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Generalized (σ,τ)-derivations on Associative Rings Satisfying Certain Identities

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ABSTRACT: Several results pertaining to the generalized (σ,τ)-derivation D connected with the derivation d of the semiprime ring and prime ring R, where the mappings σ and τ operate as automorphism mappings, are the primary focus of this work. On R, D and d have zero power values.
There are two sections in this article. The generalized (σ,τ)-derivation D related to the derivation d of the semiprime ring and prime ring R is the focus of the first part. In the second part, we examine how the compositions of generalized (σ,τ)-derivations of the semiprime ring and prime ring R affect R, for some positive integer n, such that D has period $(n-1)$.

Key Words: Generalized (σ,τ)-derivation, semiprime ring, automorphism mapping, zero power-valued mapping, compositions of generalized (σ,τ)-derivations.

Contents

1 Introduction **1**

2 Some preliminaries **2**

3 On Generalized (σ,τ)-derivation of Semiprime Rings **3**

4 The Compositions of Generalized (σ,τ)-derivations with Their Applications **13**

1. Introduction

One of the most basic concerns in ring theory is how to find circumstances that imply the ring's commutativity. Several authors have focused on the commutativity of associative rings with derivations over the last two decades, and significant research has been conducted in this area. The concept of derivations and automorphisms of associated rings represents a significant step forward in the development of classical Galois theory and invariant theory. Commutative ring theory is crucial to analysis, algebraic geometry, and algebra.
The study of the Krull dimension of rings and modules, where the idea of the Krull dimension of commutative rings was first conceived by E. Noether and W. Krull in the 1920s, is where the study of generalized derivations of partially ordered sets had its start. Indeed, there are a number of uses for (σ,τ)-derivations that can aid in the development of a method for deforming Lie algebras and that have several uses in the analysis of complex systems and the modeling of quantum processes. In pure algebra, the map has been studied in great detail. Banach algebra theory has recently been studied [1].

The research now under publication has a number of findings pertaining to centralizing and commuting maps on rings. In essence, derivation research got its start in the 1950s and 1960s. The existence of a non-zero centralizing derivation on a prime ring necessitates the ring to be commutative (known as Posner's Second Theorem), according to E. C. Posner [2], who was the first to examine centralizing mappings. The aforementioned conclusion was attempted to be generalized by J. Vukman [3], who confirmed that R is commutative if R is a 2-torsion free prime ring and $d\colon R \to R$ is a non-zero derivation such that the map $x \to [d(x), x]$ is commuting on R.

A derivation d that satisfies the criteria $[d(x^2), d(y^2)] - [x, y] \in Z(R)$ is admitted if R is a 2-torsion free semiprime ring and U is a non-zero ideal of R, as demonstrated by Atteya [4], then R has a non-zero central ideal for every $x, y \in U$. Under specific conditions on a prime ring R, any Jordan (σ,τ)-higher

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derivation of R is a (σ,τ)-higher derivation of R, as demonstrated by M. Ashraf, A. Khan, and C. Haetinger [5]. B. Dhara and A. Pattanayak [6] demonstrated that an additive mapping $D\colon R\to R$ is a semiprime ring if R is a semiprime ring, U is a non-zero ideal of R, and σ and τ are two epimorphisms of R. is a generalized (σ,τ)-derivation of R if there exists a (σ,τ)-derivation $d\colon R\to R$ such that $D(xy)=D(x)\sigma(y)+\tau(x)d(y)$ for all $x,y\in R$. If $\tau(U)d(U)\neq 0$, then R contains a non-zero central ideal of R if the condition $D[x,y]=\pm(xoy)_{\sigma,\tau}$ holds.

Mehsin J. Atteya's findings [7] also focused on the action of an additive mapping derivation over the prime and semiprime rings, which are associative rings.

On the other hand, a number of findings linking derivations, (σ,τ)-derivations, and generalized derivations to the generalized (σ,τ)-derivation of R were reported by Marubayashi *et al* [8]. The authors specifically examined the commutativity of a prime ring R that admits a generalized (σ,τ)-derivation F, meeting particular requirements like $[F(x),x]_{\sigma,\tau}=0$ for every x in a suitable subset of R, where σ,τ are automorphisms of R. Finally, through their investigation of the concept of an additive mapping, known as zero-power valued mapping with a dependent element of a semiprime ring R, Horana and Atteya [9] produced a number of findings. Here we obtain some results about the generalized (σ,τ)-derivations on prime rings, semiprime rings, and associative rings under specific conditions.

2. Some preliminaries

In this study, an associative ring with centre $Z(R)$ shall be denoted by R. Let x, y, and z be in R. In addition to using the identities $[xy,z]=[x,z]y+x[y,z]$ and $[x,yz]=[x,y]z+y[x,z]$, we write the notation $[y,x]$ for the commutator $yx-xy$ and $x\circ y$ for the anticommutator $xy+yx$. Keep in mind that R is prime if $xRy=0$ suggests $x=0$ or $y=0$, and R is semiprime if $xRx=0$ indicates $x=0$. All prime rings are semiprime rings, but this isn't always the case. If for each $x,y\in R$ $xy=yx$, then R is commutative. The concept of anticommutativity of rings is comparable. When $xy=-yx$ for every $x,y\in R$, the ring R is said to be anticommutative. If, for $x\in R$, $nx=0$ implies $x=0$, then a ring R is n-torsion free. If for every $x\in R$ $[d(x),x^n]=0$, then a map $d\colon R\to R$ is n-commuting on R.

If the Leibniz's rule $d(xy)=d(x)y+xd(y)$ holds for every $x,y\in R$, then an additive map $d\colon R\to R$ is referred to as a derivation. Additionally, if there is an additive mapping d on R such that $D(xy)=D(x)y+xd(y)$ for any $x,y\in R$, then an additive mapping $D\colon R\to R$ is called a generalized derivation.

As an additional incentive, we define an additive mapping $d\colon R\to R$, which is known as a (σ,τ)-derivation, if there are automorphisms $\sigma,\tau\colon R\to R$ such that $d(xy)=d(x)\sigma(y)+\tau(x)d(y)$ for all $x,y\in R$. Additionally, if there are automorphisms $\sigma,\tau\colon R\to R$ and d functions as a (σ,τ)-derivation such that $D(xy)=D(x)\sigma(y)+\tau(x)d(y)$ holds for any $x,y\in R$.
, then $D\colon R\to R$ is referred to as a generalized (σ,τ)-derivation.

A mapping $d\colon R\to R$ preserves S when $S\subseteq R$, provided that $d(S)\subseteq S$. If d preserves S and if there is a positive integer $n(x)>1$ such that $d^{n(x)}(x)=0$ for each $x\in S$, then a mapping $d\colon R\to R$ is zero-power valued on S. If $[x,y]=[d(x),d(y)]$ for all $x,y\in S$, then a mapping $d\colon R\to R$ is strong commutativity-preserving (SCP) on S.

