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Spectra of M-generalized corona of graphs constrained by vertex subsets*

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ABSTRACT: In this paper, we define a new type of corona operation which generalizes almost all the variants of corona of graphs defined in the literature. As particular cases of this construction, we define several variants of corona of graphs and some new unary graph operations. We determine the generalized characteristic polynomial of this constructed graph. Consequently, we derive the characteristic polynomials of the adjacency matrix and Laplacian matrix of the graphs constructed by the newly defined, and almost all the existing variants of corona of graphs. As applications of these results, we construct infinite families of integral graphs and cospectral graphs.

Key Words: Corona of graphs, Generalized characteristic polynomial, Adjacency spectrum, Laplacian spectrum.

Contents

1. Introduction

1.1. Basic definitions and notations

All the graphs considered in this paper are finite and simple. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. The *adjacency matrix of* G, denoted by $A(G) = [a_{ij}]$, is the $n \times n$ matrix defined as $a_{ij} = 1$, if $i \neq j$ and v_i and v_j are adjacent in G; 0, otherwise. The *vertex-edge incident matrix of* G is the $n \times m$ matrix, denoted by $B(G) = [b_{ij}]$, is defined as $b_{ij} = 1$, if v_i is incident with e_j; 0, otherwise. The *degree matrix of* G, denoted by $D(G)$, is the

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diagonal matrix $diag(d_1, d_2, \ldots, d_n)$, where d_i denotes the degree of v_i in G. The *Laplacian matrix of* G is $L(G) = D(G) - A(G)$ and the *signless Laplacian matrix of* G is $\mathcal{Q}(G) = D(G) + A(G)$. The characteristic polynomial of $A(G)$ (resp. $L(G)$ and $\mathcal{Q}(G)$) is denoted by $P_G(x)$ (resp. $L_G(x)$ and $\mathcal{Q}_G(x)$), and the multi set of eigenvalues of $A(G)$ (resp. $L(G)$ and $\mathcal{Q}(G)$) is said to be the *$A-$spectrum* (resp. *$L-$spectrum* and *$\mathcal{Q}-$spectrum) of* G. The $A-$spectrum of G is denoted by $\lambda_i(G)$ $(i = 1, 2, \ldots n)$ and $L-$spectrum of G is denoted by $\mu_i(G)$ $(i = 1, 2, \ldots n)$. We shall assume that $\lambda_1(G) \geq \lambda_2(G) \geq \ldots \geq \lambda_n(G)$, and $0 = \mu_1(G) \leq \mu_2(G) \leq \ldots \leq \mu_n(G)$.

Two graphs are said to be *$A-$cospectral* (resp. *$L-$cospectral and $\mathcal{Q}-$cospectral*) if they have same $A-$spectrum (resp. $L-$spectrum and $\mathcal{Q}-$spectrum). A graph G is said to be *$A-$integral (resp. $L-$integral, $\mathcal{Q}-$integral)* if all the eigenvalues of $A(G)$ (resp. $L(G)$, $\mathcal{Q}(G)$) are integers.

In [12], Cvetković et al. introduced the generalized characteristic polynomial $\phi_G(x, \beta)$ of G, which is defined as $\phi_G(x, \beta) = |xI - (A(G) - \beta D(G))|$. Notice that $P_G(x)$, $L_G(x)$ and $\mathcal{Q}_G(x)$ are equal to $\phi_G(x, 0)$, $(-1)^n \phi_G(-x, 1)$ and $\phi_G(x, -1)$, respectively.

The complete graph on n vertices is denoted by K_n and the complete bipartite graph whose vertex set partition having m and n vertices is denoted by $K_{m,n}$. The complement graph of G is denoted by $\overline{G}$. For a vertex v of G, the *neighbourhood of* v, denoted by $N_G(v)$, is the set of all vertices in G that are adjacent to v, and the *closed neighbourhood of* v is $N_G(v) \cup \{v\}$. $J_{n\times m}$ denotes the matrix of size $n \times m$ in which all the entries are 1. We denote $J_{n\times n}$ by J_n.

$M_n(\mathbb{R})$ denotes the set of all $n \times n$ real matrices. Let $\mathcal{R}_{n\times m}(s)$ denote the set of all $n \times m$ real matrices M such that the sum of the entries in each row of M are equal to s. Let $\mathcal{C}_{n\times m}(c)$ denote the set of all $n \times m$ real matrices M such that the sum of the entries in each column of M are equal to c. Let $\mathcal{RC}_{n\times m}(s, c)$ denote the set of all $n \times m$ matrices such that $M \in \mathcal{R}_{n\times m}(s)$ and $M \in \mathcal{C}_{n\times m}(c)$. Let $c_i(M)$ denote the sum of the entries in the $i-$th column of the matrix M. For a subset S of an ordered set $A = \{a_1, a_2, \ldots, a_n\}$, the *indicator vector of S (with respect to A)* is a vector $\mathbf{r}_S = (r_1, r_2, \ldots, r_n)$ in which $r_i = 1$ or 0, according as $a_i \in S$ or $a_i \notin S$. We denote by $R_S := diag(r_1, r_2, \ldots, r_n)$.

1.2. Spectra of graphs constructed by graph operations

The study of various spectra of graphs is an active research topic in spectral graph theory as several structural properties a graph can be explored through the knowledge of its spectra. Moreover, there are numerous applications of the study of spectra of graphs in various branches of science such as quantum physics, chemistry, computer science, etc. have been found (see [4,8,12,13,14]).

Due to this significance, finding the spectrum of a graph is an inevitable problem in spectral graph theory. In this direction, a natural question arise is "to what extent the spectrum of a given graph can be expressed in terms of the spectrum of some other graphs?". In this point of view, to construct graphs from the given graphs, several graph operations were defined in the literature such as the union, the complement, the subdivision, the Cartesian product, the Kronecker product, the NEPS, the corona, the rooted product, the edge-rooted product, the join, deletion of a vertex, insertion/deletion of an edge and the graph operations mentioned below. We refer the reader to [7,13,19,36,38,39,41] and the references therein for the results on the spectra of these graphs. These shows the necessity and importance of defining graph operations in spectral graph theoretic point of view.

Now, we recall the definition of some unary graph operations which are used in this paper. The *subdivision graph $S(G)$ of G* is the graph obtained by inserting a new vertex into every edge of G. The *$R-$graph $R(G)$ of G* is the graph obtained by taking one copy of $S(G)$ and joining two vertices of G if and only if they are adjacent in G. The *$Q-$graph $Q(G)$ of G* is the graph obtained by taking one copy of $S(G)$, and joining the new vertices which lie on the adjacent edges of G. The *total graph $T(G)$ of G* is the graph whose vertex set is $V(G) \cup E(G)$, with two vertices of $T(G)$ are adjacent if and only if the corresponding elements are adjacent or incident in G. The *duplication graph $Du(G)$ of G* is the graph obtained by taking new vertices corresponding to each vertex of G and joining the new vertex to the vertices in G which are adjacent to the corresponding vertex in G of the new vertex and deleting the edges of G [40]. The set of new vertices in $S(G)$, $R(G)$, $Q(G)$ and $T(G)$ are commonly denoted by $I(G)$.

1.2.1. Corona of graphs and its variants. The corona of graphs is another well-known graph operation that has received a lot of attention from researchers. In 1970, Frucht and Harary introduced this graph operation to construct a graph whose automorphism group is the wreath product of the automorphism group of their components [16].

Let G be a graph with n vertices and m edges. Let H and H_i $(i = 1, 2, \ldots, n$ (or $m))$ be graphs. The *corona of* G *and* H is obtained by taking one copy of G and n copies of H and joining the $i-$th vertex of G to all the vertices of the $i-$th copy of H for $i = 1, 2, \ldots, n$. In the same paper, the following variant of corona of graphs was defined. The *cluster of* G *and* H, denoted by $G\{H\}$ is the graph obtained by taking one copy of G and n copies of H and joining the $i-$th vertex of G to the root vertex of the $i-$th copy of H for $i = 1, 2, \ldots, n$. In 2007, Barik et al. [5] investigated the spectral properties of the corona of graphs. They determined the $A-$spectrum (resp. $L-$spectrum) of the corona of G and H for any graph G and a regular graph H (resp. for any graph G and H), in terms of the $A-$spectrum (resp. $L-$spectrum) of G and H. Since then several variants of the corona of graphs have been introduced and their spectra were studied by many researchers. The definitions of all the variants of corona of graphs defined in the literature are given below for the convenience of the reader.

In 2010, Y. Hou and W- C. Shiu [22] introduced the edge corona of graphs: The *edge corona of* G *and* H is the graph obtained by taking one copy of G and m copies of H and joining the end vertices of the $i-$th edge of G to all the vertices of the $i-$th copy of H for $i = 1, 2, \ldots, m$. In 2011, G. Indulal [23] defined the following variant of corona of graphs. The *neighbourhood corona of* G *and* H is the graph obtained by taking one copy of G and n copies of H and joining the vertices in the neighbourhood of the $i-$th vertex of G to all the vertices of the $i-$th copy of H for $i = 1, 2, \ldots, n$. In the same year, C. McLeman and Eris McNicholas [35] computed the $A-$spectrum of the corona of any pair of graphs using a new graph invariant called the coronal value.

In 2013, the following four variants of corona were introduced: the first two are due to X. Liu and P. Lu [25] and the rest are due to P.L. Lu and Y.F. Miao [30]. The *subdivision vertex corona of* G *and* H is the graph obtained by taking one copy of $S(G)$ and n copies of H and joining the $i-$th vertex of $V(G)$ to all the vertices of the $i-$th copy of H for $i = 1, 2, \ldots, n$. The *subdivision edge corona of* G *and* H is the graph obtained by taking one copy of $S(G)$ and m copies of H and joining the $i-$th vertex of $I(G)$ to all the vertices of the $i-$th copy of H for $i = 1, 2, \ldots, m$. The *subdivision vertex neighbourhood corona of* G *and* H is the graph obtained by taking one copy of $S(G)$ and n copies of H and joining the vertices in the neighbourhood of the $i-$th vertex of $V(G)$ to all the vertices of the $i-$th copy of H for $i = 1, 2, \ldots, n$. The *subdivision edge neighbourhood corona of* G *and* H is the graph obtained by taking one copy of $S(G)$ and m copies of H and joining the $i-$th vertex of $I(G)$ to all the vertices of the $i-$th copy of H for $i = 1, 2, \ldots, m$.

In 2014, P. L. Lu and Y. F. Miao [32] introduced the following two variants of corona of graphs. The *corona-vertex of subdivision graph of* G *and* H is the graph obtained by taking one copy of G and n copies of $S(H)$ and joining the $i-$th vertex of G to all the vertices of the $i-$th copy of $V(H)$ for $i = 1, 2, \ldots, n$. The *corona-edge of subdivision graph of* G *and* H is the graph obtained by taking one copy of G and n copies of $S(H)$ and joining the $i-$th vertex of G to all the vertices of the $i-$th copy of $I(H)$ for $i = 1, 2, \ldots, n$.

The variants of corona of graphs defined in 2015: In [24], J. Lan et al. defined four graphs using $R-$graphs namely, *the* $R-$*vertex corona, the* $R-$*edge corona, the* $R-$*vertex neighbourhood corona* and *the* $R-$*edge neighbourhood corona of* G *and* H in which they replaced the $S(G)$ by $R(G)$ in the definitions of the subdivision vertex corona, the subdivision edge corona, the subdivision vertex neighbourhood corona and the subdivision edge neighbourhood corona, respectively. C. Adiga et al. [1] defined the $C-$*graph* $C(G)$ (the $N-$*graph* $N(G)$) which is the corona of G and K_1 (the neighbourhood corona of G and K_1). By using these graphs, they defined the $C-$*vertex neighbourhood corona* and the $N-$*vertex corona of* G *and* H in which they replaced $S(G)$ by $C(G)$, $N(G)$, respectively in the definitions of the subdivision vertex corona and the subdivision vertex neighbourhood corona, respectively. Further, they defined the $C-$*edge corona* and the $N-$*edge corona of* G *and* H, in which they replaced $S(G)$ by $C(G)$, $N(G)$, respectively in the definition of the subdivision edge corona.

The variants of corona of graphs defined in 2016: X. Q. Zhu et al. defined the *total corona of* G *and* H [43], in which they replaced the subdivision graph of G by the total graph of G in the definition of the subdivision vertex corona. F. Laali et al. [15] defined the *generalized corona of graphs*, in which they replaced the graphs H by H_i's in the definition of the corona of G and H. S. Caixia et al. [9] defined the following graph. The *subdivision vertex-edge corona of graphs* G, H_1 and H_2 is the graph obtained by taking one copy of $S(G)$, n copies of H_1 and m copies of H_2 and joining the $i-$th vertex of $V(G)$ to all the vertices of H_1 and joining the $j-$th vertex of of $I(G)$ to all the vertices of H_2 for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. In [6] (see also [7]), S. Barik and G. Sahoo defined the *subdivision double corona of graphs*, which is same as the definition of the subdivision vertex-edge corona of graphs. Also, there in, they defined the following graphs. The *subdivision double neighbourhood corona of* G*,* H_1 *and* H_2 is the graph obtained by taking one copy of $S(G)$, n copies of H_1 and m copies of H_2 and joining the new vertices in the neighbourhood of the $i-$th vertex of of $V(G)$ to all the vertices of H_1 and joining vertices in the neighbourhood of the $j-$th vertex of $I(G)$ to all the vertices of H_2 for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. Similarly, they defined the $R-$*graph (*$Q-$*graph, total graph,* respectively) *double corona of* G*,* H_1 *and* H_2 and the $R-$*graph (*$Q-$*graph, total graph*, respectively) *double neighbourhood corona of* G*,* H_1 *and* H_2.

