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General and Optimal Decay Result for a viscoelastic Timoshenko beam fixed into a moving base

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ABSTRACT: In this paper we consider a multidimensional thermoviscoelastic system of Bresse type where the heat conduction is given by Green and Naghdi theories. For a wider class of relaxation functions, We show that the dissipation produced by the memory effect is strong enough to produce a general decay results. We establish a general decay results, from which the usual exponential and polynomial decay rates are only special cases.

Key Words: Bresse system, thermoelasticity, relaxation function, general decay, viscoelastic.

Contents

1. Introduction

In this paper, we consider a cantilevered Euler-Bernoulli beam. It is fixed to a base in a translational motion at one end and to a tip mass at its free end. The beam is made of a viscoelastic material. The governing equations of the system are

$$\begin{cases} m\xi_{tt}(t)+\rho_1\int_0^L\left(\xi_{tt}(t)+\omega_{tt}(x,t)\right)dx+m_E\left(\xi_{tt}(t)+\omega_{tt}(L,t)\right)=\Phi(t), \quad t\geq 0,\\ \rho_1\left(\xi_{tt}(t)+\omega_{tt}(x,t)\right)-k\left(\omega_{xx}(x,t)+\psi_x(x,t)\right)=0, \quad \forall(x,t)\in(0,L)\times[0,\infty),\\ \rho_2\psi_{tt}(x,t)-b\psi_{xx}(x,t)+k\left(\omega_x(x,t)+\psi(x,t)\right)+\int_0^t g(t-\tau)\psi_{xx}(\tau)d\tau=0, \quad \forall(x,t)\in(0,L)\times[0,\infty), \end{cases} \tag{1.1}$$

subject to the boundary conditions

$$\begin{cases} \omega(0,t)=\psi(0,t)=0, \quad t\geq 0,\\ -k\left(\omega_x(L,t)+\psi(L,t)\right)=m_E\left(\omega_{tt}(L,t)+\xi_{tt}(L,t)\right), \quad t\geq 0,\\ b\psi_x(L,t)-\int_0^t g(t-\tau)\psi_x(L,\tau)d\tau=-J\psi_{tt}(L,t), \quad t\geq 0, \end{cases} \tag{1.2}$$

and the initial data

$$\begin{cases} \xi(0)=\xi^0, \ \xi_t(0)=\xi^1, \omega(x,0)=\omega^0(x), \ \psi(x,0)=\psi^0(x),\\ \omega_t(x,0)=\omega^1(x), \psi_t(x,0)=\psi^1, \quad x\in(0,L). \end{cases} \tag{1.3}$$

where t denotes the times variable and x is the space variable along the beam of length L, ξ is the displacement of the translational base, w is the beam transversal displacement and ψ is the rotational angle of the beam. The constants ρ_1, ρ_2, b and k are the mass density, the moment mass inertia, the rigidity coefficient (of the crosssection) and the shear modulus of elasticity, respectively. The coefficients m and m_E denote, respectively, the mass of the translational base and the mass attached at the free end of the beam with rotational J. $\Phi(t)$ is the input control.

In the third equation of (1.1), the convolution term represents the viscoelastic damping. This term appears in the constitutive relationship between the stress and the strain according to the Boltzmann

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Principle [23,15,16]. The kernel g is called the relaxation function and will be specified later.

Recently, several stability results of the Euler-Bernoulli beam have been obtained [38,40,44,45]. [21] investigated a similar problem to (1.1)-(1.3), with viscous damping:

$$\begin{cases} MS''(t)+\int_0^L \rho\left(S''(t)+u_{tt}(x,t)\right)dx+m_E\left(S''(t)+u_{tt}(L,t)\right) \\ =f(t), \quad t\geq 0, \\ \rho\left(S''(t)+u_{tt}(x,t)\right)+EIu_{xxxx}(x,t)+Bu_t(x,t) \\ +Cu_{xt}(x,t)+Du_{xxxxt}(x,t)=0, \quad (x,t)\in(0,L)\times\mathbb{R}^+, \end{cases} \tag{1.4}$$

with the following boundary conditions:

$$\begin{cases} u(0,t)=u_x(0,t)=0, \quad t\geq 0, \\ Du_{xxxt}(L,t)+EIu_{xxx}(L,t)=m_E\left(u_{tt}(L,t)+S''(t)\right), \quad t\geq 0, \\ Du_{xxt}(L,t)+EIu_{xx}(L,t)=-Ju_{xtt}(L,t), \quad t\geq 0, \end{cases} \tag{1.5}$$

where $f(t)$ is an external force acting on the base. They obtained exponential stability results.

Regarding the works on translational Euler-Bernoulli beam with internal memory feedback, we would like to mention here, the contribution of [10], [9] who studied the following system:

$$\begin{cases} MS''(t)+\int_0^L \rho\left(S''(t)+u_{tt}(x,t)\right)dx+m_E\left(S''(t)+u_{tt}(L,t)\right) \\ =f(t), \quad t\geq 0, \\ \rho\left(S''(t)+u_{tt}(x,t)\right)+EIu_{xxxx}(x,t)-EI\int_0^t g(t-s)\,u_{xxxx}(s)\,ds \\ +Cu_{xt}(x,t)+Du_{xxxxt}(x,t)=0, \quad (x,t)\in(0,L)\times\mathbb{R}^+, \end{cases} \tag{1.6}$$

with the boundary conditions

$$\begin{cases} u(0,t)=u_x(0,t)=0, \quad t\geq 0, \\ Du_{xxxt}(L,t)+EIu_{xxx}(L,t)-EI\int_0^t g(t-s)\,u_{xxx}(L,s)\,ds \\ =m_E\left(u_{tt}(L,t)+S''(t)\right), \quad t\geq 0, \\ Du_{xxt}(L,t)+EIu_{xx}(L,t)-EI\int_0^t g(t-s)\,u_{xx}(L,s)\,ds \\ =-Ju_{xtt}(L,t), \quad t\geq 0, \end{cases} \tag{1.7}$$

and obtained the exponential decay.

The reader is referred to [46,47,63,27,28,29,26,18,20,19] and the references therein for results related to stabilization of translational Euler-Bernoulli beam using different types of damping terms.