Additionally, if $d^2(x)=x$ for every $x\in R$, then a mapping $d\colon R\to R$ is termed period 2 on R. Without specifically stating them, we will utilize the following fundamental identities:

$$[xy,z]_{\sigma,\tau}=x[y,z]_{\sigma,\tau}+[x,\tau(z)]y=x[y,\sigma(z)]+[x,z]_{\sigma,\tau}y,$$

$$[x,yz]_{\sigma,\tau}=\sigma(y)[x,z]_{\sigma,\tau}+[x,y]_{\sigma,\tau}\sigma(z),$$

$$(xo(yz))_{\sigma,\tau}=(xoy)_{\sigma,\tau}\sigma(z)-\tau(y)(xoz)_{\sigma,\tau}=\tau(y)(xoz)_{\sigma,\tau}+(xoy)_{\sigma,\tau}\sigma(z).$$

where $[x,y]_{(\sigma,\tau)}$ for the commutator $x\sigma(y)-\tau(y)x$ and $(x\circ y)_{(\sigma,\tau)}$ for anti-commutator $x\sigma(y)+\tau(y)x$.

The composition of a generalized (σ,τ)-derivation D, some examples of its applications, and a number of findings on the generalized (σ,τ)-derivation D related to the derivation d of the semiprime ring and prime ring R are presented in this work.

We assume the composition $\sigma \circ D = D \circ \sigma, \tau \circ D = D \circ \tau, \sigma \circ d = d \circ \sigma$ and $\tau \circ d = d \circ \tau$ of R.
Also, we used the well-known fact about the center of semiprime rings:
The center of semiprime ring contains no non-zero nilpotent elements.

We begin with the following known results, on which our derivation subsequently depends:

Lemma 2.1 [10, Proposition 8.5.3, Page 330,] *Let R be a ring. Then every intersection of prime ideals is semiprime. Conversely every semiprime ideal is an intersection of prime ideals.*

Lemma 2.2 [11, Lemma 2.1,] *Let R be a semiprime ring, U a non-zero two-sided ideal of R and $a \in R$ such that $axa = 0$ for all $x \in U$, then $a = 0$.*

Lemma 2.3 [12, Lemma 2.4,] *Let R be a semiprime ring and $a \in R$. Then $[a, [a, x]] = 0$ holds for all $x \in R$ if and only if $a^2, 2a \in Z(R)$.*

Lemma 2.4 [13, Lemma 2,] *Let R be a prime ring. If $a, b, c \in R$ such that $axb = cxa$ for all $x \in R$, then either $a = 0$ or $c = b$.*

Lemma 2.5 [14, Lemma 1.1,] *Let R be a semiprime ring. If $a, b \in R$ such that $axb = 0$ for all $x \in R$ then $ab = ba = 0$.*

3. On Generalized (σ, τ)-derivation of Semiprime Rings

The generalized (σ, τ)-derivation D associated with the derivation d of the semiprime ring and prime ring R has the property of torsion free restricted, where the mappings σ and τ act as automorphisms mappings. We highlight several results in this section.

Theorem 3.1 *Let R be a 2 and 3-torsion free semiprime ring and σ, τ be automorphism mappings of R. If D is a generalized (σ, τ)-derivation which is zero power valued index 2 on R then $d = 0$, where R satisfies the relation $aRb \subset Z(R), a, b \in R$ and $\sigma^2 = \sigma$.*

Proof. From our hypothesis, we have D is zero power valued on R. Then there exists an integer $n(r) > 1$ such that $D^{n(r)}(r) = 0$ for all $r \in R$. Since D is zero power valued index 2 on R, we deduce that $D^2(r) = 0$ for all $r \in R$. Replacing r with rs for all $r, s \in R$, we find that

$$D(D(rs)) = D(D(r)\sigma(s) + \tau(r)d(s)) = 0.$$

We rewrite the above relation as

$$D(D(rs)) = D(D(r)\sigma(s)) + D(\tau(r)d(s)) = 0.$$

After simple calculation, we see that

$$D^2(r)\sigma^2(s) + \tau(D(r))d(\sigma(s)) + D(\tau(r))\sigma(d(s)) + \tau^2(r)d^2(s) = 0.$$

Where $\sigma \circ d = d \circ \sigma$ and $\tau \circ D = D \circ \tau$ of R, we obtain

$$D^2(r)\sigma(s) + D(\tau(r))d(\sigma(s)) + D(\tau(r))d(\sigma(s)) + \tau(r)d^2(s) = 0. \tag{3.1}$$

Due to D is zero power valued over R, we conclude that

$$D(\tau(r))d(\sigma(s)) + D(\tau(r))d(\sigma(s)) + \tau(r)d^2(s) = 0.$$

The knowledge that σ and τ are automorphisms of R is implied by this. $\sigma, \tau \colon R \to R$ are 1-1 and onto in this instance. $(\tau(R) = R; \sigma(R) = R)$: Specifically, we employ $\sigma(s) = q, \tau(r) = p$ in the above relation as σ, τ are automorphisms over R. We discover that

$$2D(p)d(q) + pd^2(q) = 0. \tag{3.2}$$

In (3.2), we substitute q via tq, $t \in R$, we see that

$$2D(p)(d(t)\sigma(q) + \tau(t)d(q)) + pd(d(t)\sigma(q) + \tau(t)d(q)) = 0.$$

Moreover, the left side of this relation imply

$$\begin{aligned} &2D(p)d(t)\sigma(q) + 2D(p)\tau(t)d(q)) + pd^2(t)\sigma^2(q) \\ &+ p\tau(d(t))d(\sigma(q)) + pd(\tau(t))\sigma(d(q)) + p\tau^2(t)d^2(q) = 0. \end{aligned} \tag{3.3}$$

In agreement with (3.2), the first term of (3.3) becomes $-pd^2(t)\sigma(q)$. This cancels with the item $pd^2(t)\sigma^2(q)$ after applying the condition $\sigma^2(q) = \sigma(q)$. (3.3) becomes σ and τ are automorphisms of R. Then (3.3) changes to

$$2D(p)\tau(t)d(q) + p\tau(d(t))d(q) + pd(\tau(t))\sigma(d(q)) + p\tau^2(t)d^2(q) = 0. \tag{3.4}$$

Applying the facts that $\sigma \circ d = d \circ \sigma$ and $\tau \circ D = D \circ \tau$ of R and σ and τ are automorphisms of R. In this case $\sigma, \tau \colon R \to R$ are 1-1 and onto. $(\sigma(R) = R; \tau(R) = R)$: In particular, since σ and τ are automorphisms of R. We employ the relations $\sigma(t) = y, \sigma(q) = e, \tau(t) = x, \tau(x) = w$ in (3.4), we find that

$$2D(p)xd(q) + pd(x)d(q) + pd(x)d(e) + pwd^2(q) = 0.$$

Replacing x by t, we obtain

$$2D(p)td(q) + pd(t)d(q) + pd(t)d(e) + pwd^2(q) = 0.$$

Now, replacing p by x, q by y, w by t and e by y, we deduce

$$2D(x)td(y) + 2xd(t)d(y) + xtd^2(y) = 0. \tag{3.5}$$

Using (3.2) in relation (3.5), we find that

$$2D(x)td(y) + 2xd(t)d(y) - 2xD(t)d(y) = 0. \tag{3.6}$$

Writing $x = D(x)$ and applying the hypothesis that $D^2(R) = 0$, we see that

$$2D(x)d(t)d(y) - 2D(x)D(t)d(y) = 0. \tag{3.7}$$

According to (3.2), we rewrite (3.7) as follows

$$-xd^2(t)d(y) + D(x)td^2(y) = 0.$$

Replacing x with $D(x)$ and using the fact $D^2(R) = 0$, we conclude that

$$D(x)d^2(t)d(y) = 0. \tag{3.8}$$

In (3.8) replacing y with ys, $s \in R$ and applying the result imply

$$D(x)d^2(t)yd(s) = 0.$$

Replacing y with $yD(x)$ and s with $d(t)$ and using the semiprimeness of R, we obtain