The variants of corona of graphs introduced in 2017: Y. Luo and W. Yan [34] defined the *generalized edge corona of graphs*, in which they replaced the m copies of H by $H_1, H_2, \ldots, H_m$ in the definition of edge corona of graphs. P. L. Lu and Y. M. Wu [33] defined the *generalized subdivision-vertex corona of graphs*, in which they replaced the m copies of H by $H_1, H_2, \ldots, H_m$ in the definition subdivision vertex corona of G and H. C. Adiga et al. [2] defined the following two variants of corona operations. The *extended neighborhood corona of* G *and* H is the graph obtained by taking the neighborhood corona of G and H and joining each vertex of the $i-$th copy of H to every vertex of $j-$th copy of H provided the $i-$th and the $j-$th vertices are adjacent in G. The *extended corona of* G *and* H is the graph obtained by taking the corona of G and H and joining each vertex of the $i-$th copy of H to every vertex of the $j-$th copy of H provided the $i-$th and the $j-$th vertices are adjacent in G.

The variants of corona of graphs introduced in 2018: P. Lu et al. [31] defined the *generalized subdivision-edge corona of graphs*, in which they replaced the m copies of H by $H_1, H_2, \ldots, H_m$ in the definition of subdivision edge corona of G and H. Q. Liu and Z. Zhang [28] also defined the generalized subdivision-vertex corona of graphs as in [31]. C. Adiga et al. [3] defined *the duplication vertex corona, the duplication edge corona of* G *and* H, the duplication vertex neighbourhood corona, in which they replaced $S(G)$ by $Du(G)$ in the definitions of the subdivision vertex corona, the subdivision edge corona, the subdivision vertex neighbourhood corona, respectively. W. Wen et al. [42] defined the following graphs. The *subdivision vertex-edge neighbourhood vertex-corona (short for SVEV- corona)* of G with H_1 and H_2 is the graph consisting of $S(G)$, $|V(G)|$ copies of H_1 and $|I(G)|$ copies of H_2, all vertex-disjoint, and joining the neighbours of the $i-$th vertex of $V(G)$ to every vertex in the $i-$th copy of H_1 and the $j-$th vertex of $I(G)$ to each vertex in the $j-$th copy of H_2 for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. The *subdivision vertex-edge neighbourhood edge-corona (short for SVEE- corona)* of G with H_1 and H_2 is the graph consisting of $S(G)$, $|V(G)|$ copies of H_1 and $|I(G)|$ copies of H_2 and joining the neighbours of the $i-$th vertex of $I(G)$ to every vertex in the $i-$th copy of H_1 and $j-$th vertex of $V(G)$ to each vertex in the $j-$th copy of H_2 for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Q. Liu [29] defined the the *generalized* $R-$*vertex (*resp. *edge) corona of graphs*, in which they replaced the m copies of H by $H_1, H_2, \ldots, H_n$ (resp. $H_1, H_2, \ldots, H_m$) in the definition of the $R-$vertex (resp. edge) corona of G and H.

In [37], the authors defined the generalized corona of graphs constrained by vertex subsets as follows: Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$. Let $\mathcal{H}_n = (H_1, H_2, \ldots, H_n)$ be a sequence of graphs. Let $\mathcal{T} = (T_1, T_2, \ldots, T_n)$ be a sequence of sets, where $T_i \subseteq V(H_i)$ for $i = 1, 2, \ldots, n$. Then the *generalized corona of* G *and* $\mathcal{H}_n$ *constrained by* $\mathcal{T}$ is the graph obtained by taking one copy of G, $H_1, H_2, \ldots, H_n$, and joining the vertex v_i to all the vertices in T_i for $i = 1, 2, \ldots, n$. In [17], the authors defined some more variants of the neighbourhood corona of graphs, namely, the subdivision (resp. $R-$graph, $Q-$graph, total) neighbourhood corona of graphs constrained by vertex subsets, the $R-$graph (resp. $Q-$graph, total) semi-edge neighbourhood corona of graphs constrained by vertex subsets, the $R-$graph (resp. total) semi-vertex neighbourhood corona of graphs constrained by vertex subsets. Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$. Let $\mathcal{H}_n = (H_1, H_2, \ldots, H_n)$ and $\mathcal{H}'_m = (H'_1, H'_2, \ldots, H'_m)$ be two sequences of graphs. Let $\mathcal{T} = (T_1, T_2, \ldots, T_n)$ and $\mathcal{T}' = (T'_1, T'_2, \ldots, T'_m)$ be two sequences of sets, where $T_i \subseteq V(H_i)$ and $T'_j \subseteq V(H'_j)$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. Let $I(G) = \{u_1, u_2, \ldots, u_m\}$. Then the *subdivision (resp.* $R-$*graph,* $Q-$*graph, total) neighbourhood corona of* G *with* $\mathcal{H}_n$ *and* $\mathcal{H}'_m$ *constrained by* $\mathcal{T}$ *and* $\mathcal{T}'$, simply we call it as *S-N* corona (resp. *R-N* corona, *Q-N* corona, T-N corona) of G with $\mathcal{H}_n$ and $\mathcal{H}'_m$ constrained by $\mathcal{T}$ and $\mathcal{T}'$, is defined as the graph obtained by taking one copy of $S(G)$ (resp. $R(G)$, $Q(G)$, $T(G)$), $H_1, H_2, \ldots, H_n$, $H'_1, H'_2, \ldots, H'_m$, and joining all the vertices in T_i to the vertices in $N_{S(G)}(v_i)$ (resp. $N_{R(G)}(v_i)$, $N_{Q(G)}(v_i)$, $N_{T(G)}(v_i)$), and joining all the vertices in T'_j to the vertices in $N_{S(G)}(u_j)$ (resp. $N_{R(G)}(u_j)$, $N_{Q(G)}(u_j)$, $N_{T(G)}(u_j)$), for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. The *$R-$graph (resp. $Q-$graph, total) semi-edge neighbourhood corona of G with $\mathcal{H}_n$ and $\mathcal{H}'_m$ constrained by $\mathcal{T}$ and $\mathcal{T}'$*, simply we call it as $R-$SEN corona (resp. $Q-$SEN corona, $T-$SEN corona), is defined as the graph obtained by taking one copy of $R(G)$ (resp. $Q(G)$, $T(G)$), $H_1, H_2, \ldots, H_n$, $H'_1, H'_2, \ldots, H'_m$, and joining all the vertices in T_i to the vertices in $N_{R(G)}(v_i) \cap I(G)$ (resp. $N_{Q(G)}(v_i) \cap I(G)$, $N_{T(G)}(v_i) \cap I(G)$), and joining all the vertices in T'_j to the end vertices of e_j for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. The *$R-$graph (resp. total) semi-vertex neighbourhood corona of G with $\mathcal{H}_n$ and $\mathcal{H}'_m$ constrained by $\mathcal{T}$ and $\mathcal{T}'$*, simply we call it as $R-$SVN corona (resp. $T-$SVN corona), is defined as the graph obtained by taking one copy of $R(G)$ (resp. $T(G)$), $H_1, H_2, \ldots, H_n$, $H'_1, H'_2, \ldots, H'_m$, and joining all the vertices in T_i to the vertices in $N_{R(G)}(v_i) \cap V(G)$ (resp. $N_{T(G)}(v_i) \cap V(G)$), $v_i \in V(G)$, and joining all the vertices in T'_j to the end vertices of e_j for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

1.3. Scope of the paper

In view of the variants of corona of graphs summarized above, it is natural to arise the following questions:

(1) Is it possible to define a new graph operation which includes all (or almost all) the corona operations mentioned above as particular cases?

(2) Is it possible to deduce various spectra of the variants of corona of graphs mentioned above from the determination of the corresponding spectra of newly defined graph operation?

While looking for the answer to the above questions, we arrive at the definition of a new graph operation, which we call "$M-$generalized corona of graphs constrained by vertex subsets". This is not only a generalization of the existing corona operations but also leads to the definition of many more variants of corona of graphs. This paper is a part of [18].

The rest of the paper is arranged as follows: In Section 2, we define the $M-$generalized corona of graphs constrained by vertex subsets. We show that this construction generalizes all the variants of corona of graphs mentioned above, except the extended neighbourhood corona and the extended corona. As particular cases of this construction, we get some more variants of corona of graphs and some new unary graph operations.

In Section 3, we obtain the generalized characteristic polynomial of the $M-$generalized corona of graphs constrained by vertex subsets expressed in terms of the characteristic polynomial of the matrices related to the constituent graphs and their coronals constrained by the corresponding vertex subsets. Moreover, we determine the characteristic polynomials of the adjacency and the Laplacian matrices of this graph and obtain the $A-$spectra and the $L-$spectra of the graphs defined in Section 2 as well as for almost all the variants of corona of graphs mentioned in Sub-subsection 1.2.1. Also, we determine the $A-$spectra and the $L-$spectra of some classes of $M-$generalized corona of graphs.

In Section 4, as applications of these results, we obtain some infinite family of $A-$integral ($L-$integral) graphs and $A-$cospectral ($L-$cospectral) graphs.

2. $M-$generalized corona of graphs constrained by vertex subsets

In this section, we define a new type of corona operation and show that this generalizes almost all the corona graph operations defined in the literature.

Definition 2.1 Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let $\mathcal{H}_k = (H_1, H_2, \ldots, H_k)$ be a sequence of k graphs and $\mathcal{T} = (T_1, T_2, \ldots, T_k)$ be a sequence of sets, where $T_j \subseteq V(H_j)$, for $j = 1, 2, \ldots, k$. Let $M = [m_{ij}]$ be a $0-1$ matrix of size $n \times k$. Then the *M-generalized corona of G and $\mathcal{H}_k$ constrained by $\mathcal{T}$*, denoted by $G \tilde{\circledast}_{[M:\mathcal{T}]} \mathcal{H}_k$, is the graph obtained by taking one copy of G, $H_1, H_2, \ldots, H_k$ and joining the vertex v_i to all the vertices in T_j if and only if $m_{ij} = 1$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k$. We call the matrix M as *the corona generating matrix of the graph $G \tilde{\circledast}_{[M:\mathcal{T}]} \mathcal{H}_k$*.

We denote the graph $G\tilde{\circledast}_{[M:\mathcal{T}]}\mathcal{H}_k$ simply as

- $G\tilde{\circledast}_M\mathcal{H}_k$, if $T_j = V(H_j)$ for $j = 1, 2, \ldots, k$ and call it as *the $M-$generalized corona of G and $\mathcal{H}_k$*;
- $G\tilde{\circledast}_{[M:\mathcal{T}]}H$, if $H_j = H$ for $j = 1, 2, \ldots, k$ and call it as *the $M-$generalized corona of G and H constrained by $\mathcal{T}$*;
- $G\tilde{\circledast}_M H$, if $H_j = H$ and $T_j = V(H_j)$ for $j = 1, 2, \ldots, k$ and call it as *the $M-$generalized corona of G and H*.

Example 2.1 Consider the graphs G, H_1, H_2, H_3, H_4 as shown in Figure 1(a), 1(b), 1(c), 1(d) and 1(e), respectively. Let $\mathcal{H}_4 = (H_1, H_2, H_3, H_4)$ and $\mathcal{T} = (T_1, T_2, T_3, T_4)$, where $T_1 = \{u_3, u_4\}$, $T_2 = \{v_1, v_2, v_3\}$, $T_3 = \{w_1, w_6, w_8\}$, $T_4 = \{t_2\}$ and

$$M = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The vertices colored with yellow represent the elements in T_j $(j = 1, 2, 3, 4)$. Then the graph $G\tilde{\circledast}_{[M:\mathcal{T}]}\mathcal{H}_4$ is as shown in Figure 1(f).

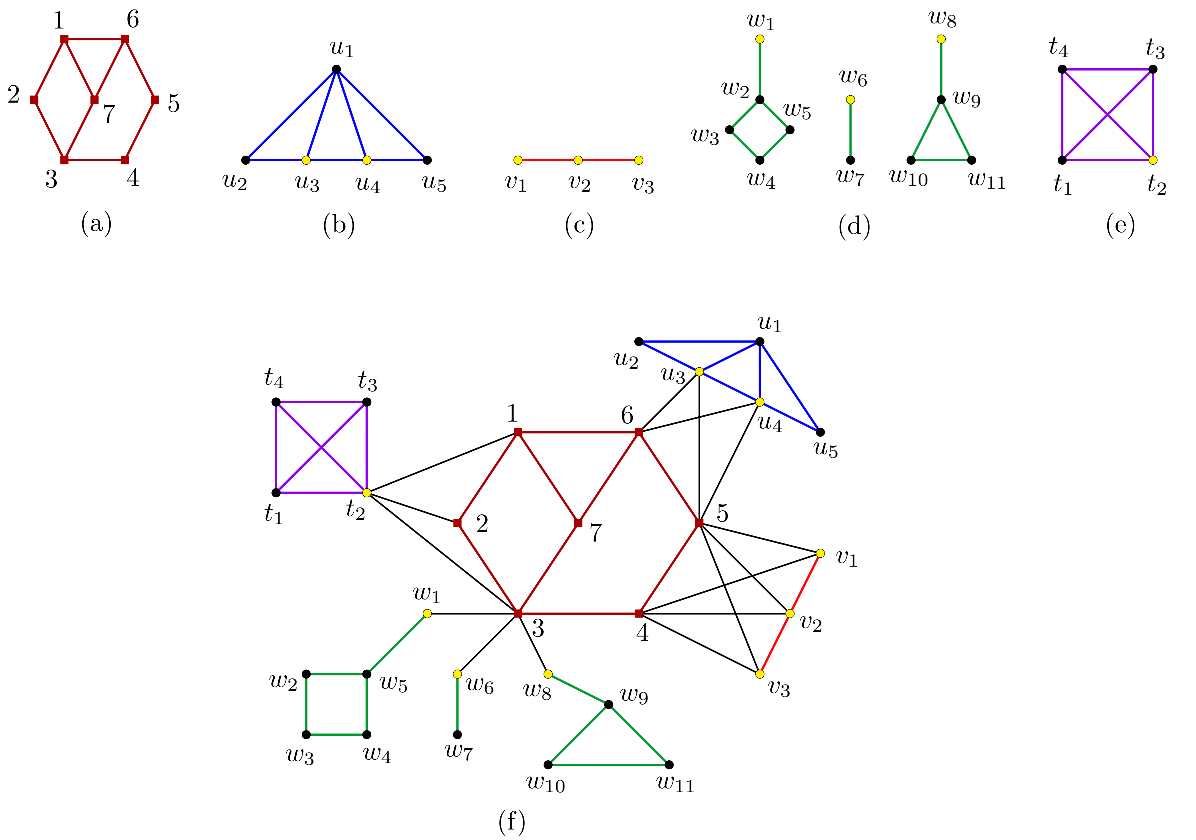

Figure 1: The graphs (a) G, (b) H_1, (c) H_2, (d) H_3, (e) H_4 and (f) $G\tilde{\circledast}_{[M:\mathcal{T}]}\mathcal{H}_4$

Example 2.2 Any bipartite graph can be viewed as a $M-$generalized corona of some graphs. For if G is a bipartite graph with bipartition (X, Y) with $|X| = m$ and $|Y| = n$. Then its adjacency matrix is of the form

$$A(G) = \begin{bmatrix} \mathbf{0} & W \\ W^T & \mathbf{0} \end{bmatrix},$$

where W is a $0-1$ matrix of size $m \times n$. Then G is the same as the graph $\overline{K}_m \tilde{\circledast}_W K_1$.