In [39], Khemmoudj and Djaidja considered the following translational viscoelastic Euler-Bernoulli beam

$$\begin{cases} M\ddot{S}(t)+\displaystyle\int_0^L \rho\Big(\ddot{S}(t)+u_{tt}(x,t)\Big)dx+m_E\Big(\ddot{S}(t)+u_{tt}(L,t)\Big) \\ =-Ku_t(0,t)-K_pu(0,t), \quad t\geq 0, \\ \rho\Big(\ddot{S}(t)+u_{tt}(t)\Big)+EIu_{xxxx}(t)-EI\displaystyle\int_0^t g(t-s)u_{xxxx}(s)ds=0, \\ \forall(x,t)\in(0,L)\times\mathbb{R}^+, \end{cases} \tag{1.8}$$

subjected to the boundary conditions

$$\begin{cases} u(0,t)=u_x(0,t)=0, \quad t\geq 0, \\ EIu_{xxx}(L,t)-EI\displaystyle\int_0^t g(t-s)u_{xxx}(L,s)ds=m_E\Big(u_{tt}(L,t)+\ddot{S}(t)\Big), \\ \quad t\geq 0, \\ EIu_{xx}(L,t)-EI\displaystyle\int_0^t g(t-s)u_{xx}(L,s)ds=-Ju_{xtt}(L,t), \quad t\geq 0, \end{cases} \tag{1.9}$$

and initial conditions

$$u(x,0)=u_0(x), \ u_t(x,0)=u_1(x), \ S(0)=S_0, \ S'(0)=S_1, \quad x\in(0,L), \tag{1.10}$$

where the dot "." denotes the derivative with respect to the time t, $S(t)$ is the base motion, $u(x,t)$ is the beam transversal displacement with respect to the base. The coefficients ρ, L, EI, M and m_E are positive constants and represent the linear density of the beam, the length, the bending stiffness, the mass of the translational base, the mass with rotational J attached at the free end of the beam. K and K_p are two positive constants. The expressions $u_0(x)$, $u_1(x)$, S_0 and S_1 are given initial data.

The authors have invistigated the problem (1.8)-(1.10) for the relaxation functions satisfying

$$g'(t) \leq -H(g(t)), \quad \forall t \geq 0, \tag{1.11}$$

where H is a nonnegative function, with $H(0) = H'(0) = 0$ and H is strictly increasing and strictly convex on $]0, k[$ for some $k_0 > 0$. The authors showed a general relation between the decay rate for the energy and that of the relaxation function g without imposing restrictive assumptions on the behavior of g at infinity. On the other hand, a condition of the form (1.11) where H is a convex function satisfying some smoothness properties, was introduced by [2] and used then by several authors with different approaches. We refer to [43] where not only general but also optimal result was established by the authors.

Under the assumptions of wider classes of kernel functions, the authors established an optimal explicit energy decay result from which they can recover the exponential and polynomial decay.

Our aim in this work, is to establish a control force $f(t)$ capable of driving the beam displacement to rest at a general decay rate for a class of kernels g satisfying the condition

$$g'(t) \leq -\zeta(t) H\left(g(t)\right), \quad \forall t \geq 0,$$

where H is an increasing and convex function near the origin and $\zeta(t)$ is a nonincreasing function with only these very general assumptions on the behavior of gat infinity, we establish optimal explicit and general energy decay results from which we can recover the optimal exponential and polynomial rates.

Our work is organized as follows. In section 2, we transform the system (1.8) − (1.10) into a simple form to study the stability of the problem. Furthermore, we state, the well posedness of our problem and we give some formules and lemmas, which will be used throughout this article. In Section 3, firstly, we introduce some notations, some assumptions on the kernel g to prouve the main result.

2. Preliminary Results

We start by defining the total deflection of the beam as follows

$$\varphi(x,t) = \xi_1(t) + \omega(x,t), \quad t \geq 0, \tag{2.1}$$

where

$$\xi_1(t) = \xi(t) - \xi_d, \quad \xi_d > 0, \tag{2.2}$$

and ξ_d is the point at which the been regulated. Thus, for all x in $(0,L)$ and $t \geq 0$, we have

$$\begin{cases} \frac{d}{dt}\xi_1(t) = \frac{d}{dt}\xi(t), \ \frac{d^2}{dt^2}\xi_1(t) = \frac{d^2}{dt^2}\xi(t), \\ \varphi(0,t) = \xi_1(t), \ \varphi_t(0,t) = \frac{d}{dt}\xi_1(t), \ \varphi_{tt}(0,t) = \frac{d^2}{dt^2}\xi_1(t), \\ \varphi_t(x,t) = \frac{d}{dt}\xi_1(t) + \omega_t(x,t), \quad \varphi_{tt}(x,t) = \frac{d^2}{dt^2}\xi_1(t) + \omega_{tt}(x,t), \\ \varphi_x(x,t) = \omega_x(x,t), \quad \varphi_{xx}(x,t) = \omega_{xx}(x,t), \\ \varphi(x,0) = \xi^0 + \omega^0(x) = \varphi^0(x), \quad \varphi_t(x,0) = \xi^1 + \omega^1(x) = \varphi^1(x). \end{cases} \tag{2.3}$$

Now, using equation (2.1)-(2.3), the problems (1.1)–(1.3) are transformed into the following one

$$\begin{cases} m\varphi_{tt}(0,t) - k\varphi_x(0,t) = \Phi(t), \quad t \geq 0, \\ \rho_1\varphi_{tt} - k(\varphi_{xx} + \psi_x) = 0, \quad \forall (x,t) \in (0,L) \times [0,\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + \int\limits_0^t g(t-\tau)\psi_{xx}(\tau)d\tau + k\left(\varphi_x + \psi\right) = 0, \quad \forall (x,t) \in (0,L) \times [0,\infty), \end{cases} \tag{2.4}$$

with the boundary conditions

$$\begin{cases} \psi(0,t)=0, \\ -k\left[\varphi_x(L,t)+\psi(L,t)\right]=m_E\varphi_{tt}(L,t), \\ b\psi_x(L,t)-\int_0^t g(t-\tau)\psi_x(L,\tau)d\tau=-J\psi_{tt}(L,t), \quad \forall t\in[0,\infty), \end{cases} \tag{2.5}$$

and the initial data

$$\varphi(x,0)=\varphi^0(x),\ \psi(x,0)=\psi^0(x),\ \varphi_t(x,0)=\varphi^1(x),\ \psi_t(x,0)=\psi^1(x), \quad x\in(0,L). \tag{2.6}$$

Throughout this paper, we denote by $\square$ the binary operator, defined by

$$(f\square v)(t):=\int\limits_0^t f(t-\tau)\left(v(x,t)-v(x,\tau)\right)^2 d\tau, \quad t\geq 0.$$

By using the Faedo-Galerkin method (see e.g. Refs. [1,4,13,14,64,57,59,60]), we can prove the existence and uniqueness of a solution for our problem. Let

$$V=\left\{z : z\in H^1(0,L),\ z(0)=0\right\},$$

where $H^1(0,L)$ is the usual Sobolev space.

