi)(\nabla h, .)\end{aligned} \tag{2.20}$$

From the equation (2.13), it follows that

$$\begin{aligned}\mathrm{Ric}(\nabla f, X) + h(\nabla^2\phi)(\nabla f, X) =&\lambda \mathrm{d}f + \frac{m}{2f}\mathrm{d}(\mid \nabla f \mid^2) + \mu(\eta\otimes\eta)(\nabla f, .)\\ &- h(\nabla^2\phi)(\nabla f, .).\end{aligned} \tag{2.21}$$

Replacing ∇f by $\nabla \phi$ in the equation (2.21), we obtain

$$\begin{aligned}\mathrm{Ric}(\nabla \phi, .) =&\lambda \mathrm{d}\phi + \frac{m}{f}(\nabla^2 f)(\nabla\phi, .)\\ &+ \mu(\eta\otimes\eta)(\nabla\phi, .) - \frac{h}{2}\mathrm{d}(\mid \nabla\phi \mid^2).\end{aligned} \tag{2.22}$$

Using equations (2.21) and (2.22) in the equation (2.20), we gain

$$\begin{aligned}\operatorname{div}(\mathrm{Ric}) =&\frac{m\lambda}{f}\mathrm{d}f + \frac{m(m-1)}{2f^2}\mathrm{d}(\mid \nabla f \mid^2) + \frac{m\mu}{f}(\eta\otimes\eta)(\nabla f, .)\\ &- \frac{mh}{f}\mathrm{d}(\nabla\phi(f)) + \frac{m}{f}\mathrm{d}(\Delta f) - h\lambda\mathrm{d}\phi - h\mu(\eta\otimes\eta)(\nabla\phi, .)\\ &+ \frac{h^2}{2}\mathrm{d}(\mid \nabla\phi \mid^2) - h\mathrm{d}(\Delta\phi) - (\nabla^2\phi)(\nabla h, .).\end{aligned} \tag{2.23}$$

Using equations (2.18) and (2.23) in the equation (2.19), we obtain

$$\begin{aligned}0 =&\frac{m}{2f^2}\Delta f\mathrm{d}f + \frac{m}{2f}\mathrm{d}(\Delta f) + \frac{1}{2}\Delta\phi\mathrm{d}h\\ &- \frac{h}{2}\mathrm{d}(\Delta\phi) + \frac{m\lambda}{f}\mathrm{d}f + \frac{m(m-1)}{2f^2}\mathrm{d}(\mid \nabla f \mid^2)\\ &+ \frac{m\mu}{f}(\eta\otimes\eta)(\nabla f, .) - \frac{mh}{f}\mathrm{d}(\nabla\phi(f)) - h\lambda\mathrm{d}\phi\\ &- h\mu(\eta\otimes\eta)(\nabla\phi, .) + \frac{h^2}{2}\mathrm{d}(\mid \nabla\phi \mid^2) - (\nabla^2\phi)(\nabla h, .).\end{aligned} \tag{2.24}$$

Multiplying the equation (2.24) by $\frac{2f^2}{m}$, we get

$$\begin{aligned}0 =&\mathrm{d}[f\Delta f + \lambda f^2 + (m-1)\mid \nabla f \mid^2] - \frac{hf^2}{m}\mathrm{d}[\Delta\phi + 2\lambda\phi - h\mid \nabla\phi \mid^2]\\ &+ \frac{f^2}{m}\Delta\phi\mathrm{d}h + 2\mu f(\eta\otimes\eta)(\nabla f, .) - 2hf\mathrm{d}(\nabla\phi(f))\\ &- \frac{2h\mu f^2}{m}(\eta\otimes\eta)(\nabla\phi, .) - \frac{2f^2}{m}(\nabla^2\phi)(\nabla h, .).\end{aligned}$$

Using the hypothesis

$$2\lambda\phi - \mid \nabla\phi \mid^2 + \Delta\phi + \frac{m}{f}\nabla\phi(f) = c,$$

we derive after some steps

$$\begin{aligned} 0 =& \mathrm{d}(f\Delta f + \lambda f^2 + (m-1) \mid \nabla f \mid^2) + hf\mathrm{d}f(\nabla\phi(f)) - h\mathrm{d}f(\nabla\phi(f)) \\ &+ \frac{hf^2}{m}\mathrm{d}(h \mid \nabla\phi \mid^2) - \frac{hf^2}{m}\mathrm{d}(\mid \nabla\phi \mid^2) + 2f\mu(\eta\otimes\eta)(\nabla f, .) \\ &+ \frac{f^2}{m}\Delta\phi\mathrm{d}h - 2hf\mathrm{d}(\nabla\phi(f)) - \frac{2h\mu f^2}{m}(\eta\otimes\eta)(\nabla\phi, .) \\ &- \frac{2f^2}{m}(\nabla^2\phi)(\nabla h, .) + \mathrm{d}hf(\nabla\phi(f)). \end{aligned} \tag{2.25}$$