In Table 1, we show that the definitions of all the variants of corona of graphs mentioned in Subsection 1.2.1, except the extended corona and the extended neighbourhood corona of G and H are particular cases of Definition 2.1 by suitably taking k, the graphs G, H_j, the subsets T_j and the matrix M.

Table 1: Variants of corona of graphs defined in the literature as particular cases of the $M-$generalized corona of G and $\mathcal{H}_k$ constrained by $\mathcal{T}$

S. No	**Name of the corona operation**	$\boldsymbol{G}$	$\boldsymbol{k}$	$\boldsymbol{H_j}$	$\boldsymbol{T_j}$	$\boldsymbol{M}$
1.	Corona of G and H	G	n	H	$V(H)$	I_n
2.	Generalized corona of G and H_1, H_2, $\ldots$, H_n	G	n	H_j	$V(H_j)$	I_n
3.	Cluster of G and H	G	n	H	{the root vertex of H}	I_n
4.	Edge corona of G and H	G	m	H	$V(H)$	$B(G)$
5.	Generalized edge corona of G and $H_1, H_2, \ldots, H_n$	G	m	H_j	$V(H_j)$	$B(G)$
6.	Neighbourhood corona of G and H	G	n	H	$V(H)$	$A(G)$
7.	Subdivision vertex corona of G and H	$S(G)$	n	H	$V(H)$	$\begin{bmatrix} I_n \\ \mathbf{0} \end{bmatrix}$
8.	Generalized subdivision vertex corona of G and $H_1, H_2, \ldots, H_n$	$S(G)$	n	H_j	$V(H_j)$	$\begin{bmatrix} I_n \\ \mathbf{0} \end{bmatrix}$
9.	Subdivision edge corona of G and H	$S(G)$	m	H	$V(H)$	$\begin{bmatrix} \mathbf{0} \\ I_m \end{bmatrix}$
10.	Generalized subdivision edge corona of G and $H_1, H_2, \ldots, H_n$	$S(G)$	m	H_j	$V(H_j)$	$\begin{bmatrix} \mathbf{0} \\ I_m \end{bmatrix}$
11.	Subdivision vertex neighbourhood corona of G and H	$S(G)$	n	H	$V(H)$	$\begin{bmatrix} \mathbf{0} \\ B(G)^T \end{bmatrix}$
12.	Subdivision edge neighbourhood corona of G and H	$S(G)$	m	H	$V(H)$	$\begin{bmatrix} B(G) \\ \mathbf{0} \end{bmatrix}$
13.	$R-$vertex corona of G and H	$R(G)$	n	H	$V(H)$	$\begin{bmatrix} I_n \\ \mathbf{0} \end{bmatrix}$

14.	Generalized $R-$vertex corona of G and $H_1, H_2, \ldots, H_n$	$R(G)$	n	H_j	$V(H_j)$	$\begin{bmatrix} I_n \\ \mathbf{0} \end{bmatrix}$
15.	$R-$edge corona of G and H	$R(G)$	m	H	$V(H)$	$\begin{bmatrix} \mathbf{0} \\ I_m \end{bmatrix}$
16.	Generalized $R-$edge corona of G and $H_1, H_2, \ldots, H_n$	$R(G)$	m	H_j	$V(H_j)$	$\begin{bmatrix} \mathbf{0} \\ I_m \end{bmatrix}$
17.	$R-$vertex neighbourhood corona of G and H	$R(G)$	n	H	$V(H)$	$\begin{bmatrix} A(G) \\ B(G)^T \end{bmatrix}$
18.	$R-$edge neighbourhood corona of G and H	$R(G)$	m	H	$V(H)$	$\begin{bmatrix} B(G) \\ \mathbf{0} \end{bmatrix}$
19.	$N-$vertex corona of G and H	$N(G)$	n	H	$V(H)$	$\begin{bmatrix} I_n \\ \mathbf{0} \end{bmatrix}$
20.	$C-$vertex neighbourhood corona of G and H	$C(G)$	n	H	$V(H)$	$\begin{bmatrix} A(G) \\ \mathbf{0} \end{bmatrix}$
21.	$N-$edge corona of G and H	$N(G)$	n	H	$V(H)$	$\begin{bmatrix} B(G) \\ \mathbf{0} \end{bmatrix}$
22.	$C-$edge corona of G and H	$C(G)$	n	H	$V(H)$	$\begin{bmatrix} B(G) \\ \mathbf{0} \end{bmatrix}$
23.	Total corona of G and H	$T(G)$	n	H	$V(H)$	$\begin{bmatrix} I_n \\ \mathbf{0} \end{bmatrix}$
24.	Duplication vertex corona of G and H	$Du(G)$	n	H	$V(H)$	$\begin{bmatrix} I_n \\ \mathbf{0} \end{bmatrix}$
25.	Duplication vertex neighbourhood corona of G and H	$Du(G)$	n	H	$V(H)$	$\begin{bmatrix} A(G) \\ \mathbf{0} \end{bmatrix}$
26.	Duplication vertex edge corona of G and H	$Du(G)$	n	H	$V(H)$	$\begin{bmatrix} B(G) \\ \mathbf{0} \end{bmatrix}$
27.	Corona-vertex of subdivision graph of G and H	G	n	$S(H)$	$V(H)$	I_n
28.	Corona-edge of subdivision graph of G and H	G	n	$S(H)$	$I(H)$	I_n
29.	Subdivision (resp. $R-$graph, $Q-$graph, total) double corona of G and H_1, H_2	$S(G)$ (resp. $R(G)$, $Q(G)$, $T(G)$)	$n+m$	H_1 for $j = 1, 2, \ldots, n$; H_2 for $j = n+1, \ldots, n+m$.	$V(H_j)$	I_{n+m}
30.	Subdivision (resp. $R-$graph, $Q-$graph, total) double neighbourhood corona of G and H_1, H_2	$S(G)$ (resp. $R(G)$, $Q(G)$, $T(G)$)	$n+m$	H_1 for $j = 1, 2, \ldots, n$; H_2 for $j = n+1, \ldots, n+m$.	$V(H_j)$	$A(S(G))$

31.	SVEV corona of G with H_1 and H_2	$S(G)$	$n+m$	H_1 for $j=1,2,\ldots,n$; H_2 for $j=n+1,\ldots,n+m$.	$V(H_j)$	$\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ B(G)^T & I_m \end{bmatrix}$
32.	SVEE corona of G with H_1 and H_2	$S(G)$	$n+m$	H_2 for $j=1,2,\ldots,n$; H_1 for $j=n+1,\ldots,n+m$.	$V(H_j)$	$\begin{bmatrix} I_n & B(G) \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$
33.	Generalized corona of G and $\mathcal{H}_n$ constrained by $\mathcal{T}$	G	n	H_j	T_j	I_n
34.	S-N corona (resp. R-N corona, Q-N corona, T-N corona) of G with $\mathcal{H}_n$ and $\mathcal{H}'_m$ constrained by $\mathcal{T}$ and $\mathcal{T}'$	$S(G)$ (resp. $R(G)$, $Q(G)$, $T(G)$)	$n+m$	H_i for $i=1,2,\ldots,n$; H'_s for $s=1,2,\ldots,m$.	T_j	$A(S(G))$ (resp. $A(R(G))$, $A(Q(G))$, $A(T(G))$)
35.	$R-$SEN corona (resp. $Q-$SEN corona, $T-$SEN corona) of G with $\mathcal{H}_n$ and $\mathcal{H}'_m$ constrained by $\mathcal{T}$ and $\mathcal{T}'$	$R(G)$ (resp. $Q(G)$, $T(G)$)	$n+m$	H_i for $i=1,2,\ldots,n$; H'_s for $s=1,2,\ldots,m$.	T_j	$A(S(G))$
36.	$R-$SVN corona (resp. $T-$SVN corona) of G with $\mathcal{H}_n$ and $\mathcal{H}'_m$ constrained by $\mathcal{T}$ and $\mathcal{T}'$	$R(G)$ (resp. $T(G)$)	$n+m$	H_i for $i=1,2,\ldots,n$; H'_s for $s=1,2,\ldots,m$.	T_j	$\begin{bmatrix} A(G) & B(G) \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$

2.1. Some more new variants of corona of graphs

As particular cases of Definition 2.1, we get some more variants of corona of graphs, which are described in Table 2. In addition to the assumptions in Definition 2.1, we assume that $m = |E(G)|$ and $m' = |E(\overline{G})|$.

Table 2: Some new variants of corona of graphs obtained as particular cases of the $M-$generalized corona of G and $\mathcal{H}_k$ constrained by $\mathcal{T}$

S. No	G	k	M	Description for M	Name of the corona operation
1.	G	n	$A(G)$	joining every vertex in the neighborhood of the vertex v_i of G to every vertex in T_i for $i=1,2,\ldots,n$	Generalized neighbourhood corona of G and $\mathcal{H}_n$ constrained by $\mathcal{T}$
2.	G	n	$I_n + A(G)$	joining every vertex in the closed neighborhood of the vertex v_i of G to every vertex in T_i for $i=1,2,\ldots,n$	Closed neighbourhood corona of G and $\mathcal{H}_n$ constrained by $\mathcal{T}$
3.	G	n	$J_n - I_n$	joining every vertex of G other than the vertex v_i to every vertex in T_i for $i = 1,2,\ldots,n$	Vertex complemented corona of G and $\mathcal{H}_n$ constrained by $\mathcal{T}$
4.	G	n	$J_n - A(G)$	joining every vertex in the neighborhood of the vertex v_i in $\overline{G}$ to every vertex in T_i for $i=1,2,\ldots,n$	Neighbourhood complemented corona of G and $\mathcal{H}_n$ constrained by $\mathcal{T}$

5.	G	n	$J_n - I_n - A(G)$	joining every vertex in the closed neighborhood of the vertex v_i in $\overline{G}$ to every vertex in T_i for $i = 1, 2, \ldots, n$	Closed neighbourhood complemented corona of G and $\mathcal{H}_n$ constrained by $\mathcal{T}$
6.	G	m	$B(G)$	joining two end vertices of the edge e_i of G to every vertex in T_i for $i = 1, 2, \ldots, m$	Generalized edge corona of G and $\mathcal{H}_m$ constrained by $\mathcal{T}$ of G and $\mathcal{H}_m$ constrained by $\mathcal{T}$
7.	G	m	$J_{n\times m} - B(G)$	joining the vertices of G other than the two end vertices of the edge e_i of G to every vertex in T_i for $i = 1, 2, \ldots, m$	Edge complemented corona of G and $\mathcal{H}_m$ constrained by $\mathcal{T}$
8.	G	m'	$B(\overline{G})$	joining two end vertices of the edge e_i of $\overline{G}$ to every vertex in T_i for $i = 1, 2, \ldots, m'$	Nonadjacent vertices corona of G and $\mathcal{H}_{m'}$ constrained by $\mathcal{T}$
9.	G	m'	$J_{n\times m'} - B(\overline{G})$	joining the vertices of G other than the two end vertices of the edge e_i of $\overline{G}$ to every vertex in T_i for $i = 1, 2, \ldots, m'$	Nonadjacent vertices complemented corona of G and $\mathcal{H}_{m'}$ constrained by $\mathcal{T}$
10.	$\overline{K}_n$	n	$A(G)$	joining every vertex in the neighborhood of the vertex v_i of G to every vertex in T_i for $i = 1, 2, \ldots, n$ and deleting all the edges in G	Duplicate neighbourhood corona of G and $\mathcal{H}_n$ constrained by $\mathcal{T}$
11.	$\overline{K}_n$	n	$A(G) + I_n$	joining every vertex in the closed neighborhood of the vertex v_i of G to every vertex in T_i for $i = 1, 2, \ldots, n$ and deleting all the edges in G	Duplicate closed neighbourhood corona of G and $\mathcal{H}_n$ constrained by $\mathcal{T}$
12.	$\overline{K}_n$	m	$J_{n\times m} - B(G)$	joining two end vertices of the edge e_i of G to every vertex in T_i for $i = 1, 2, \ldots, m$ and deleting all the edges in G	Duplicate edge corona of G and $\mathcal{H}_m$ constrained by $\mathcal{T}$
13.	$\overline{K}_n$	m	$J_{n\times m} - B(G)$	joining the vertices of G other than the two end vertices of the edge e_i of G to every vertex in T_i for $i = 1, 2, \ldots, m$ and deleting all the edges in G	Duplicate edge complemented corona of G and $\mathcal{H}_m$ constrained by $\mathcal{T}$

Taking all H_js as K_1 in the definitions of corona operations of graphs given in Table 2, we get some existing unary graph operations such as subdivision graph, $R-$graph, duplicate graph, $C-$graph and $N-$graph, and some new unary graph operations as mentioned in Table 3.