Proposition 2.1 *For $\left(\varphi^0,\varphi^1\right)\in H^1(0,L)\times L^2(0,L)$ and $\left(\psi^0,\psi^1\right)\in V\times L^2(0,L)$, under the control $\Phi(t)$ defined in (2.12), the closed-loop systems (2.4) and (2.5) has a unique global (weak) solution (φ,ψ), such that*

$$\varphi\in C(\mathbb{R}^+;H^1(0,L))\times C^1(\mathbb{R}^+;L^2(0,L)),$$
$$\psi\in C(\mathbb{R}^+;V)\times C^1(\mathbb{R}^+;L^2(0,L)).$$

Lemma 2.1 *We have*

$$\delta\gamma\leq\eta\delta^2+\frac{\gamma^2}{4\eta}, \quad \delta,\ \gamma\in\mathbb{R}, \quad \eta>0. \tag{2.7}$$

We shall use the following inequalities.

Lemma 2.2 *Let u be a function in $H^1(0,L)$ that satisfies the boundary condition $u(0)=0$, then, the following inequalities hold:*

$$\|u\|_2^2\leq L^2\|u_x\|_2^2 \tag{2.8}$$

and

$$u^2(x)\leq L\|u_x\|_2^2 \quad \forall x\in[0,L], \tag{2.9}$$

where $\|.\|_2$ is the norm in $L^2(0,L)$.

Lemma 2.3 *The classical energy of the problem (2.4)-(2.6) defined by*

$$\begin{aligned} 2E(t) \;&=\; m\varphi_t^2(0,t)+m_E\varphi_t^2(L,t)+J\psi_t^2(L,t)+\rho_1\|\varphi_t\|_2^2+\rho_2\|\psi_t\|_2^2 \\ &\quad+\left(b-\int\limits_0^t g(\tau)d\tau\right)\|\psi_x\|_2^2+k\|\varphi_x+\psi\|_2^2+\int\limits_0^L(g\square\psi_x)dx, \end{aligned} \tag{2.10}$$

satisfies

$$E'(t)=\frac{1}{2}\int\limits_0^L(g'\square\psi_x)dx-\frac{g(t)}{2}\|\psi_x\|_2^2+\Phi(t)\varphi_t(0,t), \quad t\geq 0. \tag{2.11}$$

Our assumptions on the kernel g are the following:

(A1) The kernel g is a continuously differentiable nonnegative function satisfying

$$b - l := b - \int_0^{+\infty} g(\tau) d\tau > 0.$$

(A2) $g'(t) \leq 0, \quad \text{a.e.} \quad t \geq 0.$

(A3) There exists a C^1 function $H : (0, \infty) \to (0, \infty)$ which is linear or it is strictly increasing and strictly convex C^2 function on $(0, r]$, $r \leq g(0)$, with $H(0) = H'(0) = 0$, such that

$$g'(t) \leq -\zeta(t) H(g(t)), \quad t \geq 0,$$

where ζ is a positive nonincreasing differentiable function.

Let $t_* > 0$ be a number such that $\int_0^{t_*} g(\tau) d\tau = g_* > 0$.

In order to stabilize the system (2.4)-(2.6) to the equilibrium state, we suggest the control $\Phi(t)$ as follows:

$$\Phi(t) = -K\varphi_t(0, t) + \varphi(0, t), \quad t \geq 0, \tag{2.12}$$

where K is a positive constant.

Remark 1 *1. The well-known Jensen's inequality will be of essential use in estabishing our main result. If F is a convex function on $[a, b]$, $f : [0, L] \to [a, b]$ and h are integrable functions on $[0, L]$, $h(x) \geq 0$, and $\int_0^L h(x) dx = k > 0$, then Jensen's inequality states that*

$$F\left[\frac{1}{k}\int_0^L f(x)h(x)dx\right] \leq \frac{1}{k}\int_0^L F\left[f(x)\right] h(x)dx.$$

2. We easily deduce, by (A), that $\lim_{t \to +\infty} g(t) = 0$. Hence, there is $t_1 \geq 0$ large enough such that

$$g(t_1) = r \Rightarrow g(t) \leq r, \quad \forall t \geq t_1. \tag{2.13}$$

As g and ζ are positive nonincreasing continuous functions and H is positive continuous function, then, for all $t \in [0, t_1]$

$$\left\{ \begin{array}{l} 0 < g(t_1) \leq g(t) \leq g(0) \\ 0 < \zeta(t_1) \leq \zeta(t) \leq \zeta(0) \end{array} \right. \Rightarrow a \leq \zeta(t) H(g(t)) \leq b,$$

for some positive constants a and b. Consequently, for all $t \in [0, t_1]$,

$$g'(t) \leq -\zeta(t) H(g(t)) \leq -\frac{a}{g(0)} g(0) \leq -\frac{a}{g(0)} g(t). \tag{2.14}$$

3. If H is a strictly increasing and strictly convex C^2 function on $]0, r]$, with $H(0) = H'(0) = 0$, then it has an extension $\overline{H}$, which is strictly increasing and strictly convex C^2 function on$]0, \infty[$. For instance, if $H(r) = a$, $H'(r) = b$, $H''(r) = c$, we can define $\overline{H}$, for $t > r$, by

$$\overline{H}(t) = \frac{c}{2}t^2 + (b - cr)t + \left(a + \frac{c}{2}r^2 - br\right). \tag{2.15}$$

3. Asymptotic Behavior

Our aim, in this section, is to prove an arbitrary decay of solutions for the system (2.4)-(2.6) under the control $\Phi(t)$ applied at the base motion of the structure. To this end, we set

$$L(t) := E\left(t\right) + \frac{1}{2}\varphi^2(0,t) + \sum_{i=1}^{5} \lambda_i I_i\left(t\right), \quad t \geq 0, \tag{3.1}$$

where λ_i, $i = 1, ..., 5$, are positive constants which will be chosen later and

$$\begin{aligned}
I_1\left(t\right) &= -\int\limits_0^L \left(\rho_1\varphi_t\varphi + \rho_2\psi_t\psi\right) dx - m\varphi_t\left(0,t\right)\varphi(0,t) - m_E\varphi_t\left(L,t\right)\varphi(L,t) - J\psi_t\left(L,t\right)\psi(L,t), \quad t \geq 0,\\
I_2\left(t\right) &= -\rho_2\int\limits_0^L \psi_t(t)\int\limits_0^t g(t-\tau)\left(\psi\left(t\right) - \psi\left(\tau\right)\right) d\tau dx - J_\psi\psi_{xt}(L,t)\int\limits_0^t g(t-\tau)\left(\psi_x\left(L,t\right) - \psi_x\left(L,\tau\right)\right) d\tau, \quad t \geq 0,\\
I_3\left(t\right) &= \rho_2\int\limits_0^L \psi_t(\varphi_x + \psi)dx + \frac{b\rho_1}{k}\int\limits_0^L \varphi_t\psi_x dx - \frac{\rho_1}{k}\int\limits_0^L \varphi_t(t)\int\limits_0^t g(t-\tau)\psi_x\left(\tau\right) d\tau dx + J\psi_t\left(L,t\right)\psi\left(L,t\right), \quad t \geq 0,\\
I_4\left(t\right) &= \rho_2\int\limits_0^L \psi_t m(x)\left(b\psi_x - \int\limits_0^t g(t-\tau)\psi_x\left(\tau\right) d\tau\right) dx, \quad t \geq 0
\end{aligned}$$

and

$$I_5\left(t\right) = \rho_1\int\limits_0^L m(x)\varphi_x\varphi_t dx, \quad t \geq 0,$$

where m is a function in $C^1\left([0,L]\right)$ satisfying $m(0) = -m(L) = 2$, e.g.; $m(x) = 2 - \frac{4}{L}x$. Now, the following proposition shows that $E(t) + \varphi^2(0,t)$ and $L(t)$ are equivalent.