$$D(x)d^2(t) = 0. \tag{3.9}$$

In (3.2) we set $y = d(y)$, we show that

$$2D(x)d^2(y) + xd^3(y) = 0.$$

Applying (3.9) and using the semiprimeness of R, we find that

$$d^3(y) = 0.$$

In this result replacing y via xy, we obtain

$$3d^2(x)d(y) + 3d(x)d^2(y) = 0,$$

for all $x, y \in R$.
Substitution x via $d(x)$ and employing R is 3-torsion free, we conclude that

$$d^3(x)d(y) + d^2(x)d^2(y) = 0,$$

Due to the result $d^3(y) = 0$ for all $x \in R$. The first term becomes zero, that means the above relation reduces to

$$d^2(x)d^2(y) = 0.$$

Applying the knowledge that the centre of the semiprime ring contains no non-zero nilpotent elements and the knowledge that $aRb \subset Z(R), a, b \in R$, we arrive at the conclusion that

$$d^2(y)Rd^2(x) = 0.$$

by multiplying right by $rd^2(x)$, $r \in, R$ and left by $d^2(y)r$, $r \in R$.

$$d^2(x) = 0,$$

as determined by the semiprimeness action of R. Using the finding

$$d^2(x) = 0,$$

and setting $x = xy$, we get

$$2d(x)d(y) = 0.$$

The centre of the semiprime ring contains no non-zero nilpotent elements, and $aRb \in Z(R)$. This is done by applying the fact that R is 2-torsion free and left-multiplying by $d(y)R$ and right-multiplying by $Rd(x)$. Consequently, $d(y)Rd(x) = 0$ results in $d = 0$. Using the same logic as in the preceding part of the proof, we obtain the required result. $\square$

The requirement $aRb \subset Z(R)$ for the results required *i.e.* is demonstrated in the following example, which means that it cannot be excluded from the hypothesis.

Example 3.1 *Let* $R = \{\begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix} \setminus x, y \in \mathbb{F}\}$ *be a ring over a field* $\mathbb{F}$ *such that* x *and* y *are nilpotent index 2 also* y *is an annihilator element. Define the mappings* $g, h\colon R \to R$ *as follows:*

$$g(t) = g(\begin{pmatrix} 0 & 0 \\ n & m \end{pmatrix}) = \begin{pmatrix} 0 & 0 \\ 0 & n \end{pmatrix} \text{ and } h(s) = h(\begin{pmatrix} 0 & 0 \\ p & q \end{pmatrix}) = \begin{pmatrix} 0 & 0 \\ q & 0 \end{pmatrix}$$

for all $t, s \in R, n, m, p, q \in \mathbb{F}$.

Obviously, $guh = \begin{pmatrix} 0 & 0 \\ 0 & n \end{pmatrix} \begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix} \begin{pmatrix} 0 & 0 \\ q & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ nyq & 0 \end{pmatrix}$, *where* $u \in R$.
Thus, we find that

$$[u, guh] = [\begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ nyq & 0 \end{pmatrix}] = \begin{pmatrix} 0 & 0 \\ y^2nq & 0 \end{pmatrix}.$$

Due to $y^2 = 0$ *for all* $y \in \mathbb{F}$ *this matrix reduces to* $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Also, with applying the relation $x^2 = 0$, *we conclude that*

$$[g(t), guh] = [\begin{pmatrix} 0 & 0 \\ 0 & n \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ nyq & 0 \end{pmatrix}] = 0 \text{ and } [h(s), guh] = [\begin{pmatrix} 0 & 0 \\ q & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ nyq & 0 \end{pmatrix}]$$

$= 0.$

Hence, we arrive to $guh \subset Z(R)$*. Let us keep the definition of* g *and* h *with* $u \in R$*. We now suppose*

$R^* = \{\begin{pmatrix} h & g \\ u & 0 \end{pmatrix}, u \in R\}$*, where* R^* *is a ring has no divisors of zero.*

Let $d\colon R^* \to R^*$ *be an additive mapping define as*

$$d(s) = \begin{pmatrix} h & g \\ u & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} h & g \\ u & 0 \end{pmatrix} = \begin{pmatrix} 0 & -g \\ u & 0 \end{pmatrix} \text{ for all } s \in R^*.$$

Clearly, d *is a derivation of* R^**.*

Suppose $\sigma, \tau\colon R^* \to R^*$ *be a mappings defined by*

$$\sigma(r_1) = \sigma\begin{pmatrix} w & z \\ u & 0 \end{pmatrix} = \begin{pmatrix} w & 0 \\ zuw & 0 \end{pmatrix} \text{ and } \tau(r_2) = \tau\begin{pmatrix} h & g \\ u & 0 \end{pmatrix} = \begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix}$$

for all $r_1, r_2 \in R^*$*.*

Moreover, we check whether d *is* (σ, τ)*-derivation on* R^**. Hence, we assume*

$$d(r_1 r_2) = d(r_1)\sigma(r_2) + \tau(r_1)d(r_2),$$

for all $r_1, r_2 \in R^*$*. We consider* $r_1 = \begin{pmatrix} h & g \\ u & 0 \end{pmatrix}$ *and* $r_2 = \begin{pmatrix} w & z \\ u & 0 \end{pmatrix}$*.*

Then from the right-side,

$$= d\begin{pmatrix} h & g \\ u & 0 \end{pmatrix}\sigma\begin{pmatrix} w & z \\ u & 0 \end{pmatrix} + \tau\begin{pmatrix} h & g \\ u & 0 \end{pmatrix} d\begin{pmatrix} w & z \\ u & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -g \\ u & 0 \end{pmatrix}\begin{pmatrix} w & 0 \\ zuw & 0 \end{pmatrix} + \begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -z \\ u & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -gzuw & 0 \\ uw & 0 \end{pmatrix} + \begin{pmatrix} 0 & -hz \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -gzuw & -gz) \\ uw & 0 \end{pmatrix},$$

since $zuw \subset Z(R)$*, then*

$= \begin{pmatrix} -zuwg & -hz \\ uw & 0 \end{pmatrix}$*, where* $z = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$ *and* $w = \begin{pmatrix} 0 & 0 \\ e & c \end{pmatrix} =$ $\begin{pmatrix} 0 & 0 \\ e & 0 \end{pmatrix}$ *for all* $a, b, e, c \in \mathbb{Z}, z, w \in R^\star$ *yields* $zuwg = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ *and* $hz = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$*.*