Table 3: Some (existing and new) unary graph operations defined using the $M-$generalized corona operation

S. No	**Definition**	**Name of the graph**	**Notation**
1.	Corona of G and K_1	$C-$graph of G	$C(G)$
2.	Neighbourhood corona of G and K_1	$N-$graph of G	$N(G)$
3.	Closed neighbourhood corona of G and K_1	$\overline{N}-$graph of G	$\overline{N}(G)$
4.	Vertex complemented corona of G and K_1	$VC-$graph of G	$VC(G)$

5.	Neighbourhood complemented corona of G and K_1	$NC-$graph of G	$NC(G)$
6.	Closed neighbourhood complemented corona of G and K_1	$\overline{N}C-$graph of G	$\overline{N}C(G)$
7.	Duplicate neighbourhood corona of G and K_1	Duplicate graph of G	$Du(G)$
8.	Duplicate closed neighbourhood corona of G and K_1	$D\overline{N}-$graph of G	$D\overline{N}(G)$
9.	Edge corona of G and K_1	$R-$graph of G	$R(G)$
10.	Edge complemented corona of G and K_1	$EC-$graph of G	$EC(G)$
11.	Duplicate edge corona of G and K_1	Subdivision graph of G	$S(G)$
12.	Duplicate edge complemented corona of G and K_1	$DEC-$graph of G	$DEC(G)$
13.	Non-adjacent vertices corona of G and K_1	$NV-$graph of G	$NV(G)$
14.	Non-adjacent vertices complemented corona of G and K_1	$NVC-$graph of G	$NVC(G)$

3. Spectra of $M-$generalized corona of graphs constrained by vertex subsets

In this section, we compute the generalized characteristic polynomial of the $M-$generalized corona of graphs G and $\mathcal{H}_k$ constrained by vertex subsets $\mathcal{T}$. By using this, we obtain the adjacency and the Laplacian spectrum of the graph $G\tilde{\circledast}_{[M:\mathcal{T}]}\mathcal{H}_k$, for some special graphs G, H_is and some special subsets T_is. Also, we obtain the adjacency and the Laplacian spectrum of the variants of corona of graphs defined in Section 2.

First, we give some additional definitions, notations, and results that are used in the rest of this paper.

C. McLeman and E. McNicholas introduced the coronal of a graph as follows:

Definition 3.1 *([35])* Let H be a graph with n vertices. The *coronal* $\Gamma_H(x)$ *of* H is defined as the sum of the entries of the matrix $(xI - A(H))^{-1}$. Note that this can be calculated as

$$\Gamma_H(x) = J_{1\times n}(xI_n - A(H))^{-1}J_{n\times 1}.$$

S.Y. Cui and G. X. Tian generalized the preceding definition as follows:

Definition 3.2 *([10])* Let G be a graph with n vertices and M be a graph matrix of G. The $M-$*coronal of* G, denoted by $\Gamma_M(x)$, is defined as the sum of the entries of the matrix $(xI_n - M)^{-1}$, that is

$$\Gamma_M(x) = J_{1\times n}(xI_n - M)^{-1}J_{n\times 1}.$$

The authors introduced the coronal of a matrix constrained by an index set as follows:

Definition 3.3 *([37])* Let $M \in M_n(\mathbb{R})$ and $\alpha \subseteq \{1, 2, \dots, n\}$ be an index set. Then *the coronal of* M *constrained by* α, denoted by $\Gamma^{\alpha}_M(x)$, is defined as the sum of all entries in the the principal submatrix formed by α. This can be calculated by

$$\Gamma^{\alpha}_M(x) = \mathbf{r}_{\alpha}(xI_n - M)^{-1}\mathbf{r}^T_{\alpha}.$$

Remark 3.1 (1) If $\alpha = \{1, 2, \dots, n\}$, then we denote $\Gamma^{\alpha}_M(x)$ simply by $\Gamma_M(x)$ and we call this simply as the *coronal of* M. Notice that $\Gamma_M(x) = J_{n\times 1}(xI_n - M)^{-1}J_{1\times n}$.

(2) If H is a graph with $V(H) = \{u_1, u_2, \dots, u_n\}$ and $T \subseteq V(H)$, then we simply denote $\Gamma^T_{A(H)}(x)$ by $\Gamma^T_H(x)$.

The following observations will be used later without mentioning them explicitly.

Observation 3.1 $\Gamma^{\alpha}_{-M}(x) = -\Gamma^{\alpha}_M(-x)$ *and* $\Gamma^{\alpha}_{M+cI_n}(x) = \Gamma^{\alpha}_M(x - c)$.

The following results are used in the subsequent sections.

Proposition 3.1 *([10, proposition 2]) If $M \in \mathcal{R}_{n\times n}(t)$, then $\Gamma_M(x) = \dfrac{n}{x-t}$.*

Theorem 3.1 *([4]) Let A be an $n \times n$ matrix partitioned as*

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix},$$

where A_1, A_4 are square matrices. If A_1, A_4 are invertible, then

$$|A| = |A_4||A_1 - A_2A_4^{-1}A_3| = |A_1||A_4 - A_3A_1^{-1}A_2|.$$

Assumptions 3.1 In the rest of this paper, we assume the following, unless we specifically mention otherwise: G is a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. $\mathcal{H}_k = (H_1, H_2, \ldots, H_k)$ is a sequence of k graphs with $V(H_j) = \{u_{j1}, u_{j2}, \ldots, u_{jn_j}\}$. $\mathcal{T} = (T_1, T_2, \ldots, T_k)$ is a sequence of sets, where $T_j \subseteq V(H_j)$ with $|T_j| = t_j$, and $\mathbf{r_j}$ is the indicator vector of T_j for $j = 1, 2, \ldots, k$. $M = [m_{ij}]$ is a $0-1$ matrix of size $n \times k$.

In the following, we obtain the generalized characteristic polynomial of $G\tilde{\circledast}_{[M:\mathcal{T}]}\mathcal{H}_k$, which is one of the main result of this paper.

Theorem 3.2 *The generalized characteristic polynomial of $G\tilde{\circledast}_{[M:\mathcal{T}]}\mathcal{H}_k$ is*

$$\phi_{G\tilde{\circledast}_{[M:\mathcal{T}]}\mathcal{H}_k}(x,\beta) = \left\{ \prod_{j=1}^{k} \left| xI_{n_j} - \mathbb{H}_j \right| \right\} \times \left| xI_n - A(G) + \beta D(G) + \beta N - MUM^T \right|,$$

where $\mathbb{H}_j = A(H_j) - \beta D(H_j) - \beta c_j(M) R_{T_j}$, $j = 1, 2, \ldots, k$,

$$N = diag\left(\sum_{j=1}^{k} m_{1j}t_j, \sum_{j=1}^{k} m_{2j}t_j, \ldots, \sum_{j=1}^{k} m_{nj}t_j \right),$$

and

$$U = \begin{bmatrix} \Gamma_{\mathbb{H}_1}^{T_1}(x) & 0 & \cdots & 0 \\ 0 & \Gamma_{\mathbb{H}_2}^{T_2}(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Gamma_{\mathbb{H}_k}^{T_k}(x) \end{bmatrix}.$$

Proof: We arrange the rows and columns of the adjacency matrix of $G\tilde{\circledast}_{[M:\mathcal{T}]}\mathcal{H}_k$ by the vertices of $G, H_1, H_2, \ldots, H_k$, respectively. Let $p = \sum_{i=1}^{k} n_i$. Then the adjacency matrix of $G\tilde{\circledast}_{[M:\mathcal{T}]}\mathcal{H}_k$ is

$$A(G\tilde{\circledast}_{[M:\mathcal{T}]}\mathcal{H}_k) = \begin{bmatrix} A(G) & C \\ C^T & E \end{bmatrix},$$

where

$$E = \begin{bmatrix} A(H_1) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & A(H_2) & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & A(H_k) \end{bmatrix}_{p\times p},$$

$$C = \begin{bmatrix} m_{11}\mathbf{r_1} & m_{12}\mathbf{r_2} & \cdots & m_{1k}\mathbf{r_k} \\ m_{21}\mathbf{r_1} & m_{22}\mathbf{r_2} & \cdots & m_{2k}\mathbf{r_k} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1}\mathbf{r_1} & m_{n2}\mathbf{r_2} & \cdots & m_{nk}\mathbf{r_k} \end{bmatrix}_{n\times p} = M \begin{bmatrix} \mathbf{r_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{r_2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{r_k} \end{bmatrix}.$$

Also,

$$D(G\tilde{\circledast}_{[M:\mathcal{T}]}\mathcal{H}_k) = \begin{bmatrix} D(G)+N & \mathbf{0} \\ \mathbf{0} & E' \end{bmatrix},$$

where

$$E' = \begin{bmatrix} D(H_1)+c_1(M)R_{T_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & D(H_2)+c_2(M)R_{T_2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & D(H_k)+c_k(M)R_{T_k} \end{bmatrix}_{p\times p}.$$

Then by using Theorem 3.1, we have

$$\begin{aligned}\Phi_{G\tilde{\circledast}_{[M:\mathcal{T}]}\mathcal{H}_k}(x,\beta) &= \begin{vmatrix} xI_n - A(G)+\beta D(G)+\beta N & -C \\ -C^T & xI_p - E+\beta E' \end{vmatrix} \\ &= |xI_p - E+\beta E'| \times \left|xI_n - A(G)+\beta D(G)+\beta N - C(xI_p-E+\beta E')^{-1}C^T\right|. \quad (3.1)\end{aligned}$$

It is not hard to see that

$$|xI_p - E+\beta E'| = \prod_{j=1}^{k}\left|xI_{n_j} - A(H_j)+\beta D(H_j)+\beta c_j(M)R_{T_j}\right|.$$

Also,

$$\begin{aligned}C(xI_p-E+\beta E')^{-1}C^T &= M\begin{bmatrix} \mathbf{r_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{r_2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{r_k} \end{bmatrix}(xI_p-E+\beta E')^{-1}\begin{bmatrix} \mathbf{r_1}^T & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{r_2}^T & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{r_k}^T \end{bmatrix}M^T \\ &= M\begin{bmatrix} \Gamma_{\mathbb{H}_1}^{T_1}(x) & 0 & \cdots & 0 \\ 0 & \Gamma_{\mathbb{H}_2}^{T_2}(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Gamma_{\mathbb{H}_k}^{T_k}(x) \end{bmatrix}M^T.\end{aligned}$$

Substituting these values in (3.1) we get the result. □

3.1. A–spectrum and L–spectrum of $M-$generalized corona of graphs constrained by vertex subsets

Theorem 3.3 *(1) The characteristic polynomial of the adjacency matrix of* $G\tilde{\circledast}_{[M:\mathcal{T}]}\mathcal{H}_k$ *is*

$$\left\{\prod_{j=1}^{k} P_{H_j}(x)\right\} \times \left|xI_n - A(G) - MU_1M^T\right|,$$

(2) The characteristic polynomial of the Laplacian matrix of $G\tilde{\circledast}_{[M:\mathcal{T}]}\mathcal{H}_k$ *is*

$$\left\{\prod_{j=1}^{k}\left|xI_{n_j} - L(H_j) - c_j(M)R_{T_j}\right|\right\} \times \left|xI_n - L(G) - N - MU_2M^T\right|,$$

where N *is as in Theorem 3.2,*

$$U_1 = \begin{bmatrix} \Gamma_{H_1}^{T_1}(x) & 0 & \cdots & 0 \\ 0 & \Gamma_{H_2}^{T_2}(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Gamma_{H_k}^{T_k}(x) \end{bmatrix},$$

and

$$U_2 = \begin{bmatrix} \Gamma^{T_1}_{L(H_1)+c_1(M)R_{T_1}}(x) & 0 & \cdots & 0 \\ 0 & \Gamma^{T_2}_{L(H_2)+c_2(M)R_{T_2}}(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Gamma^{T_k}_{L(H_k)+c_k(M)R_{T_k}}(x) \end{bmatrix}.$$

Proof:

(1) Taking $\beta = 0$ in Theorem 3.2, we obtain the result. Notice that here $\mathbb{H}_j = A(H_j)$ and so $\Gamma^{T_j}_{\mathbb{H}_j}(x) = \Gamma^{T_j}_{H_j}(x)$ for $j = 1, 2, \ldots, k$.

(2) For $\beta = 1$, $\mathbb{H}_j = -(L(H_j) + c_j(M)R_{T_j})$, $j = 1, 2, \ldots, k$. By applying these in Theorem 3.2, we get,

$$\begin{aligned} L_{G \tilde{\circledast}_{[M:\mathcal{T}]} \mathcal{H}_k}(x) &= (-1)^{n+p} \left\{ \prod_{j=1}^{k} \left| -xI_{n_j} + L(H_j) + c_j(M)R_{T_j} \right| \right\} \\ & \quad \times \left| -xI_n + L(G) + N - MUM^T \right| \\ &= \left\{ \prod_{j=1}^{k} \left| xI_{n_j} - L(H_j) - c_j(M)R_{T_j} \right| \right\} \times \left| xI_n - L(G) - N + MUM^T \right|, \end{aligned}$$

where $p = \sum_{j=1}^{k} n_j$ and $U = diag\left(\Gamma^{T_1}_{\mathbb{H}_1}(-x), \Gamma^{T_2}_{\mathbb{H}_2}(-x), \ldots, \Gamma^{T_k}_{\mathbb{H}_k}(-x)\right)$.

Since, $\Gamma^{T_j}_{\mathbb{H}_j}(-x) = -\Gamma^{T_1}_{L(H_j)+c_j(M)R_{T_j}}(x)$, for $j = 1, 2, \ldots, k$, we have

$$U = -diag\left(\Gamma^{T_1}_{L(H_1)+c_1(M)R_{T_1}}(x), \ldots, \Gamma^{T_k}_{L(H_k)+c_k(M)R_{T_k}}(x)\right) = -U_2.$$

So the proof follows.

□

To express the characteristic polynomial of the adjacency matrix and the Laplacian matrix of the $M-$generalized corona of G and $\mathcal{H}_k$ constrained by $\mathcal{T}$ more explicitly for some special corona generating matrices M, we define and formulate the following:

Definition 3.4 Let $A_1, A_2, \ldots, A_m \in M_n(\mathbb{R})$. Then $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{R}$ are said to be *co-eigenvalues of* $A_1, A_2, \ldots, A_m$, if there exists a vector $X \in \mathbb{R}^n$ such that $A_i X = \lambda_i X$ for $i = 1, 2, \ldots, m$.