Proposition 3.1 *There exist two positive constants σ_1, σ_2 such that*

$$\sigma_1\left(E\left(t\right) + \varphi^2(0,t)\right) \leq L(t) \leq \sigma_2\left(E\left(t\right) + \varphi^2(0,t)\right), \quad t \geq 0,$$

provided that λ_i, $i = 1, ..., 5$, are sufficiently small.

Proof: By using (2.7), (2.8), (2.9) and Hölder's inequality, we get

$$\begin{aligned}
I_1\left(t\right) \leq{}& \frac{\rho_1}{2}\left\|\varphi_t\right\|_2^2 + \frac{\rho_1}{2}\left\|\varphi\right\|_2^2 + \frac{\rho_2}{2}\left\|\psi_t\right\|_2^2 + \frac{\rho_2 L^2}{2}\left\|\psi_x\right\|_2^2 + \frac{m}{2}\varphi_t^2(0,t) + \frac{m}{2}\varphi^2(0,t)\\
&+ \frac{m_E}{2}\varphi_t^2(L,t) + \frac{m_E}{2}\varphi^2(L,t) + \frac{J}{2}\psi_t^2(L,t) + \frac{J}{2}\psi^2(L,t), \quad t \geq 0.
\end{aligned}$$

In view of (2.1), we see that $\varphi(x,t) = \xi_1(t) + \omega(x,t) = \varphi(0,t) + \omega(x,t)$. Therefore, using (2.8) and (2.9) we get

$$\left\|\varphi\right\|_2^2 \leq 2L\varphi^2(0,t) + 2\left\|\omega\right\|_2^2 \leq 2L\varphi^2(0,t) + 2L^2\left\|\omega_x\right\|_2^2, \quad t \geq 0$$

and

$$\varphi^2(L,t) \leq 2\varphi^2(0,t) + 2\omega^2(L,t) \leq 2\varphi^2(0,t) + 2L\left\|\omega_x\right\|_2^2, \quad t \geq 0.$$

Since $\varphi_x(x,t) = \omega_x(x,t)$,using (2.8) and (2.9) and the fact

$$\left\|\varphi_x\right\|_2^2 \leq 2\left\|\varphi_x + \psi\right\|_2^2 + 2L^2\left\|\psi_x\right\|_2^2, \quad t \geq 0,$$

we obtain

$$\begin{aligned} I_1(t) \leq\; & \frac{\rho_1}{2}\|\varphi_t\|_2^2 + \frac{\rho_2}{2}\|\psi_t\|_2^2 + \left(\frac{\rho_2 L}{2} + 2\rho_1 L^3 + m_E L^2 + \frac{J}{2}\right) L\psi_x^2(0,t) \\ & + \frac{m}{2}\varphi_t^2(0,t) + \frac{J}{2}\psi_t^2(L,t) + \left(\frac{m}{2} + \rho_1 L + m_E\right)\varphi^2(0,t) + \frac{m_E}{2}\varphi_t^2(L,t) \\ & + (L\rho_1 + m_E)\,2L\,\|\varphi_x + \psi\|_2^2 . \end{aligned} \tag{3.2}$$

Similarly, using (2.8) and (2.9), we arrive at

$$I_2(t) \leq \frac{\rho_2}{2}\|\psi_t\|_2^2 + \frac{J}{2}\psi_t^2(L,t) + \frac{L}{2}(\rho_2 L + J)\left(\int\limits_0^t g(\tau)d\tau\right)\int\limits_0^L (g\Box\psi_x)\,dx, \quad t \geq 0. \tag{3.3}$$

Now, we pass to $I_3(t)$,

$$\begin{aligned} I_3(t) =\; & \rho_2\int\limits_0^L \psi_t(\varphi_x + \psi)\,dx + \frac{b\rho_1}{k}\int\limits_0^L \varphi_t\psi_x dx - \frac{\rho_1}{k}\int\limits_0^L \varphi_t\int\limits_0^t g(t-\tau)\Big(\psi_x(\tau)d\tau dx \\ & + J\psi_t(L,t)\psi(L,t) \\ =\; & \rho_2\int\limits_0^L \psi_t(\varphi_x + \psi)\,dx + \frac{\rho_1}{k}\int\limits_0^L \varphi_t\int\limits_0^t g(t-\tau)\Big((\psi_x(t) - \psi_x(\tau))\,d\tau dx \\ & + J\psi_t(L,t)\psi(L,t) + \frac{\rho_1}{k}\left(b - \int\limits_0^t g(\tau)d\tau\right)\int\limits_0^L \varphi_t\psi_x dx, \quad t \geq 0. \end{aligned}$$

Therefore, (2.8) and (2.9) imply that

$$\begin{aligned} I_3(t) \leq\; & \frac{\rho_2}{2}\|\psi_t\|_2^2 + \frac{\rho_2}{2}\|\varphi_x + \psi\|_2^2 + \frac{\rho_1}{2k}\left(\int\limits_0^t g(\tau)d\tau\right)\int\limits_0^L (g\Box\psi_x)\,dx + \frac{J}{2}\psi_t^2(L,t) \\ & + \frac{\rho_1}{2k}\left[1 + \left(b - \int\limits_0^t g(\tau)d\tau\right)^2\right]\|\varphi_t\|_2^2 + \left(\frac{\rho_1}{2k} + \frac{JL}{2}\right)\|\psi_x\|_2^2, \quad t \geq 0. \end{aligned} \tag{3.4}$$