Then, this matrix reduces to $\begin{pmatrix} 0 & 0 \\ uw & 0 \end{pmatrix}$*.*

While the left-side

$$d(r_1 r_2) = d(\begin{pmatrix} h & g \\ u & 0 \end{pmatrix}\begin{pmatrix} w & z \\ u & 0 \end{pmatrix}) = d(\begin{pmatrix} hw + ug & hz \\ uw & uz \end{pmatrix}) = \begin{pmatrix} 0 & -hz \\ uw & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ uw & 0 \end{pmatrix}.$$

Thus, d *is* (σ, τ)*-derivation of* R^**.*

We now investigate on generalized (σ, τ)*-derivation of* R^**. Let* D *be additive mapping on* R^* *defined*

$D(t) = D(\begin{pmatrix} h & g \\ u & 0 \end{pmatrix}) = \begin{pmatrix} g & 0 \\ u & 0 \end{pmatrix}$*.Then, we check*

$D(r_1 r_1) = D(r_1)\sigma(r_2) + \tau(r_1)d(r_2)$*, for all* $r_1, r_1 \in R^*$*.*

Take $r_1 = \begin{pmatrix} h & g \\ u & 0 \end{pmatrix}$ *and* $r_2 = \begin{pmatrix} w & z \\ u & 0 \end{pmatrix}$,

where $\tau(r_1) = \tau \begin{pmatrix} h & 0 \\ u & 0 \end{pmatrix} = \begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix}$ *and* $\sigma(r_2) = \sigma \begin{pmatrix} w & z \\ u & 0 \end{pmatrix} = \begin{pmatrix} w & 0 \\ zuw & 0 \end{pmatrix}$ *for all* $r_1, r_2 \in R^*$.

The left-side give us

$$D(r_1 r_2) = D(\begin{pmatrix} h & g \\ u & 0 \end{pmatrix} \begin{pmatrix} w & z \\ u & 0 \end{pmatrix}) = D \begin{pmatrix} hw + gu & hz \\ uw & uz \end{pmatrix} = \begin{pmatrix} hz & 0 \\ uw & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ uw & 0 \end{pmatrix}.$$

Furthermore, the right-side provide

$$D(r_1)\sigma(r_2) + \tau(r_1)d(r_2) = D \begin{pmatrix} h & g \\ u & 0 \end{pmatrix} \sigma \begin{pmatrix} w & z \\ u & 0 \end{pmatrix} + \tau \begin{pmatrix} h & g \\ u & 0 \end{pmatrix} d \begin{pmatrix} w & z \\ u & 0 \end{pmatrix}$$

Applying the definitions of the mappings, we find that

$$= \begin{pmatrix} g & 0 \\ u & 0 \end{pmatrix} \begin{pmatrix} w & o \\ zuw & 0 \end{pmatrix} + \begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -z \\ u & 0 \end{pmatrix}$$

$$= \begin{pmatrix} gw & 0 \\ uw & 0 \end{pmatrix} + \begin{pmatrix} 0 & hz \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ uw & 0 \end{pmatrix}.$$

Thus, D is generalized (σ,τ)-derivation of R^*.

We have enough information to determine whether $D^2 = 0$. Therefore,

$$D^2(r_1 r_1) = D^2(r_1)\sigma^2(r_2) + \tau(D(r_1))d(\sigma(r_2)) + D(\tau(r_1))\sigma(d(r_2)) + \tau^2(r_1)d^2(r_2)....(*)$$

$$= D(D \begin{pmatrix} h & g \\ u & 0 \end{pmatrix})\sigma(\sigma \begin{pmatrix} w & z \\ u & 0 \end{pmatrix}) + \tau(D \begin{pmatrix} h & g \\ u & 0 \end{pmatrix})d(\sigma \begin{pmatrix} w & z \\ u & 0 \end{pmatrix}) + D(\tau \begin{pmatrix} h & g \\ u & 0 \end{pmatrix})$$

$$\sigma(d \begin{pmatrix} w & z \\ u & 0 \end{pmatrix}) + \tau(\tau \begin{pmatrix} h & g \\ u & 0 \end{pmatrix})d(d \begin{pmatrix} w & z \\ u & 0 \end{pmatrix})$$

$$= D \begin{pmatrix} g & 0 \\ u & 0 \end{pmatrix} \sigma \begin{pmatrix} w & 0 \\ zuw & 0 \end{pmatrix} + \tau \begin{pmatrix} g & 0 \\ u & 0 \end{pmatrix} d \begin{pmatrix} w & 0 \\ zuw & 0 \end{pmatrix} + D \begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix}$$

$$\sigma \begin{pmatrix} 0 & -z \\ u & 0 \end{pmatrix} + \tau \begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix} d \begin{pmatrix} 0 & -z \\ u & 0 \end{pmatrix}.$$

Consequently, we conclude that

$$= \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix} \begin{pmatrix} w & 0 \\ 0(zuw) & 0 \end{pmatrix} + \begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ zuw & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & z \\ u & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ uw & 0 \end{pmatrix} + \begin{pmatrix} 0 & hz \\ 0 & 0 \end{pmatrix}.$$

$uw = 0$ as a result of the actions of the entries of u and w. Because of this outcome, this matrix becomes zero i.e. $D^2 = 0$.
Now let's demonstrate that $d^2 = 0$ as well.

$$d^2(r_1 r_1) = d^2(r_1)\sigma^2(r_2) + \tau(d(r_1))d(\sigma(r_2)) + d(\tau(r_1))\sigma(d(r_2)) + \tau^2(r_1)d^2(r_2)$$

$$= d(d \begin{pmatrix} h & g \\ u & 0 \end{pmatrix})\sigma(\sigma \begin{pmatrix} w & z \\ u & 0 \end{pmatrix}) + \tau(d \begin{pmatrix} h & g \\ u & 0 \end{pmatrix})d(\sigma \begin{pmatrix} w & z \\ u & 0 \end{pmatrix}) + d(\tau \begin{pmatrix} h & g \\ u & 0 \end{pmatrix})$$

$\sigma(d\begin{pmatrix} w & z \\ u & 0 \end{pmatrix}) + \tau(\tau\begin{pmatrix} h & g \\ u & 0 \end{pmatrix})d(d\begin{pmatrix} w & z \\ u & 0 \end{pmatrix})$.

$= d\begin{pmatrix} 0 & -g \\ u & 0 \end{pmatrix}\sigma\begin{pmatrix} w & 0 \\ zuw & 0 \end{pmatrix} + \tau\begin{pmatrix} 0 & -g \\ u & 0 \end{pmatrix}d\begin{pmatrix} w & 0 \\ zuw & 0 \end{pmatrix} + d\begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix}\sigma\begin{pmatrix} 0 & -z \\ u & 0 \end{pmatrix}$

$+\tau\begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix}d\begin{pmatrix} 0 & -z \\ u & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & g \\ u & 0 \end{pmatrix}\begin{pmatrix} w & 0 \\ 0(zuw) & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ zuw & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & z \\ u & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & g \\ u & 0 \end{pmatrix}\begin{pmatrix} w & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & z \\ u & 0 \end{pmatrix}$.

$= \begin{pmatrix} 0 & 0 \\ uw & 0 \end{pmatrix} + \begin{pmatrix} 0 & hz \\ 0 & 0 \end{pmatrix}$.