If we consider that

$$\begin{aligned} 0 =& -hf\mathrm{d}(\nabla\phi(f)) + \frac{hf^2}{m}\mathrm{d}(h \mid \nabla\phi \mid^2) - \frac{hf^2}{m}\mathrm{d}(\mid \nabla\phi \mid^2) \\ &+ 2f\mu(\eta\otimes\eta)(\nabla f, .) + \frac{f^2}{m}\Delta\phi\mathrm{d}h - \frac{2h\mu f^2}{m}(\eta\otimes\eta)(\nabla\phi, .) \\ &- \frac{2f^2}{m}(\nabla^2\phi)(\nabla h, .) + \mathrm{d}hf(\nabla\phi(f)). \end{aligned}$$

Then the equation (2.25) becomes

$$\mathrm{d}\left(f\Delta f + \lambda f^2 + (m-1) \mid \nabla f \mid^2 - hf(\nabla\phi(f))\right) = 0, \tag{2.26}$$

which is sufficient to complete the proof. □

Theorem 2.4. *Let $M = B^n \times_f F^m$ be a warped product and ϕ is a smooth function on B such that $(M, g, \nabla\tilde{\phi}, h, \eta, \lambda)$ is a steady or expanding gradient h-almost η-Ricci soliton. Also, suppose that fiber F^m of this warped product is of dimension ≥ 2 and the warping function f of it attains minimum as well as maximum with the condition (2.26). Then M will definitely be a Riemannian product if $(h-1)\nabla\phi(f) \geq \frac{(1-m)}{f} \mid \nabla f \mid^2$.*

Proof. Let $M = B^n \times_f F^m$, $m > 1$, be a gradient h-almost η-Ricci soliton satisfying

$$\mathrm{Ric} + h\nabla^2\tilde{\phi} = \lambda g + \mu(\eta\otimes\eta)$$

Then Proposition 2.2 indicates

$$\mathrm{Ric}_F = \beta g_F + \mu(\eta\otimes\eta),$$

where

$$\beta = \lambda f^2 + f\Delta f + (m-1) \mid \nabla f \mid^2 - hf(\nabla\phi(f)).$$

From Proposition 2.3, it is clear that β is a constant. Equations (2.13) and (2.14) are guaranteed from the equations (2.4) and (2.7) satisfying the condition (2.26). Suppose that $p, q \in B^n$ are the points where the warping function f reaches its minimum as well as maximum in B^n. Hence

$$\nabla f(p) = 0 = \nabla f(q),\ \ \nabla f(p) \leq 0 \leq \nabla f(q). \tag{2.27}$$

As, $\lambda \leq 0$ and $f > 0$, we obtain

$$-\lambda(f(p))^2 \geq -\lambda(f(q))^2$$

and plugging this with the equation (2.27), we get

$$0 \geq f(p)\Delta f(p) = \beta - \lambda(f(p))^2 \geq \beta - \lambda(f(q))^2 = f(q)\Delta f(q) \geq 0. \tag{2.28}$$

Equation (2.28) now implies

$$\beta - \lambda(f(p))^2 = \beta - \lambda(f(q))^2 = 0.$$

Hence, $\lambda < 0$ implies that $f(p) = f(q)$. That is, the warping function f is a constant function.

When $\lambda = 0$, we obtain that $\beta = 0$ and the equation (2.27) becomes

$$\begin{aligned} Lf =& (\Delta - \nabla\phi)f, \ \text{[where } L = \Delta - \nabla\phi] \\ =& \frac{(1-m)}{f} \mid \nabla f \mid^2 + (h-1)\nabla\phi(f) \end{aligned}$$

Clearly, $\frac{(1-m)}{f} \mid \nabla f \mid^2 \leq 0$. It is also seen that $Lf \leq 0$, if $(h-1)\nabla\phi(f) \geq \frac{(1-m)}{f} \mid \nabla f \mid^2$.

Hence, if $(h-1)\nabla\phi(f) \geq \frac{(1-m)}{f} \mid \nabla f \mid^2$, then by using strong maximum principle, it is obvious that f is a constant. Therefore, in both cases M is a Riemannian product. ☐

Theorem 2.5. *Let $M = B^n \times_f F^m$ be a warped product and ϕ is a smooth function on B such that $(M, g, \nabla\tilde{\phi}, h, \eta, \lambda)$ is a shrinking gradient h-almost η-Ricci soliton having compact base and fiber of dimension greater than or equal to two. Then M will definitely be a compact manifold if* $\int_{B^n}(1-h)f(\nabla\phi(f))\mathrm{d}B > 0$.