For $I_4(t)$, we can write

$$\begin{aligned} I_4(t) \leq\; & \rho_2\|\psi_t\|_2^2 + \rho_2\int\limits_0^L \left(b\psi_x(t) - \int\limits_0^t g(t-\tau)\psi_x(\tau)d\tau\right)^2 dx \\ \leq\; & \rho_2\left\{\|\psi_t\|_2^2 + \int\limits_0^L \left[\int\limits_0^t g(t-\tau)(\psi_x(t) - \psi_x(\tau))\,d\tau + \left(b - \int\limits_0^t g(\tau)d\tau\right)\psi_x\right]^2 dx\right\} \\ \leq\; & \rho_2\|\psi_t\|_2^2 + 2\rho_2\left[\left(\int\limits_0^t g(\tau)d\tau\right)\int\limits_0^L (g\Box\psi_x)\,dx + \left(b - \int\limits_0^t g(\tau)d\tau\right)^2\|\psi_x\|_2^2\right], \quad t \geq 0. \end{aligned} \tag{3.5}$$

Finally, for $I_5(t)$, it holds that

$$\begin{aligned} I_5(t) \leq\; & \rho_1\left(\|\varphi_t\|_2^2 + \|\varphi_x\|_2^2\right) \\ \leq\; & \rho_1\|\varphi_t\|_2^2 + 2\rho_1\left(\|\varphi_x + \psi\|_2^2 + L^2\|\psi_x\|_2^2\right), \quad t \geq 0. \end{aligned} \tag{3.6}$$

Gathering equation (3.2)-(3.6), we end up with

$$\begin{aligned}
L(t) \leq{} & \left\{ 1+\lambda_1+\frac{\lambda_3}{k}\left[1+\left(b-\int\limits_0^t g(\tau)d\tau\right)^2\right]+2\lambda_5\right\}\frac{\rho_1}{2}\left\|\varphi_t\right\|_2^2 \\
&+\left(1+\lambda_1+\lambda_2+\lambda_3+2\lambda_4\right)\frac{\rho_2}{2}\left\|\psi_t\right\|_2^2+\left[\frac{1}{2}+\lambda_1\left(\frac{m}{2}+\rho_1 L+m_E\right)\right]\varphi^2(0,t) \\
&+\left(1+\lambda_1\right)\frac{m}{2}\varphi_t^2\left(0,t\right)+\left(1+\lambda_1\right)\frac{m_E}{2}\varphi_t^2\left(L,t\right)+\left(1+\lambda_1+\lambda_2+\lambda_3\right)\frac{J}{2}\psi_t^2\left(L,t\right) \\
&+\left[\frac{1}{2}\left(b-\int\limits_0^t g(\tau)d\tau\right)+B_1\right]\left\|\psi_x\right\|_2^2 \\
&+\left[\frac{1}{2}+\lambda_2\left(\frac{\rho_2 L^2}{2}+\frac{JL}{2}\right)l+\lambda_3\frac{\rho_1}{2k}l+2\lambda_4\rho_2 l\right]\int\limits_0^L\left(g\square\psi_x\right)dx \\
&+\left[\frac{k}{2}+\lambda_1\left(L\rho_1+m_E\right)2L+\lambda_3\frac{\rho_2}{2}+2\lambda_5\rho_1\right]\left\|\varphi_x+\psi\right\|_2^2, \quad t\geq 0,
\end{aligned}$$

where

$$\begin{aligned}
B_1 ={} & \lambda_1\left(\frac{\rho_2 L}{2}+2\rho_1 L^3+2m_E L^2+\frac{J}{2}\right)L+\lambda_3\left(\frac{\rho_1}{2k}+\frac{JL}{2}\right) \\
&+2\lambda_4\rho_2\left(b-\int\limits_0^t g(\tau)d\tau\right)^2+2\lambda_5\rho_1 L^2.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
2L(t) \geq{} & \left\{1-\lambda_1-\frac{\lambda_3}{k}\left[1+\left(b-\int\limits_0^t g(\tau)d\tau\right)^2\right]-2\lambda_5\right\}\rho_1\left\|\varphi_t\right\|_2^2 \\
&+\left(1-\lambda_1-\lambda_2-\lambda_3-2\lambda_4\right)\rho_2\left\|\psi_t\right\|_2^2+\left[1-\lambda_1\left(m+2\rho_1 L+2m_E\right)\right]\varphi^2(0,t) \\
&+\left(1-\lambda_1\right)m\varphi_t^2\left(0,t\right)+\left(1-\lambda_1\right)m_E\varphi_t^2\left(L,t\right)+\left(1-\lambda_1-\lambda_2-\lambda_3\right)J\psi_t^2\left(L,t\right) \\
&+\left(l-B_1\right)\left\|\psi_x\right\|_2^2+\left[1-\lambda_2\left(\rho_2 L^2+JL\right)l-\lambda_3\frac{\rho_1}{k}l+4\lambda_4\rho_2 l\right]\int\limits_0^L\left(g\square\psi_x\right)dx \\
&+\left[k-4\lambda_1 L\left(L\rho_1+m_E\right)-\lambda_3\rho_2-4\lambda_5\rho_1\right]\left\|\varphi_x+\psi\right\|_2^2, \quad t\geq 0.
\end{aligned}$$

Therefore,

$$\sigma_1\left(E\left(t\right)+\varphi^2(0,t)\right)\leq L(t)\leq\sigma_2\left(E\left(t\right)+\varphi^2(0,t)\right), \quad t\geq 0,$$

for some constants σ_1, $\sigma_2>0$, provided that λ_i, $i=1,...,5$ are sufficiently small.
This completes the proof.

$\square$

Lemma 3.1 *Under the assumption (A1)-(A3), the functional* $I_1(t)$ *satisfies*

$$\begin{aligned} I_1'(t) \quad \leq \quad & -\left[(g_* - b) - \eta\right] \|\psi_x\|_2^2 - \rho_1 \|\varphi_t\|_2^2 - \rho_2 \|\psi_t\|_2^2 - m_E \varphi_t^2(L,t) \\ & - \left(m - \frac{1}{4\eta}\right) \varphi_t^2(0,t) - J\psi_t^2(L,t) - \left(1 - \eta K^2\right) \varphi^2(0,t) \\ & + k \|\varphi_x + \psi\|_2^2 + \frac{C_\alpha}{4\eta} \int_0^L (h \Box \psi_{xx})\, dx, \quad \eta > 0, \quad t \geq 0, \end{aligned} \tag{3.7}$$

for any $0 < \alpha < 1$*, and where*

$$C_\alpha = \int_0^\infty \frac{g^2(\tau)}{\alpha g(\tau) - g'(\tau)} d\tau$$

and

$$h(t) = \alpha g(t) - g'(t).$$

Proof: Using the equations of (2.4) and (2.12), we get

$$\begin{aligned} I_1'(t) \quad = \quad & A_1 + A_2 + k \int_0^L \psi(\varphi_x + \psi)\, dx - m\varphi_t^2(0,t) - m_E \varphi_t^2(L,t) - \rho_1 \|\varphi_t\|_2^2 \\ & - \rho_2 \|\psi_t\|_2^2 - J\psi_t^2(L,t) - \varphi^2(0,t) + K\varphi_t(0,t)\varphi(0,t) - k\varphi_x(0,t)\varphi(0,t) \\ & - m_E \varphi_{tt}(L,t)\, \varphi(L,t) - J\psi_{tt}(L,t)\, \psi(L,t), \quad t \geq 0, \end{aligned} \tag{3.8}$$

where

$$A_1 = -k \int_0^L \varphi(\varphi_{xx} + \psi_x)\, dx \text{ and } A_2 = -\int_0^L \psi(t) \left(b\psi_{xx} - \int_0^t g(t-\tau)\psi_{xx}(\tau)\, d\tau\right) dx.$$