Since $uw = 0$ and $hz = 0$, this matrix becomes zero i.e $d^2 = 0$.
Substituting the values of D^2 and d^2 in relation $*$, we find that

$$\tau(D(r_1))d(\sigma(r_2)) + D(\tau(r_1))\sigma(d(r_2)) = 0.$$

Due to σ commute with d this relation modify to

$$(\tau(D(r_1)) + D(\tau(r_1)))d(\sigma(r_2)) = 0$$

for all $r_1, r_2 \in R^*$. Then

$(\tau\begin{pmatrix} g & 0 \\ u & 0 \end{pmatrix} + D\begin{pmatrix} -h & 0 \\ 0 & 0 \end{pmatrix})d\begin{pmatrix} w & 0 \\ zuw & 0 \end{pmatrix} = 0$. *Moreover,*

$(\begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix})d\begin{pmatrix} w & 0 \\ zuw & 0 \end{pmatrix} = \begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix}d\begin{pmatrix} w & 0 \\ zuw & 0 \end{pmatrix} = 0.$

Basically, R^* has no divisors of zero. Hence, we arrive to either

$\begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix} = 0$ *yields contradiction or* $d\begin{pmatrix} w & 0 \\ zuw & 0 \end{pmatrix} = 0$. *Thus $d = 0$.*

Theorem 3.2 *Let R be a 2-torsion free semiprime ring, σ and τ be two automorphisms of R. Suppose that there exists a generalized (σ,τ)-derivation D such that $[D(x), x]_{\sigma,\tau} = 0$ for all $x \in R$. If d is central period 2 on R then D is zero power valued on R.*

Proof. Let us deliver a mapping $\gamma\colon R \times R \to R$ by the relation

$$\gamma(r_1, r_2) = [D(r_1), r_2]_{\sigma,\tau} + [D(r_2), r_1]_{\sigma,\tau}$$

for all $r_1, r_2 \in R$.
It is symmetric and additive in both arguments. Notice that for all $r_1, r_2, z \in R$,

$$\gamma(r_1 r_2, z) = [D(r_1 r_2), z]_{\sigma,\tau} + [D(z), r_1 r_2]_{\sigma,\tau}.$$

Using the definition of (σ,τ)-generalized derivation, we expand the right-hand side as

$$\gamma(r_1r_2,z) = D(r_1)[\sigma(r_2),z]_{\sigma,\tau} + [D(r_1),z]_{\sigma,\tau}\sigma(r_2) + \tau(r_1)[d(r_2),z]_{\sigma,\tau} + [\tau(r_1),z]_{\sigma,\tau}d(r_2) + r_1[D(z),r_2]_{\sigma,\tau} + [D(z),r_1]_{\sigma,\tau}r_2. \tag{3.10}$$

Applying the formulas $\sigma\circ d = d\circ\sigma$ and $\tau\circ D = D\circ\tau$ of R, which are automorphisms of R, $\sigma,\tau\colon R\to R$ are 1-1 and onto in this instance. $\tau(R)=R; \sigma(R)=R$:
In particular, we employ $\sigma(r_2)=y, \tau(r_1)=x$ in the above relation as σ,τ are automorphisms of R. We discover that

$$\gamma(xy,z) = D(x)[y,z]_{\sigma,\tau} + [D(x),z]_{\sigma,\tau}y + x[d(y),z]_{\sigma,\tau} + [x,z]_{\sigma,\tau}d(y) + x[D(z),y]_{\sigma,\tau} + [D(z),x]_{\sigma,\tau}y.$$

Replacing y with xy in the main relation, we find that

$$\gamma(x,xy) = [D(x),xy]_{\sigma,\tau} + [D(xy),x]_{\sigma,\tau} = 0.$$

Further, we conclude that

$$[D(x),xy]_{\sigma,\tau} + [D(x)\sigma(y)+\tau(x)d(y),x]_{\sigma,\tau} = 0.$$

Expanding the left-hand side, we obtain

$$x[D(x),y]_{\sigma,\tau} + [D(x),x]_{\sigma,\tau}y + D(x)[\sigma(y),x]_{\sigma,\tau} + [D(x),x]_{\sigma,\tau}\sigma(y) + \tau(x)[d(y),x]_{\sigma,\tau} + [\tau(x),x]_{\sigma,\tau}d(y) = 0. \tag{3.11}$$

Applying that $\sigma\circ d = d\circ\sigma$ and $\tau\circ D = D\circ\tau$ of R and σ and τ are automorphisms of R. In this case $\sigma,\tau\colon R\to R$ are 1-1 and onto. $(\sigma(R)=R;\tau(R)=R)$: In particular, since σ,τ are automorphisms of R, we use $\sigma(y)=t, \tau(x)=s$ in the (3.11) becomes

$$x[D(x),y]_{\sigma,\tau} + [D(x),x]_{\sigma,\tau}y + D(x)[t,x]_{\sigma,\tau} + [D(x),x]_{\sigma,\tau}t + s[d(y),x]_{\sigma,\tau} + [s,x]_{\sigma,\tau}d(y) = 0$$

for all $t,s\in R$.
Writing $y=d(t)$, $s=d(s)$ and $t=d(t)$ with employing d acts as central mapping, we find that

$$[D(x),x]_{\sigma,\tau}d(t) + [D(x),x]_{\sigma,\tau}d(t) = 0$$

for all $t,s\in R$. Due to R is 2-torsion free, we deduce

$$[D(x),x]_{\sigma,\tau}d(t) = 0. \tag{3.12}$$

Writing this relation as

$$([D(x)\sigma(x) - \tau(x)D(x))d(t) = 0.$$

According to σ,τ are automorphisms of R, we use $\sigma(x)=y$ and $\tau(x)=t$ in this relation, we conclude that

$$(D(x)y - tD(x))d(t) = 0.$$

Now, by substituting y with $-y$ and combining this relation with σ,τ, which are automorphisms of R, we discover that

$$2tD(x)d(t) = 0.$$

Given that R is free of 2-torsion, we deduce that

$$tD(x)d(t) = 0.$$

Substituting this result in the relation $(D(x)y - tD(x))d(t) = 0$. It modifies to

$$D(x)yd(t) = 0.$$

Taking $t = d(t)$ and using d is a period 2, we see that

$$D(x)yt = 0.$$

In this relation, replacing x by $D(x)$ and t by $D^2(x)$ with using the semiprimeness of R, we arrive to $D^2(x) = 0$ for all $x \in R$. This completes the proof. □

When d is non-zero, we can get the same result using the same prime ring method without the condition period 2 of d.

Proposition 3.1 *Let R be a 2-torsion free prime ring, σ and τ be two automorphisms of R. Suppose that there exists a generalized (σ,τ)-derivation D such that $[D(x), x]_{\sigma,\tau} = 0$ for all $x \in R$. If d is non-zero central on R then D is zero power valued index 2 on R.*

Proposition 3.2 *Let σ and τ be two ring automorphisms of R. Suppose and there exists a generalized (σ,τ)-derivation D such that $D(x)[x, y]_{\sigma,\tau} = 0$. Then*

(i) if R is semiprime and U is a maximal ideal of R then either $D(R)$ is commuting on R or $[t, r]_{\sigma,\tau} = 0$ for all $t, r \in R$,

(ii) if R is a 2-torsion free prime ring then either D has zero power valued index 2 on R or $D(R) = 0$,

(iii) if D is a period 2 on prime ring R then $[D(x), x]_{\sigma,\tau} = 0$..