Proof. Let $M = B^n \times_f F^m$, $m > 1$, be a gradient h-almost η-Ricci soliton with $\mathrm{Ric} + h\nabla^2\tilde{\phi} = \lambda g + \mu(\eta\otimes\eta)$. From Theorem 2.4, it follows that $\mathrm{Ric}_F = \beta g_F + \mu(\eta\otimes\eta)$, where β is a constant which is given by the equation (2.27) or equivalently

$$\begin{aligned} \beta =& \lambda f^2 + f\Delta f + (m-1) \mid \nabla f \mid^2 - hf(\nabla\phi(f)) \\ =& \lambda f^2 + f(\Delta f - \nabla\phi(f)) + (m-1) \mid \nabla f \mid^2 + (1-h)f\nabla\phi(f) \\ =& \lambda f^2 + fLf + (m-1) \mid \nabla f \mid^2 + (1-h)f\nabla\phi(f). \end{aligned}$$

Integrating on both sides, we have

$$\begin{aligned} \beta.\mathrm{vol}_\phi(B^n) =& \lambda \int_{B^n} f^2 e^{-\phi}\mathrm{d}B + (m-2)\int_{B^n} \mid \nabla f \mid^2 e^{-\phi}\mathrm{d}B \\ &+ \int_{B^n}(1-h)f(\nabla\phi(f))\mathrm{d}B. \end{aligned}$$

As $m > 1$ and $\lambda > 0$, hence we can conclude that $\beta > 0$ if $\int_{B^n}(1-h)f(\nabla\phi(f))\mathrm{d}B > 0$. Therefore, by using Bonnet-Myers Theorem, it is obvious that F^m is compact and consequently $B^n \times_f F^m$ becomes a compact manifold. ☐

Theorem 2.6. *Let $\bar{M} = I \times_f M$ be a generalized Robertson-walker space time furnished by a metric $\bar{g} = -\mathrm{d}t^2 \oplus f^2 g$, where (M, g) is a Riemannian manifold and I is an open connected interval with the usual flat metric $-\mathrm{d}t^2$. If $(\bar{M}, \bar{g}, u, h, \eta, \lambda)$ be a gradient h-almost η-Ricci soliton, for $u = \int_a^t f(r)\mathrm{d}r$, where $a \in I$ is a constant, then* $\mathrm{Ric} = (\lambda - h\dot{f})\bar{g} + \mu(\eta\otimes\eta)$.

Proof. Assume that $\zeta = \mathrm{grad}\ u$, hence $\zeta = f(t)\partial_t$. Clearly, the vector field is orthogonal to M. Let ∂_t, ∂_1, ∂_2, ..., ∂_m are orthogonal bases of $\chi(\bar{M})$, then the Hessian tensor of u is given as follows.

$$H^u(\partial_t, \partial_t) = \bar{g}(\nabla_X \mathrm{grad}\ u, Y).$$

Now, the following cases may arise. The first case when $X = Y = \partial_t$. For this, we get

$$\begin{aligned} H^u(\partial_t, \partial_t) =& \bar{g}(\nabla_{\partial_t}\mathrm{grad}\ u, \partial_t) \\ =& \dot{f}\bar{g}(\partial_t, \partial_t). \end{aligned}$$

The second case when $X = \partial_t$ and $Y = \partial_i$, $i = 1, 2, 3, ..., m$. For this , we get

$$\begin{aligned} H^u(\partial_t, \partial_i) =& \bar{g}(\nabla_{\partial_t} \text{grad } u, \partial_i) \\ =& \dot{f}\bar{g}(\partial_t, \partial_i). \end{aligned}$$

At last, when $X = \partial_t$ and $Y = \partial_i$, $i = 1, 2, 3, ..., m$. For this , we obtain

$$\begin{aligned} H^u(\partial_i, \partial_j) =& \bar{g}(\nabla_{\partial_i} \text{grad } u, \partial_j) \\ =& f\bar{g}(\nabla_{\partial_i}\partial_t, \partial_j) \\ =& f\bar{g}\left(\frac{\dot{f}}{f}\partial_i, \partial_j\right) \\ =& \dot{f}\bar{g}(\partial_i, \partial_j). \end{aligned}$$

Hence, $H^u(X, Y) = \dot{f}\bar{g}(X, Y)$ and consequently

$$\begin{aligned} (\pounds_\xi \bar{g})(X, Y) =& \bar{g}(\nabla_X \text{grad } u, Y) + \bar{g}(\nabla_Y \text{grad } u, X) \\ =& 2H^u(X, Y) \\ =& 2\dot{f}\bar{g}(X, Y). \end{aligned}$$

Let $(\bar{\bar{M}}, \bar{g}, u, h, \eta, \lambda)$ be a gradient h-almost η-Ricci soliton, then

$$\begin{aligned} \text{Ric} + \frac{h}{2}\pounds_\xi \bar{g} = \lambda\bar{g} + \mu(\eta \otimes \eta) \\ i.e., \text{ Ric} = (\lambda - h\dot{f})\bar{g} + \mu(\eta \otimes \eta) \end{aligned}$$

This completes the proof of Theorem 2.6. □

Now we give the following definition for proving the next theorems.