Integrating by parts twice, we get

$$A_1 = -k\left(\varphi_x(L,t) + \psi(L,t)\right)\varphi(L,t) + k\varphi_x(0,t)\,\varphi(0,t) + k \int_0^L \varphi_x(\varphi_x + \psi)\, dx, \quad t \geq 0. \tag{3.9}$$

Similarly,

$$\begin{aligned} A_2 \quad = \quad & -\left(b\psi_x(L,t) - \int_0^t g(t-\tau)\psi_x(L,\tau)\, d\tau\right) \psi(L,t) \\ & + \int_0^L \psi_x(t) \left(b\psi_x - \int_0^t g(t-\tau)\psi_x(\tau)\, d\tau\right) dx. \end{aligned} \tag{3.10}$$

Next, substituting equations (3.9) and (3.10) in (3.8) and using the boundary conditions (2.5), we obtain

$$\begin{aligned} I_1'(t) \quad = \quad & -m\varphi_t^2(0,t) - m_E \varphi_t^2(L,t) - \rho_1 \|\varphi_t\|_2^2 - \rho_2 \|\psi_t\|_2^2 - J\psi_t^2(L,t) - \varphi^2(0,t) \\ & + k \|\varphi_x + \psi\|_2^2 + \int_0^L \psi_x(t) \int_0^t g(t-\tau)\left(\psi_x(t) - \psi_x(\tau)\right) d\tau dx \\ & + \left(b - \int_0^t g(\tau) d\tau\right) \|\psi_x\|_2^2 + K\varphi_t(0,t)\varphi(0,t), \quad t \geq 0. \end{aligned} \tag{3.11}$$

Now, we estimate the terms in the right-hand side of expression (3.11). We start with the seventh

$$\begin{aligned}&\int_0^L \psi_x(t)\int_0^t g(t-\tau)\left(\psi_x(t)-\psi_x(\tau)\right)d\tau dx\\ &\leq \eta\|\psi_x\|_2^2+\frac{1}{4\eta}\int_0^L\left(\int_0^t g(t-\tau)\left(\psi_x(\tau)-\psi_x(t)\right)d\tau\right)^2 dx,\quad t\geq 0.\end{aligned}$$

Using Cauchy-Schwarz inequality, we get

$$\begin{aligned}&\int_0^L\left(\int_0^t g(t-\tau)\left(\psi_x(\tau)-\psi_x(t)\right)d\tau\right)^2 dx \qquad (3.12)\\ =\ &\int_0^L\left(\int_0^t \frac{g(t-\tau)}{\sqrt{\alpha g(t-\tau)-g'(t-\tau)}}\sqrt{\alpha g(t-\tau)-g'(t-\tau)}\left(\psi_x(\tau)-\psi_x(t)\right)d\tau\right)^2 dx\\ \leq\ &\left(\int_0^t \frac{g^2(\tau)}{\alpha g(\tau)-g'(\tau)}d\tau\right)\int_0^L\int_0^t\left[\alpha g(t-\tau)-g'(t-\tau)\right]\left(\psi_x(\tau)-\psi_x(t)\right)^2 d\tau dx\\ \leq\ & C_\alpha\int_0^L\left(h\square\psi_x\right)(t)\,dx,\quad t\geq 0.\end{aligned}$$

For the eighth term, using (2.7), we see

$$K\varphi_t(0,t)\varphi(0,t)\leq\frac{1}{4\eta}\varphi_t^2(0,t)+\eta K^2\varphi^2(0,t),\quad \eta>0,\quad t\geq 0.$$

Combining all above estimates gives (3.7). □

Lemma 3.2 *Under the assumption (A1)-(A3), for* $I_2(t)$ *of (2.4)-(2.6), it holds that*

$$\begin{aligned}I_2'(t)\ \leq\ &-\left(\int_0^t g(\tau)d\tau-\eta\right)\left(\rho_2\|\psi_t\|_2^2+J\psi_t^2(L,t)\right)+\eta\|\varphi_x+\psi\|_2^2 \qquad (3.13)\\ &+\left(b-\int_0^t g(\tau)d\tau\right)^2\eta\|\psi_x\|_2^2+C\int_0^L(h\square\psi_x)dx,\end{aligned}$$

for all $t\geq 0$ *and position constants* $\eta>0$, *where*

$$C:=C_\alpha\left[1+\frac{cL}{\eta}(1+L)+\frac{1}{4\eta}+\frac{k^2L^2}{4\eta}\right]+\frac{cL}{\eta}(1+L).$$

Proof: The differentiation of $I_2(t)$, yields

$$\begin{aligned} I_2'(t) &= -\rho_2\int_0^L \psi_{tt}\int_0^t g(t-\tau)\left(\psi(t)-\psi(\tau)\right)d\tau dx \\ &\quad -\rho_2\int_0^L \psi_{t}\int_0^t g'(t-\tau)\left(\psi(t)-\psi(\tau)\right)d\tau dx \\ &\quad -J\psi_{tt}(L,t)\int_0^t g(t-\tau)\left(\psi(L,t)-\psi(L,\tau)\right)d\tau \\ &\quad -J\psi_{t}(L,t)\int_0^t g'(t-\tau)\left(\psi(L,t)-\psi(L,\tau)\right)d\tau \\ &\quad -\left(\int_0^t g(\tau)d\tau\right)\Big(\rho_2\left\|\psi_t\right\|_2^2+J\psi_t^2(L,t)\Big), \quad t\geq 0. \end{aligned}$$

$$\begin{aligned} I_2'(t) &= A_3+A_4-\left(\int_0^t g(\tau)d\tau\right)\Big(\rho_2\left\|\psi_t\right\|_2^2+J\psi_t^2(L,t)\Big) \\ &\quad +k\int_0^L(\varphi_x+\psi)\int_0^t g(t-\tau)\left(\psi(t)-\psi(\tau)\right)d\tau dx \\ &\quad -\rho_2\int_0^L \psi_{t}\int_0^t g'(t-\tau)\left(\psi(t)-\psi(\tau)\right)d\tau dx \\ &\quad -J\psi_{tt}(L,t)\int_0^t g(t-\tau)\left(\psi(L,t)-\psi(L,\tau)\right)d\tau \\ &\quad -J\psi_{t}(L,t)\int_0^t g'(t-\tau)\left(\psi(L,t)-\psi(L,\tau)\right)d\tau, \quad t\geq 0, \end{aligned} \tag{3.14}$$

where

$$A_3=-b\int_0^L \psi_{xx}(t)\int_0^t g(t-\tau)\Big(\psi(t)-\psi(\tau)\Big)d\tau dx, \quad t\geq 0,$$

and

$$A_4=\int_0^L\left(\int_0^t g(t-\tau)\psi_{xx}(\tau)d\tau\right)\left(\int_0^t g(t-\tau)\left(\psi(t)-\psi(\tau)\right)d\tau\right)dx, \quad t\geq 0.$$