Proof. (i) In the main relation $D(x)[x, y]_{\sigma,\tau} = 0$ for all $x, y \in R$, replacing y by yt, $t \in R$. That gives

$$D(x)\sigma(y)[x, t]_{\sigma,\tau} + D(x)[x, t]_{\sigma,\tau}\sigma(t) = 0. \tag{3.13}$$

Obviously, the second term vanishes in view of the main relation. This leads to

$$D(x)y[x, t]_{\sigma,\tau} = 0$$

for all $x, y, t \in R$. By reason of R is a semiprime ring, so in this expression we write $y = xR$ and $t = D(x)$. Consequently, we arrive to

$$D(x)R[x, t]_{\sigma,\tau} = 0. \tag{3.14}$$

We examine the set $\{P_\alpha\}$ of prime ideals of R such that $\cap P_\alpha = \{0\}$ since R is semiprime. We demonstrate that the intersection $\{P_\alpha\}$ of prime ideals of R is a semiprime ideal, which is consistent with Lemma 2.1. There are no other ideals that fall between U and R since U is a maximal ideal of R.
Hence, we find that $\cap P_\alpha \subseteq U$.
$[w, x] \in P$ or $D(x) \in P$ follows if P is a typical member of $\cap P_\alpha$ and $x \in U$.

$$T_1 = \{x \in U \mid [x, t]_{\sigma,\tau} \in P\}$$

and

$$T_2 = \{x \in U \mid D(x) \in P\},$$

are two additive subgroups that may be constructed, where each ideal of a ring R is a subgroup of the additive group of R. Consequently, $T_1 \cup T_2 = U$. Since two of a group's proper subgroups cannot form a union, either $T_1 = U$ or $T_2 = U$, that is, either

$$[x, t]_{\sigma,\tau} \in P$$

or

$$D(x) \in P.$$

Thus, both situations result in $[x, t]_{\sigma,\tau} \in \cap P_\alpha$ or $D(x) \in \cap P_\alpha$. In other fashion,

$$[x, t]_{\sigma,\tau} \in \cap P_\alpha \subseteq U$$

or

$$D(x) \in \cap P_\alpha \subseteq U.$$

In what follows, we harvest either

$$[x,t]_{\sigma,\tau} \in U$$

for all $x \in U, w \in R$ or

$$D(x) \in U$$

for all $x \in U$.
We divide the proof into two cases.
Case 1: If $[x,t]_{\sigma,\tau} \in U$ for all $x \in U, w \in R$ then

$$[x,t]_{\sigma,\tau} = 0$$

for all $x \in U, t \in R$.
Replacing x with $xr, r \in R$, we find that

$$x[t,r]_{\sigma,\tau} + [t,x]_{\sigma,\tau} r = 0$$

for all $x \in U, t, r \in R$. Applying the relation $[t,x]_{\sigma,\tau} = 0$ to this result, we conclude that

$$x[t,r]_{\sigma,\tau} = 0$$

for all $x \in U, w, r \in R$. We write this relation as follows

$$U[t,r]_{\sigma,\tau} = (0)$$

According to Lemma 2.2, we find that

$$[t,r]_{\sigma,\tau} = 0$$

for all $t, r \in R$.
Case 2: If $D(x) \in U$ for all $x \in R$ then $D(x) = 0$ for all $x \in R$, we arrive to D is commuting on R.
(ii) Suppose R is a prime, we have the relation

$$D(x)y[x,t]_{\sigma,\tau} = 0, \quad x, y, t \in R.$$

Substituting of y by R, we see that

$$D(x)R[x,t]_{\sigma,\tau} = 0.$$

Placing $D(x)$ rather than x, we find that

$$D^2(x)R[D(x),t]_{\sigma,\tau} = 0.$$

Since R is prime, we come to the following results: either $D^2(x) = 0$ that is mean has zero power valued index 2 on R or $[D(x),t]_{\sigma,\tau} = 0$.
Given that σ and τ are automorphisms of R. In this case $\sigma, \tau \colon R \to R$ are 1-1 and onto. $(\sigma(R) = R; \tau(R) = R)$: In particular, since σ, τ are automorphisms of R, we use $\sigma(t) = w, \tau(t) = y$, we conclude that

$$D(x)w = yD(x)$$

for all $w, y \in R$. Putting $y = -y$ and combine with the previous result, we deduce

$$2D(x)w = 0$$

for all $w, x \in R$. Applying the hypothesis that R is a 2-torsion free yields $D(R) = 0$.
(iii) Using the same technique of Branch(ii), we arrive to

$$D^2(x)R[D(x),t]_{\sigma,\tau} = 0.$$

In ducat to D is period 2 on R this relation modifies to

$$xR[D(x), x]_{\sigma,\tau} = 0.$$

By reason of R acts as a prime ring then

$$[D(x), x]_{\sigma,\tau} = 0.$$

Hence, we find the required result. □

Theorem 3.3 *Let R be a 2-torsion free semiprime ring and σ and τ be two automorphisms of R. Suppose that there exists a generalized (σ, τ)-derivation D such that d has zero power valued index 2 on R and $D(xy) = D(yx)$ for all $x, y \in R$. Then $d([x, y]_{\sigma,\tau}) = 0$ for all $x, y \in R$.*

Proof. Suppose $c \in R$ is a constant, *i.e.*, an element such that $D(c) = 0$, and let c be an arbitrary element of R. According to our hypothesis, we see that

$$D(rw) = D(wr)$$

for all $r, w \in R$. Replacing r with c and w with z, we arrive to

$$D(cz) = D(zc)$$

for all $z \in R$. Then

$$D(c)\sigma(z) + \tau(c)d(z) = D(z)\sigma(c) + \tau(z)d(c). \tag{3.15}$$

Applying the fact that $D(c) = 0$ to (3.15), we find that

$$\tau(c)d(z) = \tau(z)d(c). \tag{3.16}$$

For all $p, q \in R$, the commutator $[p, q]_{\sigma,\tau}$ is a constant. Hence from (3.16), we obtain

$$\tau([p, q]_{\sigma,\tau})d(z) = \tau(z)d([p, q]_{\sigma,\tau}), \quad \text{for all } p, q, z \in R.$$

Since τ is automorphism of R. In this case $\tau\colon R \to R$ is 1-1 and onto. $\tau(R) = R$): In particular, since τ is automorphism of R, we use $\tau([p, q]) = [x, y], \tau(z) = t$, this equation becomes

$$[x, y]_{\sigma,\tau}d(t) = td([x, y]_{\sigma,\tau}). \tag{3.17}$$

Replacing t by $d(t)$ and using d has zero power valued index 2 on R, we see that

$$d(t)d([x, y]_{\sigma,\tau}) = 0.$$

Writing $[x, y]_{\sigma,\tau}$ for t, we find that

$$d([x, y]^2_{\sigma,\tau}) = 0.$$

In agreement with Lemma 2.3, we obtain

$$2[d([x, y]_{\sigma,\tau}), r]_{\sigma,\tau} = 0$$

for all $x, y, r \in R$.
Based on the hypothesis that R is 2-torsion free, this relation modifies to

$$[d([x, y]_{\sigma,\tau}), r]_{\sigma,\tau} = 0$$

Furthermore,

$$d([x, y]_{\sigma,\tau}) \in Z(R)$$

Indicate to the center of semiprime ring contains no non-zero nilpotent element, the relation