The following are some easy observations that will be used later.

Observation 3.2 (1) If $A_1, A_2 \in M_n(\mathbb{R})$, then for each eigenvalue λ_1 of A_1, there need not exist an eigenvalue λ_2 such that λ_1, λ_2 are co-eigenvalues of A_1, A_2.

(2) If $A_1, A_2, \ldots, A_m \in M_n(\mathbb{R})$ are symmetric and commutes with each other, then for each eigenvalue λ_1 of A_1, [21, Proposition 2.3.2] ensures the existence of $\lambda_2, \lambda_3, \ldots, \lambda_m$ such that $\lambda_1, \lambda_2, \ldots, \lambda_m$ are co-eigenvalues of $A_1, A_2, \ldots, A_m$.

(3) If λ is an eigenvalue of $A \in M_n(\mathbb{R})$, then $\lambda, 1$ are co-eigenvalues of A, I_n.

(4) Let $A \in M_n(\mathbb{R})$ and $f(x) \in \mathbb{R}[x]$. If λ is an eigenvalue of A, then $\lambda, f(\lambda)$ are co-eigenvalues of A, $f(A)$.

(5) If G is an $r-$regular graph with n vertices, then $\lambda_i(G), \mu_i(G)$ are co-eigenvalues of $A(G)$, $L(G)$, where $\mu_i(G) = r - \lambda_i(G)$, for each $i = 1, 2, \ldots, n$.

(6) If $f(x), g(x) \in \mathbb{R}[x]$ and λ_1, λ_2 are co-eigenvalues of $A_1, A_2 \in M_n(\mathbb{R})$, then $f(\lambda_1), g(\lambda_2)$ are co-eigenvalues of $f(A_1), g(A_2)$. In particular, if G is an $r-$regular graph, $M \in M_n(\mathbb{R})$ and $\lambda(G), \lambda(M)$ are co-eigenvalues of $A(G)$, M, then $r - \lambda(G), \lambda(M)$ are co-eigenvalues of $L(G)$, M.

(7) If $f(x), g(x) \in \mathbb{R}[x]$ and λ_1, λ_2 are co-eigenvalues of $A_1, A_2 \in M_n(\mathbb{R})$, then $\lambda_1, f(\lambda_1) + g(\lambda_2)$ are co-eigenvalues of $A_1, f(A_1) + g(A_2)$.

(8) If λ_1, λ_2 are co-eigenvalues of $A_1, A_2 \in M_n(\mathbb{R})$, then $\lambda_1 + \lambda_2$ is an eigenvalue of $A_1 + A_2$; $\lambda_1\lambda_2$ is an eigenvalue of A_1A_2.

Lemma 3.1 *If $M \in \mathcal{RC}_{n\times n}(s, s)$, then s, n are co-eigenvalues of M, J_n. Also, $\lambda, 0$ are co-eigenvalues of M, J_n, where λ is an eigenvalue of M corresponding to an eigenvector X such that $X, J_{n\times 1}$ are linearly independent.*

Proof: Notice that $MJ_{n\times 1} = sJ_{n\times 1}$, since $M \in \mathcal{R}_{n\times n}(s)$. Also, $J_nJ_{n\times 1} = nJ_{n\times 1}$. So, we have s, n are co-eigenvalues of M, J_n. Since $M \in \mathcal{RC}_{n\times n}(s, s)$, M commutes with J_n. So, by [21, Proposition 2.3.2], there exists orthonormal vectors $x_1, x_2, \ldots, x_n$, which are eigenvectors of both M and J_n. Since n is an eigenvalue of J_n with eigenvector $J_{n\times 1}$ whose multiplicity is 1, we have $x_i = kJ_{n\times 1}$, for some $i = 1, 2, \ldots, n$ and some $k \in \mathbb{R}$. Without loss of generality, we assume that $x_1 = J_{n\times 1}$. Let λ be the eigenvalue of M corresponding to the eigenvector x_j, $j = 2, 3, \ldots, n$. Then x_j is also an eigenvector of J_n corresponding to the eigenvalue 0. Consequently, we have $\lambda, 0$ are co-eigenvalues of M, J_n. □

The next result follows from the fact that $L(G) \in \mathcal{RC}_{n\times n}(0, 0)$, for any graph G and $A(G) \in \mathcal{RC}_{n\times n}(r, r)$, for an $r-$regular graph G.

Corollary 3.1 *(1) If G is a graph with n vertices, then the pair $0, n$ and for each $i = 2, 3, \ldots, n$ the pairs $\mu_i(G), 0$ are co-eigenvalues of $L(G)$, J_n;*

(2) If G is $r-$regular, then the pair r, n and for each $i = 2, 3, \ldots, n$, the pairs $\lambda_i(G), 0$, are co-eigenvalues of $A(G)$, J_n.

Next, we find the characteristic polynomial of the adjacency and the Laplacian matrices of the $M-$generalized corona of G and $\mathcal{H}_k$ constrained by $\mathcal{T}$ under some special constraints.

Corollary 3.2 *(1) If $\Gamma^{T_1}_{H_1}(x) = \Gamma^{T_2}_{H_2}(x) = \cdots = \Gamma^{T_k}_{H_k}(x)$ and $A(G)$ commutes with MM^T, then the characteristic polynomial of the adjacency matrix of $G\tilde{\circledast}_{[M:\mathcal{T}]}\mathcal{H}_k$ is*

$$\left\{\prod_{j=1}^{k} P_{H_j}(x)\right\} \times \left\{\prod_{i=1}^{n}\left(x - \lambda_i(G) - \Gamma^{T_1}_{H_1}(x)\lambda_i(MM^T)\right)\right\},$$

where $\lambda_i(G), \lambda_i(MM^T)$ are co-eigenvalues of $A(G)$, MM^T for each $i = 1, 2, \ldots, n$.

(2) If $M \in \mathcal{R}_{n\times k}(s)$ is such that MM^T commutes with $L(G)$, $|T_1| = |T_2| = \ldots = |T_k| = t$ and $\Gamma^{T_1}_{L(H_1)+c_1(M)R_{T_1}}(x) = \Gamma^{T_2}_{L(H_2)+c_2(M)R_{T_2}}(x) = \cdots = \Gamma^{T_k}_{L(H_k)+c_k(M)R_{T_k}}(x)$, then the characteristic polynomial of the Laplacian matrix of $G\tilde{\circledast}_{[M:\mathcal{T}]}\mathcal{H}_k$ is

$$\left\{\prod_{j=1}^{k} \left|xI_{n_j} - L(H_j) - c_j(M)R_{T_j}\right|\right\} \times \left\{\prod_{i=1}^{n}\left(x - ts - \mu_i(G) - \Gamma^{T_1}_{L(H_1)+c_1(M)R_{T_1}}(x)\lambda_i(MM^T)\right)\right\},$$

where $\mu_i(G), \lambda_i(MM^T)$ are co-eigenvalues of $L(G)$, MM^T for each $i = 1, 2, \ldots, n$.

Proof:

(1) Since $\Gamma_{H_1}^{T_1}(x) = \Gamma_{H_2}^{T_2}(x) = \cdots = \Gamma_{H_k}^{T_k}(x)$, we have $U_1 = \Gamma_{H_1}^{T_1}(x) I_k$. Applying these in Theorem 3.3(1), we get

$$P_{G\tilde{\circledast}_{[M:\mathcal{T}]}\mathcal{H}_k}(x) = \left\{ \prod_{j=1}^{k} P_{H_j}(x) \right\} \times \left| xI_n - A(G) - \Gamma_{H_1}^{T_1}(x) MM^T \right|.$$

Since $A(G)$ commutes with MM^T, for each $\lambda_i(G)$, $i = 1, 2, \ldots, n$, there exists $\lambda_i(MM^T)$ such that $\lambda_i(G), \lambda_i(MM^T)$ are co-eigenvalues of $A(G)$ and MM^T. So, the proof follows from Observation 3.2(8).

(2) Since $|T_1| = |T_2| = \ldots = |T_k| = t$ and $M \in \mathcal{R}_{n\times k}(s)$, we have $\sum_{j=1}^{k} m_{ij} t_j = t \sum_{j=1}^{k} m_{ij} = ts$ for $i = 1, 2, \ldots, n$ and so $N = tsI_n$. Since $\Gamma_{L(H_1)+c_1(M)R_{T_1}}^{T_1}(x) = \Gamma_{L(H_2)+c_2(M)R_{T_2}}^{T_2}(x) = \cdots = \Gamma_{L(H_k)+c_k(M)R_{T_k}}^{T_k}(x)$, we have $U_2 = \Gamma_{L(H_1)+c_1(M)R_{T_1}}^{T_1}(x) I_k$. Applying these in Theorem 3.3, we get

$$L_{G\tilde{\circledast}_{[M:\mathcal{T}]}\mathcal{H}_k}(x)$$
$$= \left\{ \prod_{j=1}^{k} \left| xI_{n_j} - L(H_j) - c_j(M) R_{T_j} \right| \right\} \times \left| xI_n - L(G) - tsI_n - \Gamma_{L(H_1)+c_1(M)R_{T_1}}^{T_1}(x) MM^T \right|.$$

Since $L(G)$ commutes with MM^T, for each $\mu_i(G)$, $i = 1, 2, \ldots, n$, there exists $\lambda_i(MM^T)$ such that $\mu_i(G), \lambda_i(MM^T)$ are co-eigenvalues of $L(G)$ and MM^T. So, the proof follows from Observation 3.2(8).

□

3.1.1. The characteristic polynomial of $M-$generalized corona for some special classes of graphs G and subsets $\mathcal{T}$. Now, we consider some special classes of graphs G and subsets $\mathcal{T}$ in Definition 2.1. Since $A(\overline{K}_n)$ (which is the zero matrix) commutes with every matrix, by using Corollary 3.2, we get the following result.

Corollary 3.3 *(1) If $\Gamma_{H_1}^{T_1}(x) = \Gamma_{H_2}^{T_2}(x) = \cdots = \Gamma_{H_k}^{T_k}(x)$, then the characteristic polynomial of the adjacency matrix of $\overline{K}_n\tilde{\circledast}_{[M:\mathcal{T}]}\mathcal{H}_k$ is*

$$\left\{ \prod_{j=1}^{k} P_{H_j}(x) \right\} \times \left\{ \prod_{i=1}^{n} \left(x - \Gamma_{H_1}^{T_1}(x) \lambda_i(MM^T) \right) \right\}.$$

(2) If $M \in \mathcal{R}_{n\times k}(s)$, $|T_1| = |T_2| = \ldots = |T_k| = t$ and $\Gamma_{L(H_1)+c_1(M)R_{T_1}}^{T_1}(x) = \Gamma_{L(H_2)+c_2(M)R_{T_2}}^{T_2}(x) = \cdots = \Gamma_{L(H_k)+c_k(M)R_{T_k}}^{T_k}(x)$, then the characteristic polynomial of the Laplacian matrix of $\overline{K}_n\tilde{\circledast}_{[M:\mathcal{T}]}\mathcal{H}_k$ is

$$\left\{ \prod_{j=1}^{k} \left| xI_{n_j} - L(H_j) - c_j(M) R_{T_j} \right| \right\} \times \left\{ \prod_{i=1}^{n} \left(x - ts - \Gamma_{L(H_1)+c_1(M)R_{T_1}}^{T_1}(x) \lambda_i(MM^T) \right) \right\}.$$

In the next result, we obtain the characteristic polynomial of $K_n\tilde{\circledast}_{[M:\mathcal{T}]}\mathcal{H}_k$ for some matrices M.

Corollary 3.4 *If $M \in \mathcal{RC}_{n\times k}(s, c)$, then we have the following:*

(1) If $\Gamma_{H_1}^{T_1}(x) = \Gamma_{H_2}^{T_2}(x) = \cdots = \Gamma_{H_k}^{T_k}(x)$, then the characteristic polynomial of the adjacency matrix of $K_n\tilde{\circledast}_{[M:\mathcal{T}]}\mathcal{H}_k$ is

$$\left(x - n + 1 - cs\Gamma_{H_1}^{T_1}(x) \right) \times \left\{ \prod_{j=1}^{k} P_{H_j}(x) \right\} \times \left\{ \prod_{i=2}^{n} \left(x + 1 - \Gamma_{H_1}^{T_1}(x) \lambda_i(MM^T) \right) \right\}.$$

(2) If $|T_1| = |T_2| = \ldots = |T_k| = t$ *and* $\Gamma^{T_1}_{L(H_1)+cR_{T_1}}(x) = \Gamma^{T_2}_{L(H_2)+cR_{T_2}}(x) = \cdots = \Gamma^{T_k}_{L(H_k)+cR_{T_k}}(x)$*, then the characteristic polynomial of the Laplacian matrix of* $K_n \tilde{\circledast}_{[M:\mathcal{T}]} \mathcal{H}_k$ *is*

$$\left(x - ts - cs\Gamma^{T_1}_{L(H_1)+cR_{T_1}}(x)\right) \times \left\{\prod_{j=1}^{k} \left|xI_{n_2} - L(H_j) - cR_{T_j}\right|\right\}$$

$$\times \left\{\prod_{i=2}^{n} \left(x - n - ts - \Gamma^{T_1}_{L(H_1)+cR_{T_1}}(x)\lambda_i(MM^T)\right)\right\}.$$

Proof: Notice that $A(K_n) = J_n - I_n$ and $L(K_n) = nI_n - J_n$. Since $M \in \mathcal{RC}_{n\times n}(s, c)$, we have $MM^T \in \mathcal{RC}_{n\times n}(cs, cs)$. So by using Lemma 3.1 and Observation 3.2(6), for each $i = 2, 3, \ldots, n$, we have $cs, n-1$ and $\lambda_i(MM^T), -1$ are co-eigenvalues of $MM^T, A(K_n)$, respectively; $cs, 0$ and $\lambda_i(MM^T), n$ are co-eigenvalues of $MM^T, L(K_n)$, respectively. So, the result follows from Corollary 3.2. □

Lemma 3.2 *Let* G *be a spanning* $r-$*regular subgraph of* $K_{p,p}$*. Then we have the following:*

(1) The co-eigenvalues of $A(G)$, $A(K_{p,p})$ *are:* r, p; $-r, -p$ *and* $\lambda_i(G), 0$ *for* $i = 1, 2, \ldots, 2p$;

(2) The co-eigenvalues of $L(G)$, $L(K_{p,p})$ *are:* $0, 0$; $2r, 2p$ *and* $\mu_i(G), p$ *for* $i = 1, 2, \ldots, 2p$.