A vector field ς on a Riemannian manifold M which satisfies $\nabla_X \varsigma = X$, for any vector field X is called a concurrent vector field [11]. ς is called gradient if there is a function u defined on M such that $\varsigma = \nabla u$

Theorem 2.7. *Let $(M, g, h, \varsigma, \lambda, \mu)$ be an h-almost η-Ricci soliton and ς be a concurrent vector field on M where $M = B^n \times_f F^m$ and $\varsigma_2 \neq 0$. Then F becomes an Einstein manifold for $U_1, U_2 \in \mathfrak{X}(B)$.*

Proof. We consider that $(M, g, h, \varsigma, \lambda, \mu)$ is an h-almost η-Ricci soliton. Then we have

$$\text{Ric}(X, Y) + \frac{h}{2}\pounds_X g(X, Y) = \lambda g(X, Y) + \mu\eta(X)\eta(Y),$$

where $\eta(X) = g(X, U)$.

Since ς is a concurrent vector field, we obtain

$$\text{Ric}(X, Y) + \frac{h}{2}(g(D_X\varsigma, Y) + g(D_Y\varsigma, X)) = \lambda g(X, Y) + \mu\eta(X)\eta(Y).$$

Hence we get

$$\text{Ric}(X, Y) = (\lambda - h)g(X, Y) + \mu\eta(X)\eta(Y),$$

Putting $X = V \in \mathfrak{X}(F)$, $Y = W \in \mathfrak{X}(F)$, and $U_1, U_2 \in \mathfrak{X}(B)$ then by using Lemma 1.4, it follows that

$$\text{Ric}_F(V, W) = (\lambda - h)f^2 g_F(V, W) + \left[\frac{\Delta f}{f} + \frac{|\nabla f|^2}{f^2}(m - 1)\right] f^2 g_F(V, W).$$

Since ς is concurrent and $\varsigma_2 \neq 0$, ς is concurrent and f is constant. Hence we have $[\frac{\Delta f}{f} + \frac{|\nabla f|^2}{f^2}(m-1)] = 0$ and also we obtain

$$\text{Ric}_F(V, W) = (\lambda - h)f^2 g_F(V, W).$$

Therefore, F is an Einstein manifold. □

Theorem 2.8. *Let $(M, g, h, u, \varsigma, \lambda, \mu)$ be a gradient h-almost η-Ricci soliton where $M = B^n \times_f F^m$. Then (B, g, u, λ) is a gradient Ricci soliton if h is a constant function and $U_1, U_2 \in \mathfrak{X}(F)$.*

Proof. Let $(M, g, h, u, \varsigma, \lambda, \mu)$ be a gradient h-almost η-Ricci soliton. Then we have

$$\mathrm{Ric}(X', X'') + hH^u(X', X'') = \lambda g(X', X'') + \mu\eta(X')\eta(X'').$$

Let $X' = Y \in \mathfrak{X}(B)$, $X'' = Z \in \mathfrak{X}(B)$ and $U_1, U_2 \in \mathfrak{X}(F)$, then it follows that

$$\mathrm{Ric}(Y, Z) + hH_B^{u_1}(Y, Z) = \lambda g(Y, Z).$$

Using Lemma 1.4 we have

$$\mathrm{Ric}_B(Y, Z) - \frac{m}{f} H^f(Y, Z) + hH_B^{u_1}(Y, Z) = \lambda g(Y, Z).$$

Then we obtain

$$\begin{aligned} &h(Y(Zu_1)) - h(\nabla_Y Z)u_1 - \frac{m}{f}(Y(Zf)) + \nabla_Y(Z(m\ln f)) - Z(Y(m\ln f)) \\ &+ \mathrm{Ric}_B(Y, Z) = \lambda g_B(Y, Z). \end{aligned}$$

Hence we get

$$Y(Z(hu_1 - m\ln f)) - (\nabla_Y Z)(hu_1 - m\ln f) + \mathrm{Ric}_B(Y, Z) = \lambda g_B(Y, Z).$$

It follows that

$$H_B^{\phi_1}(Y, Z) + \mathrm{Ric}_B(Y, Z) = \lambda g_B(Y, Z),$$

where $\phi_1 = hu_1 - m\ln f$, $h =$ constant and $u_1 = u$ at a fixed point on F. Hence we establish that (B, g, u, λ) is a gradient Ricci soliton. □

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