$d([x, y]^2_{\sigma,\tau}) = 0$ yields $d([x, y]_{\sigma,\tau}) = 0$. □

Theorem 3.4 *Let R be a prime ring, σ and τ be two automorphisms of R. Suppose that there exists a generalized (σ,τ)-derivation D such that d is period 2 of R commute with D and $[D(r_1),D(r_2)]_{\sigma,\tau}=0$ for all $r_1,r_2\in R$. Then either $[D(r_1),r_1]_{\sigma,\tau}=0$ or $d(R)=0$.*

Proof. Replacing r_2 with r_1r_2 in the main relation $[D(r_1),D(r_2)]_{\sigma,\tau}=0$, we obtain

$$[D(r_1),D(r_1)\sigma(r_2)+\tau(r_1)d(r_2)]_{\sigma,\tau}=0$$

for all $r_1,r_2\in R$. Moreover, we find that

$$D(r_1)[D(r_1),\sigma(r_2)]_{\sigma,\tau}+\tau(r_1)[D(r_1),d(r_2)]_{\sigma,\tau}+[D(r_1),\tau(r_1)]_{\sigma,\tau}d(r_2)=0. \tag{3.18}$$

In (3.18), we substitute r_2 with $D(z)$, $z\in R$. Due to σ and τ are automorphisms of R. Applying the same previous technique which used in the proof of Theorem 3.1 and thanks to $[D(x),D(z)]_{\sigma,\tau}=0$, we find that

$$x[D(x),d(D(z))]_{\sigma,\tau}=-[D(x),x]d(D(z)),$$

for all $x,z\in R$. Putting $b=[D(x),d(D(z))]_{\sigma,\tau}$ and $a=-[D(x),x]d(D(z))$ yields $xb=-a$. Left-multiplying by a and right-multiplying by xa, we conclude the following equation

$$ax(bxa)=-a^2xa.$$

According to Lemma 2.4, we conclude that either $a=[D(x),x]_{\sigma,\tau}d(D(z))$ equals to zero for all $x,z\in R$ or $(bxa)=-a^2$. Now we focus on the term

$$[D(x),x]_{\sigma,\tau}d(D(z))=0.$$

Due to d and D commute with each other, we have

$$[D(x),x]_{\sigma,\tau}D(d(z))=0$$

Replacing z by $d(z)$ and using d is period 2 of R, we find that

$$[D(x),x]_{\sigma,\tau}D(z)=0$$

for all $x,z\in R$. Replacing z by yz, we deduce that

$$[D(x),x]_{\sigma,\tau}yd(z)=0.$$

Applying the primeness of R, we complete the proof. □

4. The Compositions of Generalized (σ,τ)-derivations with Their Applications

In [15], Ajda and Mehsin derived a Leibniz's formula for the compositions of generalized (σ,τ)-derivations and some results based on it.

Definition 4.1 *Let D be a generalized (σ,τ)-derivation of a ring R, σ and τ be automorphisms of R such that σ and τ commute with D and d. Then we define the compositions of D as*

$$D^n(xy)=\sum_{r=0}^{n}\binom{n}{r}D^{n-r}(\sigma^{n-r}(x))d^r(\tau^r(y))$$

for all $x,y\in R$, where n and r are a positive integers (we adopt the convention $D^0=d^0=id$).

Theorem 4.1 *Let R be a 2-torsion free prime ring, σ and τ be two automorphisms of R. For some positive integer n, suppose that D is a non zero generalized (σ,τ)-derivation satisfying $D^n(x)\in Z(R)$ for all $x\in R$ and has period $n-1$ of R. Then $[x,y]_{\sigma,\tau}\in Z(R)$ for all $x,y\in R$.*

Proof. From the hypothesis, we have $[D^n(x), r] = 0$ for all $x, r \in R$. Basically we have the relation

$$[D^n(x), x^n]_{\sigma,\tau} = 0$$

for all $x \in R$. Placing x rather than yx, we find that

$$[D^n(xy), (xy)^n]_{\sigma,\tau} = 0$$

for all $x, y \in R$. Now applying the previous definition to this relation, we find that

$$\left[\sum_{r=0}^{n} \binom{n}{r} D^{n-r}(\sigma^{n-r}(x)) d^r(\tau^r(y)), (xy)^n\right]_{\sigma,\tau} = 0,$$

for all $x, y \in R$. Then

$$\Bigg[\binom{n}{0} D^n(\sigma^n(x)) d^0(\tau^0(y)) + \binom{n}{1} D^{n-1}(\sigma^{n-1}(x)) d(\tau(y))$$
$$+\binom{n}{2} D^{n-2}(\sigma^{n-2}(x)) d^2(\tau^2(y)) + \cdots + \binom{n}{n} D^{n-n}(\sigma^{n-n}(x))$$
$$d^n(\tau^n(y)), (xy)^n]_{\sigma,\tau} = 0.$$

From this relation, we obtain

$$[D^n(\sigma^n(x))y + nD^{n-1}(\sigma^{n-1}(x))d(\tau(y)) + \frac{n(n-1)!}{2} D^{n-2}(\sigma^{n-2}(x))d^2(\tau^2(y))$$
$$+ \cdots + xd^n(\tau^n(x)), (xy)^n]_{\sigma,\tau} = 0.$$

We rewrite this relation as a sum of two commutators

$$\Bigg[nD^{n-1}(\sigma^{n-1}(x))d(\tau(y)) + \frac{n(n-1)!}{2} D^{n-2}(\sigma^{n-2}(x))d^2(\tau^2(y))$$
$$+ \cdots + xd^n(\tau^n(x)), (xy)^n]_{\sigma,\tau} + \left[D^n(\sigma^n(x))y, (xy)^n\right]_{\sigma,\tau} = 0.$$

Furthermore,

$$\sum_{r=1}^{n} \binom{n}{r} [D^{n-r}(\sigma^{n-r}(x))d^r(\tau^r(y)), (xy)^n]_{\sigma,\tau} + [D^n(\sigma^n(x))y, (xy)^n]_{\sigma,\tau} = 0. \tag{4.1}$$

Multiplying (4.1) by $t \in R$ on the left and right, we see that

$$\sum_{r=1}^{n} \binom{n}{r} t[D^{n-r}(\sigma^{n-r}(x))d^r(\tau^r(y)), (xy)^n]_{\sigma,\tau} t + t[D^n(\sigma^n(x))y, (xy)^n]_{\sigma,\tau} t = 0.$$