Proof:

(1) Notice that the spectrum of $K_{p,p}$ is $p, 0^{2p-2}, -p$. Since G is an $r-$regular spanning subgraph of $K_{p,p}$, it is a $r-$regular bipartite graph. By [21, Proposition 2.3.6], $A(K_{p,p})$ and $A(G)$ commutes with each other. So, by [21, Proposition 2.3.2], there exists orthonormal eigenvectors $x_1, x_2, \ldots, x_{2p}$, which are eigenvectors of both $A(G)$ and $A(K_{p,p})$. Since G (resp. $K_{p,p}$) is $r-$regular (resp. $p-$regular), we have $x_1 = J_{2p\times 1}$. By [4, Proof of Lemma 3.13], $-r$ (resp. $-p$) is also an eigenvalue of $A(G)$ (resp. $A(K_{p,p})$) corresponding to the eigenvector $x_{2p} = [J_{1\times p} \;\; -J_{1\times p}]^T$. Then x_i is an eigenvector corresponding to $\lambda_i(G)$ for some $i = 2, 3, \ldots, 2p-1$. But for each $j = 2, 3, \ldots, 2p-1$, x_i is an eigenvector of $A(K_{p,p})$ corresponding to 0. So the result follows.

(2) Since G is $r-$regular, the result follows from Observation 3.2(6) and part (1) of this lemma.

□

By using Lemma 3.2 in Corollary 3.2, we get the following results.

Corollary 3.5 *Let* G *be a spanning* $r-$*regular subgraph of* $K_{p,p}$ *and let* $M = A(G)$*. Then we have the following:*

(1) If $\Gamma^{T_1}_{H_1}(x) = \Gamma^{T_2}_{H_2}(x) = \cdots = \Gamma^{T_{2p}}_{H_{2p}}(x)$*, then the characteristic polynomial of the adjacency matrix of* $K_{p,p} \tilde{\circledast}_{[M:\mathcal{T}]} \mathcal{H}_{2p}$ *is*

$$\left(x - p - r^2\Gamma^{T_1}_{H_1}(x)\right)\left(x + p - r^2\Gamma^{T_1}_{H_1}(x)\right) \times \left\{\prod_{j=1}^{2p} P_{H_j}(x)\right\} \times \left\{\prod_{i=2}^{2p-1} \left(x - \lambda_i(G)^2\Gamma^{T_1}_{H_1}(x)\right)\right\}.$$

(2) If $\Gamma^{T_1}_{L(H_1)+c_1(M)R_{T_1}}(x) = \Gamma^{T_2}_{L(H_2)+c_2(M)R_{T_2}}(x) = \cdots = \Gamma^{T_{2p}}_{L(H_{2p})+c_{2p}(M)R_{T_{2p}}}(x)$ *and* $|T_1| = |T_2| = \cdots = |T_{2p}| = t$*, then the characteristic polynomial of the Laplacian matrix of* $K_{p,p} \tilde{\circledast}_{[M:\mathcal{T}]} \mathcal{H}_{2p}$ *is*

$$\left(x - tr - p - r^2\Gamma^{T_1}_{L(H_1)+rR_{T_1}}(x)\right)\left(x - tr - 2p - 4r^2\Gamma^{T_1}_{L(H_1)+rR_{T_1}}(x)\right)$$

$$\times \left\{\prod_{j=1}^{2p} \left|xI_{n_j} - L(H_j) - rR_{T_j}\right|\right\} \times \left\{\prod_{i=2}^{2p-1} \left(x - tr - p - \lambda_i(G)^2\Gamma^{T_1}_{H_1}(x)\right)\right\}.$$

Corollary 3.6 *Let G be a spanning $r-$regular subgraph of $K_{p,p}$ and let $M = A(K_{p,p})$. Then we have the following:*

(1) If $\Gamma_{H_1}^{T_1}(x) = \Gamma_{H_2}^{T_2}(x) = \cdots = \Gamma_{H_{2p}}^{T_{2p}}(x)$, then the characteristic polynomial of the adjacency matrix of $G\tilde{\circledast}_{[M:\mathcal{T}]}\mathcal{H}_{2p}$ is

$$x^{2p-2}\left(x - r - p^2\Gamma_{H_1}^{T_1}(x)\right)\left(x + r - p^2\Gamma_{H_1}^{T_1}(x)\right) \times \left\{\prod_{j=1}^{2p} P_{H_j}(x)\right\} \times \left\{\prod_{i=2}^{2p-1}(x - \lambda_i(G))\right\}.$$

(2) If $\Gamma_{L(H_1)+c_1(M)R_{T_1}}^{T_1}(x) = \Gamma_{L(H_2)+c_2(M)R_{T_2}}^{T_2}(x) = \cdots = \Gamma_{L(H_k)+c_k(M)R_{T_{2p}}}^{T_{2p}}(x)$ and $|T_1| = |T_2| = \ldots = |T_{2p}| = t$, then the characteristic polynomial of the Laplacian matrix of $G\tilde{\circledast}_{[M:\mathcal{T}]}\mathcal{H}_{2p}$ is

$$\left(x - tp - r - p^2\Gamma_{L(H_1)+pR_{T_1}}^{T_1}(x)\right)\left(x - tp - 2r - 4p^2\Gamma_{L(H_1)+pR_{T_1}}^{T_1}(x)\right)$$

$$\times\left\{\prod_{i=2}^{2p-1}(x - tr - \mu_i(G))\right\} \times \left\{\prod_{j=1}^{2p}\left|xI_{n_j} - L(H_j) - pR_{T_j}\right|\right\}.$$

In the following results, we obtain the $A-$spectrum and the $L-$spectrum of the graph $G\tilde{\circledast}_M\mathcal{H}_k$ for some families of graphs H_i and G.

Corollary 3.7 *Let $\mathcal{H}_k = (H_1, H_2, \ldots, H_k)$ be a sequence of graphs with n_2 vertices. Then we have the following:*

(1) If H_j $(j = 1, 2, \ldots, k)$ are r_2-regular and $A(G)$ commutes with MM^T, then the $A-$spectrum of $G\tilde{\circledast}_M\mathcal{H}_k$ is

(i) r_2 with multiplicity $k - n$;

(ii) $\lambda_l(H_j)$ for $j = 1, 2, \ldots, k$ and $l = 2, 3, \ldots, n_2$;

(iii) $\frac{1}{2}\left(r_2 + \lambda_i(G) \pm \sqrt{(r_2 - \lambda_i(G))^2 + 4n_2\lambda_i(MM^T)}\right)$ for $i = 1, 2, \ldots, n$.

(2) If $M \in \mathcal{RC}_{n\times k}(s, c)$ is such that MM^T commutes with $L(G)$, then the $L-$spectrum of $G\tilde{\circledast}_M\mathcal{H}_k$ is

(i) c with multiplicity $k - n$;

(ii) $c + \mu_l(H_j)$ for $j = 1, 2, \ldots, k$ and $l = 2, 3, \ldots, n_2$;

(iii) $\frac{1}{2}\left(c + sn_2 + \mu_i(G) \pm \sqrt{\{c - sn_2 - \mu_i(G)\}^2 + 4n_2\lambda_i(MM^T)}\right)$ for $i = 1, 2, \ldots, n$.

Proof:

(1) By Proposition 3.1, we have $\Gamma_{H_j}(x) = \frac{n_2}{x - r_2}$ for $j = 1, 2, \ldots, k$ and so the proof follows from Corollary 3.2(1).

(2) Since $T_j = V(H_j)$, we have $R_{T_j} = I_{n_2}$. So by using Proposition 3.1, we have $\Gamma_{L(H_j)+cR_{T_j}}(x) = \frac{n_2}{x - c}$ for $j = 1, 2, \ldots, k$. By applying these values in Corollary 3.2(2), we get the result.

□

3.1.2. $A-$spectra and $L-$spectra of new variants of corona of graphs. First, we determine the eigenvalues of MM^T for the corona generating matrices M given in Table 2. Let G be an $r-$regular graph and $m' = E(\overline{G})$. Notice that for each of the matrix M given in Table 2, $M \in \mathcal{RC}_{n\times n}(s,c)$ for some integers c and s. This implies that $MM^T \in \mathcal{RC}_{n\times n}(cs,cs)$. So, by using Observations 3.2(3)-(4), Lemma 3.1 and Observation 3.2(8), we obtain the entries $\lambda_i(MM^T)$ as mentioned in the last column of Table 4.

Table 4: The eigenvalues of MM^T for various matrices M listed in Table 2.

S. No	M	c	s	MM^T	$\lambda_i(MM^T)$
1.	$A(G)$	r	r	$A(G)^2$	$\lambda_i(G)^2$ for $i=1,2,\ldots,n$
2.	$I_n + A(G)$	$r+1$	$r+1$	$\{I_n + A(G)\}^2$	$(\lambda_i(G)+1)^2$ for $i=1,2,\ldots,n$
3.	$J_n - I_n$	$n-1$	$n-1$	$\{J_n - I_n\}^2$	$(n-1)^2$ for $i=1$; 1 for $i=2,3,\ldots,n$.
4.	$J_n - A(G)$	$n-r$	$n-r$	$\{J_n - A(G)\}^2$	$(n-r)^2$ for $i=1$; $\lambda_i(G)^2$ for $i=2,3,\ldots,n$.
5.	$J_n - I_n - A(G)$	$n-r-1$	$n-r-1$	$\{J_n - A(G) - I_n\}^2$	$(n-r-1)^2$ for $i=1$; $(\lambda_i(G)+1)^2$ for $i=2,3,\ldots,n$
6.	$B(G)$	2	r	$rI_n + A(G)$	$r+\lambda_i(G)$ for $i=1,2,\ldots,n$
7.	$J_{n\times m} - B(G)$	$n-2$	$m-r$	$(m-2r)J_n + rI_n + A(G)$	$(m-2r)n+2r$ for $i=1$; $r+\lambda_i(G)$ for $i=2,3,\ldots,n$
8.	$B(\overline{G})$	2	$n-r-1$	$J_n + (n-r-2)I_n - A(G)$	$2n-2r-2$ for $i=1$; $n-r-\lambda_i(G)-2$ for $i=2,3,\ldots,n$
9.	$J_{n\times m'} - B(\overline{G})$	$n-2$	$m'-n+r+1$	$(m'-2n+2r+3)J_n + (n-r-2)I_n - A(G)$	$nm'-(2n-2)(n-r-1)$ for $i=1$; $n-r-\lambda_i(G)-2$ for $i=2,3,\ldots,n$

In the next result, we obtain the characteristic polynomial of the adjacency and the Laplacian matrices of the graphs defined in Table 2.

Corollary 3.8 *If G is $r-$regular, then we have the following:*

(1) If $\Gamma^{T_1}_{H_1}(x) = \Gamma^{T_2}_{H_2}(x) = \cdots = \Gamma^{T_k}_{H_k}(x)$, then the characteristic polynomial of the adjacency matrices of the graphs obtained from the corona operations defined in Definitions (1)–(9) in Table 2 can be obtained from Corollary 3.2 by substituting $\lambda_i(MM^T)$, $i=1,2,\ldots,n$ as in Table 4, for suitable corona generating matrices M. (for Definitions (1)–(3) in Table 2, the result holds for any graph G).

(2) If $\Gamma^{T_1}_{L(H_1)+c_1(M)R_{T_1}}(x) = \Gamma^{T_2}_{L(H_2)+c_2(M)R_{T_2}}(x) = \cdots = \Gamma^{T_k}_{L(H_k)+c_k(M)R_{T_k}}(x)$ and $|T_1| = |T_2| = \ldots = |T_k| = t$, then the characteristic polynomial of the Laplacian matrices of the graphs obtained from the corona operations defined in Definitions (1)–(10) in Table 2 can be obtained from Corollary 3.2 by substituting $\lambda_i(MM^T)$, $i=1,2,\ldots,n$ as in Table 4, for suitable corona generating matrices M. (For Definitions (4) in Table 2, the result holds for any graph G).

Proof:

(1) If M is either $A(G)$ or $A(G)+I_n$, then MM^T commutes with $A(G)$. So, by using Observation 3.2(4), we have $\lambda_i(G), \lambda_i(MM^T)$ are co-eigenvalues of $A(G), MM^T$ for each $i=1,2,\ldots,n$.

For the remaining corona generating matrices M listed in Table 4, MM^T commutes with $A(G)$, when G is regular. So, by using Corollary 3.1(1) and Observation 3.2(7), we have $\lambda_i(G), \lambda_i(MM^T)$ are co-eigenvalues of $A(G)$, MM^T, where $\lambda_i(MM^T)$ are as mentioned in Table 4, for each $i=1,2,\ldots,n$. So the proof follows.

(2) If $M = J_n - I_n$, then $M \in \mathcal{RC}_{n\times n}(n-1, n-1)$ and MM^T commutes with $L(G)$. So, by using Corollary 3.1(2) and Observation 3.2(7), we have $\mu_i(G), \lambda_i(MM^T)$ are co-eigenvalues of $L(G)$, MM^T, where $\lambda_i(MM^T)$ are as mentioned in Table 4, for each $i = 1, 2, \ldots, n$.

For the remaining corona generating matrices M listed in Table 2, MM^T commutes with $L(G)$, if G is regular. So by using the proof of part (1) and Observation 3.2(6), we have $\mu_i(G), \lambda_i(MM^T)$ are co-eigenvalues of $L(G)$, MM^T, where $\lambda_i(MM^T)$ are as mentioned in Table 4, for each $i = 1, 2, \ldots, n$. So the proof follows.

□

In the following result, we obtain the characteristic polynomials of the adjacency and the Laplacian matrices of the graphs obtained by the corona operations defined in Definitions (11)–(14) in Table 2.