Integration by parts in A_3 and A_4, we get

$$\begin{aligned} A_3 &= -b\psi_x(L,t)\int_0^t g(t-\tau)\Big(\psi(L,t)-\psi(L,\tau)\Big)d\tau \\ &\quad +b\int_0^L \psi_x(t)\int_0^t g(t-\tau)\Big(\psi_x(t)-\psi_x(\tau)\Big)d\tau dx, \end{aligned} \tag{3.15}$$

$$\begin{aligned} A_4 &= \left(\int_0^t g(t-\tau)\psi_x(L,\tau)d\tau\right)\left(\int_0^t g(t-\tau)\left(\psi(L,t)-\psi(L,\tau)\right)d\tau\right) \\ &\quad -\int_0^L \left(\int_0^t g(t-\tau)\psi_x(\tau)d\tau\right)\left(\int_0^t g(t-\tau)\left(\psi_x(t)-\psi_x(\tau)\right)d\tau\right)dx, \quad t\geq 0. \end{aligned} \tag{3.16}$$

Substituting (3.15) and (3.16) in (3.14) using the boundary conditions (2.5), we arrive at

$$\begin{aligned} I_2^{'}(t) &= -\left(\int_0^t g(\tau)d\tau\right)\left(\rho_2\left\|\psi_t\right\|_2^2 + J\psi_t^2(L,t)\right) \\ &\quad +k\int_0^L (\varphi_x+\psi)\int_0^t g(t-\tau)\left(\psi(t)-\psi(\tau)\right)d\tau dx \\ &\quad +b\int_0^L \psi_x(t)\int_0^t g(t-\tau)\left(\psi_x(t)-\psi_x(\tau)\right)d\tau dx \\ &\quad -\rho_2\int_0^L \psi_t\int_0^t g'(t-\tau)\left(\psi(t)-\psi(\tau)\right)d\tau dx \\ &\quad -\int_0^L \left(\int_0^t g(t-\tau)\psi_x(\tau)d\tau\right)\left(\int_0^t g(t-\tau)\left(\psi_x(t)-\psi_x(\tau)\right)d\tau\right)dx \\ &\quad -J\psi_t(L,t)\int_0^t g'(t-\tau)\left(\psi(L,t)-\psi(L,\tau)\right)d\tau, \quad t\geq 0. \end{aligned}$$

Or

$$\begin{aligned} I_2^{'}(t) &= -\left(\int_0^t g(\tau)d\tau\right)\left(\rho_2\left\|\psi_t\right\|_2^2 + J\psi_t^2(L,t)\right) \\ &\quad +k\int_0^L (\varphi_x+\psi)\int_0^t g(t-\tau)\left(\psi(t)-\psi(\tau)\right)d\tau dx \\ &\quad +\left(b-\int_0^t g(\tau)d\tau\right)\int_0^L \psi_x(t)\int_0^t g(t-\tau)\left(\psi_x(t)-\psi_x(\tau)\right)d\tau dx \\ &\quad +\int_0^L \left[\int_0^t g(t-\tau)\left(\psi_x(t)-\psi_x(\tau)\right)d\tau\right]^2 dx \\ &\quad -\rho_2\int_0^L \psi_t\int_0^t g'(t-\tau)\left(\psi(t)-\psi(\tau)\right)d\tau dx \\ &\quad -J\psi_t(L,t)\int_0^t g'(t-\tau)\left(\psi(L,t)-\psi(L,\tau)\right)d\tau, \quad t\geq 0. \end{aligned} \tag{3.17}$$

For $\eta > 0$, we estimate the terms on the right-hand side of expression (3.17).We start with the third term

$$k\int_0^L (\varphi_x+\psi)\int_0^t g(t-\tau)\left(\psi(t)-\psi(\tau)\right)d\tau dx \tag{3.18}$$

$$\le \eta\|\varphi_x+\psi\|_2^2+\frac{k^2L^2}{4\eta}C_\alpha\int_0^L (h\Box\psi_x)dx, \quad t\ge 0.$$

Similar to equation (3.12), we have

$$\left(b-\int_0^t g(\tau)d\tau\right)\int_0^L \psi_x(t)\int_0^t g(t-\tau)\left(\psi_x(t)-\psi_x(\tau)\right)d\tau dx \tag{3.19}$$

$$\le \left(b-\int_0^t g(\tau)d\tau\right)^2\eta\|\psi_x\|_2^2+\frac{C_\alpha}{4\eta}\int_0^L (h\Box\psi_x)dx, \quad t\ge 0.$$

Using (2.8), the last two terms on the right-hand side of equation (3.17) are estimated by

$$\begin{aligned}
&-\int_0^L \psi_t\int_0^t g'(t-\tau)\Big(\psi(t)-\psi(\tau)\Big)d\tau dx \qquad (3.20)\\
=\;& \int_0^L \psi_t\int_0^t h(t-\tau)\Big(\psi(t)-\psi(\tau)\Big)d\tau dx\\
&-\int_0^L \psi_t\int_0^t \alpha g(t-\tau)\Big(\psi(t)-\psi(\tau)\Big)d\tau dx\\
\le\;& \eta\|\psi_t\|_2^2+\frac{1}{2\eta}\int_0^L\left(\int_0^t \sqrt{h(t-\tau)}\sqrt{h(t-\tau)}\Big(\psi(t)-\psi(\tau)\Big)d\tau\right)^2dx\\
&+\frac{\alpha^2}{2\eta}\int_0^L\left(\int_0^t g(t-\tau)\Big(\psi(t)-\psi(\tau)\Big)d\tau\right)^2dx\\
\le\;& \eta\|\psi_t\|_2^2+\frac{\left(\int_0^t h(\tau)d\tau\right)}{2\eta}\int_0^L (h\Box\psi)\,dx+\frac{\alpha^2C_\alpha}{2\eta}\int_0^L (h\Box\psi)\,dx\\
\le\;& \eta\|\psi_t\|_2^2+\frac{cL^2}{\eta}\left(1+C_\alpha\right)\int_0^L (h\Box\psi_x)(t)\,dx, \quad t\ge 0
\end{aligned}$$