This relation has the form

$$at + tb = 0, \tag{4.2}$$

where we set

$$a = \sum_{r=1}^{n} \binom{n}{r} t[D^{n-r}(\sigma^{n-r}(x))d^r(\tau^r(y)), (xy)^n]_{\sigma,\tau}$$

and

$$b = [D^n(\sigma^n(x))y, (xy)^n]_{\sigma,\tau} t.$$

Multiplying (4.2) by $s \in R$ on the left, we arrive to

$$sat + stb = 0. \tag{4.3}$$

Writing $t = st$ yields in (4.2) gives the identity

$$ast + stb = 0$$

for all $s, t \in R$. Subtracting this result from (4.3), we obtain

$$[s, a]_{\sigma,\tau} t = 0. \tag{4.4}$$

We replace t by st in the definition of a, we find that

$$a = \sum_{r=1}^{n} \binom{n}{r} st[D^{n-r}(\sigma^{n-r}(x))d^r(\tau^r(y)), (xy)^n]_{\sigma,\tau},$$

Hence, (4.4) give us

$$\sum_{r=1}^{n} \binom{n}{r} [s, st[D^{n-r}(\sigma^{n-r}(x))d^r(\tau^r(y)), (xy)^n]_{\sigma,\tau}, (xy)^n]_{\sigma,\tau}]_{\sigma,\tau} t = 0.$$

Replacing s with t and setting

$$h = \sum_{r=1}^{n} \binom{n}{r} [D^{n-r}(\sigma^{n-r}(x))d^r(\tau^r(y)), (xy)^n]_{\sigma,\tau},$$

this relation becomes

$$t^2[t, h]_{\sigma,\tau} t = 0.$$

Multiplying by $t[t, h]_{\sigma,\tau}$ on the left, we conclude that

$$(t[t, h]_{\sigma,\tau} t)^2 = 0.$$

Applying Lemma 2.3, we find that

$$2(t[t, h]_{\sigma,\tau} t) \in Z(R)$$

which implies

$$2[(t[t, h]_{\sigma,\tau} t), r] = 0$$

for all $r \in R$. The fundamental quality of R is that it is devoid of torsion. Clearly,

$$(t[t, h]_{\sigma,\tau} t) \in Z(R).$$

Since there are no non-zero nilpotent elements in the centre of the semiprime ring, we get to

$$t[t, h]_{\sigma,\tau} t = 0.$$

Right-multiplying by $[t, h]_{\sigma,\tau}$, we see that

$$(t[t, h]_{\sigma,\tau})^2 = 0.$$

Repeating the same technique as before to this result, we find that

$$t[t, h]_{\sigma,\tau} = 0. \tag{4.5}$$

Multiplying (4.5) by $[s, r]_{\sigma,\tau}$ on the left, we conclude that

$$[s, r]_{\sigma,\tau} t[t, h]_{\sigma,\tau} = 0$$

for all $s, r \in R$. Employing Lemma 2.5 with replacing s by t and r by h, we find that

$$[t, h]^2_{\sigma,\tau} = 0$$

for every $t \in R$.
Based on Lemma 2.3 and the fact that the center of semiprime ring contains no non-zero nilpotent elements with R has a 2-torsion free property, we obtain

$$[t,h]_{\sigma,\tau} = 0$$

for all $t \in R$.
Clearly, we find that $h \in Z(R)$ yields

$$\sum_{r=1}^{n} \binom{n}{r} [D^{n-r}(\sigma^{n-r}(x))d^r(\tau^r(y)), (xy)^n]_{\sigma,\tau} \in Z(R). \tag{4.6}$$

From (4.6), we obtain

$$\sum_{r=1}^{n} \binom{n}{r} [D^{n-r}(\sigma^{n-r}(x))d^r(\tau^r(y)), r]_{\sigma,\tau} \in Z(R).$$

Again, in the same manner we find from (4.1) that

$$[D^n(\sigma^n(x))y, r]_{\sigma,\tau} \in Z(R).$$

Moreover, applying $D^n(\sigma^n(x)) \in Z(R)$ to this commutator, we find that

$$D^n(\sigma^n(x))[y,r]_{\sigma,\tau} \in Z(R).$$

Consequently,

$$[D^n(\sigma^n(x))[y,r]_{\sigma,\tau}, s]_{\sigma,\tau} = 0$$

for all $x, y, r, s \in R$. Since $D^n(\sigma^n(x)) \in Z(R)$, this relation becomes

$$D^{n-1}(D(\sigma^n(x)))[[y,r]_{\sigma,\tau}, s]_{\sigma,\tau} = 0$$

for each $x, y, r, s \in R$. Due to D is period $n-1$ of R, we conclude that

$$D(\sigma^n(x)))[[y,r]_{\sigma,\tau}, s]_{\sigma,\tau} = 0.$$

According to D is a nonzero generalized (σ,τ)-derivation and using the primeness of R, we find that

$$[[y,r]_{\sigma,\tau}, s]_{\sigma,\tau} = 0.$$

Clearly, this option imply that

$$[y,r]_{\sigma,\tau} \in Z(R).$$

□

We close our paper with the following theorem.

Theorem 4.2 *Let R be a semiprime ring, σ and τ be two automorphisms of R. Suppose D is zero power valued index 2 on R. Then $\prod_{i=0}^{n+1} d^i = 0$, for some positive integer n.*

Proof. From the hypothesis, we have $D^2(R) = 0$. For every $x, y \in R$, we find that

$$D(D(x)\sigma(y) + \tau(x)d(y)) = 0.$$

Moreover,

$$D(x)^2\sigma^2(y) + \tau(D(x))d(\sigma(y)) + D(\tau(x))\sigma(d(y)) + \tau^2(x)d^2(y) = 0.$$

Due to the assumption that D plays the role of zero power valued index 2 on R, this relation reduces to

$$\tau(D(x))d(\sigma(y)) + D(\tau(x))\sigma(d(y)) + \tau^2(x)d^2(y) = 0. \tag{4.7}$$

Where σ and τ commute with D and d then (4.7) supply

$$D(\tau(x))d(\sigma(y)) + D(\tau(x))d(\sigma(y)) + \tau^2(x)d^2(y) = 0.$$

Obviously, we find that

$$2D(\tau(x))d(\sigma(y)) + \tau^2(x)d^2(y) = 0. \tag{4.8}$$

Putting $x = D(x)$ and using the facts $D^2 = 0$ and σ and τ are automorphisms of R. From (4.8), we find that

$$D(x)d^2(y) = 0.$$

Replacing x by $xr, r \in R$ in this relation and using the fact σ and τ are automorphisms of R, we conclude that

$$D(x)rd^2(y) + xd(r)d^2(y) = 0.$$

Writing $r = d^2(y)$ and using $D(x)d^2(y) = 0$, we deduce

$$xd(d^2(y))d^2(y) = 0.$$

Left-multiplying by $d(d^2(y))d^2(y)$ and employing the information that semiprimeness of R, we arrive to

$$d(d^2(y))d^2(y) = 0.$$

Right-multiplying by $d(y)y$ and left-multiplying by $(d^{n+1}(y)d^n(y)d^{n-1}(y)\cdots)$, we conclude that

$$\prod_{i=0}^{n+1} d^i(R) = 0.$$

By reason of R acts as a semiprime ring. This is the required result. □

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