Corollary 3.9 *Let G be an $r-$regular graph. Then we have the following:*

(1) If $\Gamma^{T_1}_{H_1}(x) = \Gamma^{T_2}_{H_2}(x) = \cdots = \Gamma^{T_k}_{H_k}(x)$, then the characteristic polynomial of the adjacency matrices of the graphs obtained from the corona operations defined in Definitions (11)–(14) in Table 2 can be obtained from Corollary 3.3 by suitably substituting the corona generating matrices M and using Table 4. (for Definitions (11) and (12), the result holds for any graph G).

(2) If $\Gamma^{T_1}_{L(H_1)+c_1(M)R_{T_1}}(x) = \Gamma^{T_2}_{L(H_2)+c_2(M)R_{T_2}}(x) = \cdots = \Gamma^{T_k}_{L(H_k)+c_k(M)R_{T_k}}(x)$ and $|T_1| = |T_2| = \ldots = |T_k| = t$, then the characteristic polynomial of the Laplacian matrices of the graphs obtained from the corona operations defined in Definitions (11)–(14) in Table 2 can be obtained from Corollary 3.3 by suitably substituting the corona generating matrices M and using Table 4.

Note 1 If G is $r-$regular, then we can obtain the $A-$spectra and the $L-$spectra of the graphs defined in Table 3 by taking $H_j = K_1$ for $j = 1, 2, \ldots, k$ and suitable corona generating matrices M and the graph G in Corollary 3.7 (For Definitions (1)–(3), the result holds for any graph G).

3.1.3. $A-$spectra and $L-$spectra of variants of corona of graphs defined in the literature. In this subsection, we determine the characteristic polynomials of the adjacency and the Laplacian matrices / the $A-$spectra and $L-$spectra of the graphs mentioned in Table 1 by using the results we have proved so far in this section.

In [7], S. Barik et al. described the $A-$spectra of the subdivision (resp. $R-$graph, $Q-$graph, total) double corona of the regular graphs G, H_1 and H_2. In the next result, we deduce the characteristic polynomials of these graphs for regular graphs G and arbitrary graphs H_1, H_2 .

Corollary 3.10 *Let G be an $r-$regular graph with n vertices and $m\left(=\frac{1}{2}nr\right)$ edges. Let H_1 and H_2 be graphs with h vertices. Then the characteristic polynomial of the adjacency matrix of the subdivision (resp. $R-$graph, $Q-$graph, total) double corona of G, H_1 and H_2 is*

$$\{x - \Gamma_H(x) + 2k_1\}^{m-n} \times \left\{ \prod_{i=1}^{n} \left(x^2 - \{\Gamma_{H_1}(x) + \Gamma_{H_2}(x) + k_1\lambda_i(G) - 2k_2\}\, x \right.\right.$$
$$\left.\left. + (\Gamma_{H_1}(x) + k_1\lambda_i(G))(\Gamma_{H_2}(x) - 2k_2) - \nu_i(G)\right)\right\},$$

where $k_1 = k_2 = 0$ (resp. $k_1 = 1$, $k_2 = 0$; $k_1 = 0$, $k_2 = 1$; $k_1 = k_2 = 1$).

Proof: Taking suitable entities, which are given in Table 1 and substituting them in Theorem 3.3 and then by using Theorem 3.1, we obtain the result. □

P. L. Lu and Y. M. Wu [33] studied the Laplacian and signless Laplacian spectrum of the generalized subdivision vertex corona of graphs. P. L. Lu et al. [31] studied the signless Laplacian spectrum of the generalized subdivision edge corona of graphs. Q. Liu and Z. Zhang [28] studied the normalized Laplacian spectrum of the generalized subdivision edge corona of graphs. Subsequently, Q. Liu [29] obtained the Kirchhoff index of the generalized $R-$vertex (resp. $R-$edge) corona of graphs. In the following result, we determine the characteristic polynomials of the adjacency and the Laplacian matrices of these graphs by taking suitable entities, which are given in Table 1 and substituting them in Theorem 3.3 and then using Theorem 3.1.

Corollary 3.11 *Let G be an $r-$regular graph with n vertices and $m\left(=\frac{1}{2}nr\right)$ edges. Let $H_1, H_2, \ldots, H_k$, be graphs with h vertices, where $k = n$ or m. Then we have the following:*

(1) If $\Gamma_{H_1}(x) = \Gamma_{H_2}(x) = \cdots = \Gamma_{H_n}(x)$, then the characteristic polynomial of the adjacency matrix of the generalized subdivision vertex (resp. subdivision edge, $R-$ vertex, $R-$edge) corona of G and $H_1, H_2, \ldots, H_k$, where $k = n$ or m is

$$\{x - k_3\Gamma_{H_2}(x)\}^{m-n} \times \left\{\prod_{i=1}^{n} \left(x^2 - \{k_1\lambda_i(G) + k_2\Gamma_{H_1}(x) + k_3\Gamma_{H_2}(x)\}\, x \right.\right.$$
$$\left.\left. + k_3\Gamma_{H_2}(x)\,\{k_1\lambda_i(G) + k_2\Gamma_{H_1}(x)\} - \nu_i(G)\right)\right\},$$

where $k_1 = 0$, $k_2 = 1$, $k_3 = 0$ (resp. $k_1 = 0$, $k_2 = 0$, $k_3 = 1$; $k_1 = 1$, $k_2 = 1$, $k_3 = 0$; $k_1 = 1$, $k_2 = 0$, $k_3 = 1$).

(2) The characteristic polynomial of the Laplacian matrix of the generalized subdivision vertex (resp. $R-$vertex) corona of G and $H_1, H_2, \ldots, H_n$ is

$$\frac{\{x-2\}^{m-n}}{(x-1)^n} \times \left\{\prod_{i=1}^{n} L_{H_i}(x-1)\right\}$$
$$\times \left\{\prod_{i=1}^{n} \left(x^3 - \{h + r + 3 + k_1\mu_i(G)\}\, x^2 + \{2h + r + 2 + k_2\mu_i(G)\}\, x - k_3\mu_i(G)\right)\right\},$$

where $k_1 = 0$, $k_2 = 1$, $k_3 = 1$ (resp. $k_1 = 1$, $k_2 = 4$, $k_3 = 3$).

(3) The characteristic polynomial of the Laplacian matrix of the generalized subdivision vertex (resp. $R-$vertex) corona of G and $H_1, H_2, \ldots, H_n$ is

$$\frac{\{x^2 - (h+3)x + 2\}^{m-n}}{(x-1)^m} \times \left\{\prod_{i=1}^{m} L_{H_i}(x-1)\right\}$$
$$\times \left\{\prod_{i=1}^{n} \left(x^3 - \{s + k_1\mu_i(G)\}\, x^2 + \{r(h+1) + k_2\mu_i(G) + 2\}\, x - k_3\mu_i(G)\right)\right\},$$

where $s = h + r + 3$, $k_1 = 0$, $k_2 = 1$, $k_3 = 1$ (resp. $k_1 = 1$, $k_2 = h + 4$, $k_3 = 3$).

In [42], F. Wen et al. computed the normalized Laplacian spectra of the SVEV-corona and the SVEE-corona of G and H. In the next results, we find the characteristic polynomial of the adjacency and the Laplacian matrices of the SVEV-corona and the SVEE-corona of G with H_1 and H_2.

Corollary 3.12 *Let G be an $r-$regular graph with n vertices and $m\left(=\frac{1}{2}nr\right)$ edges. Let H_1 and H_2 be graphs with h_1 and h_2 vertices, respectively. Then we have the following: .*

(1) The characteristic polynomial of the adjacency matrix of the SVEV-corona (resp. SVEE-corona) of G with H_1 and H_2 is

$$\{P_{H_1}(x)\}^n \times \{P_{H_2}(x)\}^m \times \{x - k_1\Gamma_{H_2}(x)\}^{m-n} \times \left\{\prod_{i=1}^{n} \left(x^2 - \{\nu_i(G)\Gamma_{H_1}(x) + \Gamma_{H_2}(x)\}\, x - \nu_i(G)\right)\right\},$$

where $k_1 = 1$ (resp. $k_1 = 0$).

(2) The characteristic polynomial of the Laplacian matrix of the SVEV-corona of G with H_1 and H_2 is

$$\frac{\left(x^2-(2h_1+h_2+3)x+2\right)^{m-n}}{(x-r)^n(x-1)^m}\times\{L_{H_1}(x-r)\}^n\times\{L_{H_2}(x-1)\}^m \times\left\{\prod_{i=1}^{n}\left(x^3-k_1x^2+\{k_2\mu_i(G)+k_3\}\,x-(h_1-1)\mu_i(G)\right)\right\},$$

where $k_1=2h_1+h_2+r+3,\ k_2=h_1+1,\ k_3=2k_2+r(h_2+1).$

(3) The characteristic polynomial of the Laplacian matrix of the SVEE-corona of G with H_1 and H_2 is

$$\frac{(x-2)^{m-n}}{(x-1)^n}\times\text{`}\{L_{H_1}(x-1)\}^n\times\{L_{H_2}(x-2)\}^m \times\left\{\prod_{i=1}^{n}\left(x^3-k_1x^2+\{k_2\mu_i(G)+k_3\}\,x-(h_1-1)\mu_i(G)\right)\right\},$$

where $k_1=rh_1+h_2+r+3,\ k_2=h_1+1,\ k_3=rk_2+2(h_2+1).$

Proof:

(1) Taking $M=\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ B(G)^T & I_m\end{bmatrix}$ and $U_1=\begin{bmatrix}\Gamma_{H_1}(x)I_n & \mathbf{0}\\ \mathbf{0} & \Gamma_{H_2}(x)I_m\end{bmatrix}$, we have

$$MU_1M^T=\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \Gamma_{H_1}(x)B(G)^TB(G)+\Gamma_{H_2}(x)I_m\end{bmatrix}.$$

Now,

$$\begin{aligned}
&\left|xI_{n+m}-A(S(G))-MU_1M^T\right|\\
&=\begin{vmatrix}xI_n & -B(G)\\ -B(G)^T & xI_m-\Gamma_{H_1}(x)B(G)^TB(G)-\Gamma_{H_2}(x)I_m\end{vmatrix}\\
&=x^n\times\left|xI_m-\Gamma_{H_1}(x)B(G)^TB(G)-\Gamma_{H_2}(x)I_m-\frac{1}{x}B(G)^TB(G)\right|\\
&=x^{n-m}\times\left|x^2I_m-\left(\Gamma_{H_1}(x)B(G)^TB(G)+\Gamma_{H_2}(x)I_m\right)x-B(G)^TB(G)\right|\\
&=x^{n-m}\times\left|\left(x^2-x\Gamma_{H_2}(x)\right)I_m-\left(x\Gamma_{H_1}(x)+1\right)B(G)^TB(G)\right|\\
&=x^{n-m}\times\left(x^2-x\Gamma_{H_2}(x)\right)^{m-n}\times\left|\left(x^2-x\Gamma_{H_2}(x)\right)I_n-\left(x\Gamma_{H_1}(x)+1\right)B(G)B(G)^T\right|\\
&=\left(x-\Gamma_{H_2}(x)\right)^{m-n}\times\left|\left(x^2-x\Gamma_{H_2}(x)\right)I_n-\left(x\Gamma_{H_1}(x)+1\right)B(G)B(G)^T\right|\\
&=\left(x-\Gamma_{H_2}(x)\right)^{m-n}\times\left\{\prod_{i=1}^{n}\left(x^2-\{\nu_i(G)\Gamma_{H_1}(x)+\Gamma_{H_2}(x)\}\,x-\nu_i(G)\right)\right\}
\end{aligned}$$

Then by using Theorem 3.3, we obtain the characteristic polynomial of the adjacency matrix of SVEV-corona of G with H_1 and H_2.
Similarly, by taking $M=\begin{bmatrix}I_n & B(G)\\ \mathbf{0} & \mathbf{0}\end{bmatrix}$ and $U_1=\begin{bmatrix}\Gamma_{H_2}(x)I_n & \mathbf{0}\\ \mathbf{0} & \Gamma_{H_1}(x)I_m\end{bmatrix}$, we can obtain the characteristic polynomial of the adjacency matrix of SVEE-corona of G with H_1 and H_2.

(2) Take $N=\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & (2h_1+h_2)I_m\end{bmatrix}$ and $s_1=2h_1+h_2+2$. Then

$$\begin{aligned}
&\left|xI_{n+m}-L(S(G))-N-MU_2M^T\right|\\
&=\begin{vmatrix}(x-r)I_n & -B(G)\\ -B(G)^T & (x-s_1)I_m-\Gamma_{L(H_1)+rI_n}(x)B(G)^TB(G)-\Gamma_{L(H_2)+I_m}(x)I_m\end{vmatrix}.
\end{aligned}$$

Then by using Theorems 3.1 and 3.3, and using the fact that $\Gamma_{L(H_1)+rI_n}(x) = \dfrac{h_1}{x-r}$ and $\Gamma_{L(H_2)+I_m}(x) = \dfrac{h_2}{x-1}$, we obtain the result.

(3) Proof is similar to the proof of part (2).

□

Remark 3.2 Almost all the results related to the characteristic polynomials of the adjacency and the Laplacian matrices of the variants of corona of graphs listed in Table 1 can be easily deduced from our results:

(1) We can deduce the characteristic polynomials of the adjacency and the Laplacian matrices of the generalized corona of G and $\mathcal{H}_n$ constrained by $\mathcal{T}$ (cf. [38, Theorems 3.1, 4.1]) and the generalized corona of G and $H_1, H_2, \ldots, H_n$ (cf. [15, Theorems 3.1, 4.1]), by taking suitable entities as in Table 1 and applying these in Theorem 3.3. Also, we can deduce the characteristic polynomials of the adjacency and the Laplacian matrices of the corona of G and H (cf. [35, Theorem 2], [27, Theorem 3.1]) and the cluster of G and H (cf. [38, Corollary 4.6]), by taking suitable entities as in Table 1 and applying these in Corollary 3.2. Further, we can deduce the $A-$spectrum (resp. the $L-$spectrum) of the corona of G and H (cf. [5, Theorems 3.1 (resp. Theorems 3.2)]), by taking suitable entities as in Table 1 and applying these in Corollary 3.7.

(2) We can deduce the characteristic polynomial of the Laplacian matrix (resp. adjacency matrix) of the generalized edge corona of G and $H_1, H_2, \ldots, H_m$, where G is r_1-regular, $H_1, H_2, \ldots, H_m$ are graphs with h vertices (resp. $H_1, H_2, \ldots, H_m$ are r_2-regular with h vertices) (cf. [34, Theorem 2 (resp. Theorem 4)]), by taking suitable entities as in Table 1 and applying these in Corollary 3.7. In a similar way, we can deduce the characteristic polynomial of the adjacency (resp. Laplacian) matrix of the edge corona of G and H, where G and H are regular (resp. arbitrary graph H) (cf. [22, Theorem 2.3] (resp. [22, Theorem 2.4] and [27, Theorem 4.1])).

(3) We can deduce the characteristic polynomials of the adjacency and the Laplacian amtrices of the neighbourhood corona of G and H, (cf. [26, Theorems 2.3, 2.4]), by taking suitable entities as in Table 1 and applying these in Corollary 3.8. Also, we can deduce the $A-$spectra (resp. $L-$spectra) of the neighbourhood corona of G and H, where G and H are regular (resp. arbitrary graph H) (cf. [23, Theorem 2.1 (resp. Theorem 3.1)]), by taking suitable entities as in Table 1 and applying these in Corollary 3.7.

(4) We can deduce the characteristic polynomials of the adjacency and the Laplacian matrices of the subdivision vertex corona of G and H (cf. [30, Theorems 2.1, 2.7]), the subdivision vertex neighbourhood corona of G and H (cf. [25, Theorems 2.1, 2.5]), the subdivision edge neighbourhood corona of G and H (cf. [25, Theorems 3.1, 3.5]), the $R-$vertex corona of G and H(cf. [24, Theorems 3.1, 3.2]), the $R-$edge corona of G and H (cf. [24, Theorems 4.1, 4.2]), the $R-$vertex neighbourhood corona of G and H (cf. [24, Theorems 5.1, 5.2]), the $R-$edge neighbourhood corona of G and H (cf. [24, Theorems 6.1, 6.2]), the $N-$vertex corona of G and H (cf. [1, Theroems 3.1, 3.2]), the $N-$vertex corona of G and H (cf. [1, Theorems 3.1, 3.2]), the $C-$vertex neighbourhood corona of G and H (cf. [1, Theorems 3.4, 3.5]), the $C-$edge corona of G and H (cf. [1, Theorems 4.1, 4.2]), the $N-$edge corona of G and H (cf. [1, Theorems 4.4, 4.5]), the total corona of G and H (cf. [43, Theorems 2.1, 2.2]), the duplication corona of G and H (cf. [3, Theorems 3.1, 3.4]), the duplication neighbourhood corona of G and H (cf. [3, Theorems 4.1, 4.4]), the duplication edge corona of G and H (cf. [3, Theorems 5.1, 5.4]), by taking suitable entities as in Table 1 and applying these in Theorem 3.3, and by using Theorem 3.1.

(5) If G is a graph with n_1 vertices and H is an r_2-regular graph with n_2 vertices and $m_2(=\frac{1}{2}n_2r_2)$ edges, then it is not hard to obtain that $\Gamma^{V(H)}_{S(H)}(x) = \dfrac{n_2x}{x^2-2r_2}$ and $\Gamma^{I(H)}_{S(H)}(x) = \dfrac{m_2x}{x^2-2r_2}$. By using

these facts and taking the entities as in Table 1 and applying these in Corollary 3.2, we can deduce the characteristic polynomials of the adjacency and the Laplacian matrices of the corona-vertex subdivision graph of G and H, and the corona-edge subdivision graph of G and H [32].

(6) To obtain the characteristic polynomials of the adjacency and the Laplacian matrices of the S-N corona (resp. R-N corona, Q-N corona, T-N corona, $R-$SEN corona, $Q-$SEN corona, $T-$SEN corona, $R-$SVN corona, $T-$SVN corona) of G with $\mathcal{H}_n$ and $\mathcal{H}'_m$ constrained by $\mathcal{T}$ and $\mathcal{T}'$, substituting the entities as in Table 1 in Theorem 3.3. Then we can obtain an equation similar to equation (3) in [17]. Then by continuing the similar procedure as in [17, Theorem 2.2], we can obtain the results.

4. Applications

4.1. Integral graphs

Corollary 4.1 *Let G be an $A-$integral graph with n_1 vertices. Then the neighbourhood corona of G and $\overline{K}_{n_2}$ is $A-$integral if and only if $n_2 = s(s+1)$, where $s \in \mathbb{N}$.*

Proof: Using Corollary 3.7, the $A-$spectrum of the neighbourhood corona of G and $\overline{K}_{n_2}$ is

(i) 0 with multiplicity $n_1(n_2-1)$,

(ii) $\dfrac{\lambda_j(G)}{2}\left(1 \pm \sqrt{4n_2+1}\right)$ for $j = 1, 2, \ldots, n_1$.

So, the neighbourhood corona of G and K_{n_2} is integral

$$\begin{aligned} &\Leftrightarrow \quad \sqrt{4n_2+1} \text{ is a non-negative integer} \\ &\Leftrightarrow \quad 4n_2+1 = t^2 \text{for some integer } t \\ &\Leftrightarrow \quad n_2 = \frac{1}{4}(t^2-1). \end{aligned}$$

Since n_2 is an integer, t must be odd, that is, $t = (2s+1), s = 1, 2, \ldots$ and so $n_2 = s(s+1), s = 1, 2, \ldots$ □

In the next result, we construct infinitely many families of $A-$integral and $L-$integral bipartite graphs by using Corollary 3.7.

Corollary 4.2 *(1) If M is a $0-1$ matrix such that all the eigenvalues of MM^T are perfect squares and n_2 is a perfect square, then the graph $\overline{K}_{n_1} \tilde{\circledast}_M \overline{K}_{n_2}$ is $A-$integral. In particular, if M is a $0-1$ symmetric matrix such that all of its eigenvalues are integers and and n_2 is a perfect square, then the graph $\overline{K}_{n_1} \tilde{\circledast}_M \overline{K}_{n_2}$ is $A-$integral.*

(2) If $M \in \mathcal{RC}_{n\times k}(r, r)$ is a $0-1$ matrix such that all the eigenvalues of MM^T are perfect squares, then the graph $\overline{K}_n \tilde{\circledast}_M K_1$ is $L-$integral. In particular, if $M \in \mathcal{R}_{n\times n}(r)$ is a $0-1$ symmetric matrix such that all of its eigenvalues are integers, then the graph $\overline{K}_n \tilde{\circledast}_M K_1$ is $L-$integral.

4.2. Cospectral graphs

Corollary 3.2 shows that, if all the coronals of H_j's constrained by their corresponding subsets T_j's are equal, then the $A-$spectrum of $G \tilde{\circledast}_{[M:\mathcal{T}]} \mathcal{H}_k$ is the same regardless of the order of H_j's in $\mathcal{H}_k$.

Corollary 4.3 *Let $\mathcal{H}_k = (H_1, H_2, \ldots, H_k)$ be a sequence of k graphs and $\mathcal{T} = (T_1, T_2, \ldots, T_k)$, where $T_j \subseteq V(H_j), j = 1, 2, \ldots, k$. let M be a $0-1$ matrix of size $n\times k$ such that MM^T commutes with $A(G)$. Let θ be a permutation on $\{1, 2, \ldots, k\}$. Let $\mathcal{H}'_k = (H_{\theta(1)}, H_{\theta(2)}, \ldots, H_{\theta(k)})$ and $\mathcal{T} = (T_{\theta(1)}, T_{\theta(2)}, \ldots, T_{\theta(k)})$. Then we have the following:*

(1) If $\Gamma^{T_1}_{H_1}(x) = \Gamma^{T_2}_{H_2}(x) = \cdots = \Gamma^{T_k}_{H_k}(x)$, then $G \tilde{\circledast}_{[M:\mathcal{T}]} \mathcal{H}_k$ and $G \tilde{\circledast}_{[M:\mathcal{T}']} \mathcal{H}'_k$ are $A-$cospectral.

(2) If $M \in \mathcal{R}_{n\times k}(s)$, $|T_1| = |T_2| = \ldots = |T_k| = t$ and $\Gamma^{T_1}_{L(H_1)+c_1(M)R_{T_1}}(x) = \Gamma^{T_2}_{L(H_2)+c_2(M)R_{T_2}}(x) = \cdots = \Gamma^{T_k}_{L(H_k)+c_k(M)R_{T_k}}(x)$, then $G \tilde{\circledast}_{[M:\mathcal{T}]} \mathcal{H}_k$ and $G \tilde{\circledast}_{[M:\mathcal{T}']} \mathcal{H}'_k$ are $L-$cospectral.

Proof:

(1) By Corollary 3.2(1),

$$\begin{aligned} P_{G\tilde{\circledast}_{[M:\mathcal{T}]}\mathcal{H}_k}(x) &= \left\{\prod_{j=1}^{k} P_{H_j}(x)\right\} \times \left\{\prod_{i=1}^{n}\Big(x - \lambda_i(G) - \Gamma_{H_1}^{T_1}(x)\lambda_i(MM^T)\Big)\right\} \\ &= \left\{\prod_{j=1}^{k} P_{H_j}(x)\right\} \times \left\{\prod_{i=1}^{n}\Big(x - \lambda_i(G) - \Gamma_{H_{\theta(1)}}^{T_{\theta(1)}}(x)\lambda_i(MM^T)\Big)\right\} \\ &= P_{G\tilde{\circledast}_{[M:\mathcal{T}']}\mathcal{H}'_k}(x). \end{aligned}$$

(2) Proof is similar to the proof of part (1).

□

In the following result, we show that, if we replace the graph H_j by some $A-$cospectral graph H'_j whose coronal is same as the coronal of H_j, for each $j = 1, 2, \dots, k$, then the $A-$spectrum of $G\tilde{\circledast}_{[M:\mathcal{T}]}\mathcal{H}_k$ remains unchanged.

Corollary 4.4 *Let $\mathcal{H}_k = (H_1, H_2, \dots, H_k)$ and $\mathcal{H}'_k = (H'_1, H'_2, \dots, H'_k)$ be two sequences of k graphs. Let $\mathcal{T} = (T_1, T_2, \dots, T_k)$ and $\mathcal{T}' = (T'_1, T'_2, \dots, T'_k)$, where $T_j \subseteq V(H_j)$, $T'_j \subseteq V(H'_j)$, for $j = 1, 2, \dots, k$ be such that H_j and H'_j are $A-$cospectral and $\Gamma_{H_j}^{T_j}(x) = \Gamma_{H'_j}^{T'_j}(x)$, for $j = 1, 2, \dots, k$. Then $G\tilde{\circledast}_{[M:\mathcal{T}]}\mathcal{H}_k$ and $G\tilde{\circledast}_{[M:\mathcal{T}']}\mathcal{H}'_k$ are $A-$cospectral.*

Proof: Since $P_{H_j}(x) = P_{H'_j}(x)$ and $\Gamma_{H_j}^{T_j}(x) = \Gamma_{H'_j}^{T'_j}(x)$, for $j = 1, 2, \dots, k$, so by Theorem 3.3, we get the result. □

The following two results directly follows from Theorem 3.3.

Corollary 4.5 *If G and G' are $A-$cospectral graphs with n vertices and m edges and M is any one of the corona generating matrices mentioned in Table 2, then the graphs $G\tilde{\circledast}_{[M:\mathcal{T}]}\mathcal{H}_k$ and $G'\tilde{\circledast}_{[M:\mathcal{T}]}\mathcal{H}_k$ are $A-$cospectral.*

Corollary 4.6 *If M is one of the corona generating matrices mentioned in Table 2, then we have the following:*

(1) Let $\mathcal{H}_k = (H_1, H_2, \dots, H_k)$ and $\mathcal{H}'_k = (H'_1, H'_2, \dots, H'_k)$ be two sequence of k graphs such that H_j and H'_j for $j = 1, 2, \dots, k$ are $L-$cospectral. Then $G\tilde{\circledast}_M\mathcal{H}_k$ and $G\tilde{\circledast}_M\mathcal{H}'_k$ are $L-$cospectral.

(2) If G and G' are $L-$cospectral graphs, then $G\tilde{\circledast}_M\mathcal{H}_k$ and $G'\tilde{\circledast}_M\mathcal{H}_k$ are $L-$cospectral.

5. Concluding remarks

One of the main contribution of this paper is that it devised a method of computing the generalized characteristic polynomial of the $M-$generalized corona of graphs constrained by vertex subsets by using the corona generating matrix M (ref. Theorem 3.3) and as a consequence it simplifies the repetitive process of determining the characteristic polynomials of the adjacency and the Laplacian matrices of all the existing corona (except extended corona and extended neighbourhood corona) as well as the newly defined corona of graphs, since these corona of graphs are particular cases of the $M-$generalized corona of graphs constrained by vertex subsets by taking suitable M.

We can derive the characteristic polynomial of the signless Laplacian matrix of the graph $G\tilde{\circledast}_{[M:\mathcal{T}]}\mathcal{H}_k$, by an analogous method used in the proof of Theorem 3.3. So the results proved in the Section 3 can be proved analogously for the signless Laplacian matrix of these graphs (with an additional constraint in Corollary 3.7 by assuming G is regular).

The number of spanning trees and the Kirchhoff index of variants of corona of graphs defined in Definitions (1)–(13) in Table 2 can be obtained from Corollary 3.7 by substituting suitable corona generating matrices M and using the entities given in Table 4.

The determination of the characteristic polynomials of the other graph matrices such as normalized Laplacian matrix and distance matrix of the variants of corona of graphs introduced in this paper are further research problems in this direction.

In [20], authors introduced a more generalized construction of graphs which includes the construction of $M-$generalized corona of graphs constrained by vertex subsets, the extended corona of graphs, and the extended neighbourhood corona of graphs, and studied its spectral properties.

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