and

$$\begin{aligned} & -\psi_t(L,t)\int_0^t g'(t-\tau)\Big(\psi(L,t)-\psi(L,\tau)\Big)d\tau \\ = \quad & \psi_t(L,t)\int_0^t h(t-\tau)\Big(\psi(L,t)-\psi(L,\tau)\Big)d\tau \\ & -\alpha\psi_t(L,t)\int_0^t g(t-\tau)\Big(\psi(L,t)-\psi(L,\tau)\Big)d\tau \\ \leq \quad & \eta\psi_t^2(L,t)+\frac{cL}{\eta}\left(1+C_\alpha\right)\int_0^L \left(h\square\psi_x\right)dx, \quad t\geq 0. \end{aligned} \tag{3.21}$$

Collecting the previous estimates (3.12) and (3.18)-(3.21) in (3.17), (3.13) is established for $t\geq t_*>0$. $\square$

Lemma 3.3 *Under the assumption (A1)-(A3), assume that the coefficients ρ_1, ρ_2, b and k are such that*

$$\frac{\rho_1}{\rho_2}=\frac{k}{b}.$$

Then, the time derivative of $I_3(t)$ satisfies the estimate

$$\begin{aligned} I_3'(t) \quad \leq \quad & \rho_2\left\|\psi_t\right\|_2^2+\eta\left(1+g(t)\right)\frac{\rho_1}{k}\left\|\varphi_t\right\|_2^2-k\left\|\varphi_x+\psi\right\|_2^2+J\psi_t^2\left(L,t\right)+\frac{\rho_1 g(t)}{4k\eta}\left\|\psi_x\right\|_2^2 \\ & +\frac{1}{4\eta_1}\left[\left(b\psi_x(L,t)-\int_0^t g(t-\tau)\psi_x(L,\tau)d\tau\right)^2\right. \\ & \left.+\left(b\psi_x(0,t)-\int_0^t g(t-\tau)\psi_x(0,\tau)d\tau\right)^2\right] \\ & +\eta_1\left[\varphi_x^2(L,t)+\varphi_x^2(0,t)\right]+\frac{c}{\eta}(1+C_\alpha)\int_0^L\left(h\square\psi_x\right)dx, \quad t\geq 0, \end{aligned} \tag{3.22}$$

for some constant η, $\eta_1>0$.

Proof: Direct computations yields

$$\begin{aligned} I_3'(t) \quad = \quad & \rho_2\int_0^L \psi_t(\varphi_{xt}+\psi_t)dx-k\left\|\varphi_x+\psi\right\|_2^2+\frac{b\rho_1}{k}\int_0^L\varphi_t\psi_{xt}dx \\ & +\int_0^L\left(b\psi_x(L,t)-\int_0^t g(t-\tau)\psi_x(L,\tau)d\tau\right)\left(\varphi_x+\psi\right)dx+b\int_0^L\psi_x\left(\varphi_x+\psi\right)_x dx \\ & -\int_0^L\left(\varphi_x+\psi\right)_x\int_0^t g(t-\tau)\psi_x(\tau)d\tau dx+J\psi_t^2\left(L,t\right)+J\psi_{tt}(L,t)\psi(L,t) \\ & -\frac{\rho_1}{k}\int_0^L\varphi_t\left(\int_0^t g'(t-\tau)\psi_x(\tau)d\tau+g(0)\psi_x\right)dx, \quad t\geq 0, \end{aligned}$$

and by integration by parts

$$\begin{aligned} I_3'(t) &= \left[\left(b\psi_x - \int_0^t g(t-\tau)\psi_x(\tau)d\tau\right)(\varphi_x+\psi)\right]_0^L - k\left\|\varphi_x+\psi\right\|_2^2 + \rho_2\left\|\psi_t\right\|_2^2 + J\psi_t^2(L,t) \\ &\quad -\frac{\rho_1}{k}g(t)\int_0^L \varphi_t\psi_x dx + \frac{\rho_1}{k}\int_0^L \varphi_t \int_0^t g'(t-\tau)\left(\psi_x(t)-\psi_x(\tau)\right)d\tau dx \\ &\quad + J\psi_{tt}(L,t)\psi(L,t) + \left(\frac{b\rho_1}{k}-\rho_2\right)\int_0^L \varphi_t\psi_{xt}dx, \quad t \geq 0, \end{aligned}$$

or

$$\begin{aligned} I_3'(t) &= \left[\left(b\psi_x - \int_0^t g(t-\tau)\psi_x(\tau)d\tau\right)\varphi_x\right]_0^L - k\left\|\varphi_x+\psi\right\|_2^2 + \rho_2\left\|\psi_t\right\|_2^2 \\ &\quad + J\psi_t^2(L,t) - \frac{\rho_1}{k}g(t)\int_0^L \varphi_t\psi_x dx + \frac{\rho_1}{k}\int_0^L \varphi_t \int_0^t g'(t-\tau)\left(\psi_x(t)-\psi_x(\tau)\right)d\tau dx. \end{aligned} \tag{3.23}$$

Now, by using the following estimates

$$\begin{aligned} \left[\left(b\psi_x - \int_0^t g(t-\tau)\psi_x(\tau)d\tau\right)\varphi_x\right]_0^L &\leq \eta_1\left[\varphi_x^2(L,t)+\varphi_x^2(0,t)\right] \\ &\quad + \frac{1}{4\eta_1}\left(b\psi_x(L,t) - \int_0^t g(t-\tau)\psi_x(L,\tau)d\tau\right)^2 \\ &\quad + \frac{1}{4\eta_1}\left(b\psi_x(0,t) - \int_0^t g(t-\tau)\psi_x(0,\tau)d\tau\right)^2, \end{aligned} \tag{3.24}$$

and for $\eta > 0$

$$\int_0^L \varphi_t\psi_x dx \leq \eta\left\|\varphi_t\right\|_2^2 + \frac{1}{4\eta}\left\|\psi_x\right\|_2^2, \quad t \geq 0, \tag{3.25}$$

$$\begin{aligned}
&\int_0^L \varphi_t \int_0^t g'(t-\tau)\left(\psi_x(t)-\psi_x(\tau)\right)d\tau dx \qquad (3.26)\\
= &-\int_0^L \varphi_t \int_0^t h(t-\tau)\left(\psi_x(t)-\psi_x(\tau)\right)d\tau dx+\int_0^L \varphi_t \int_0^t \alpha g(t-\tau)\left(\psi_x(t)-\psi_x(\tau)\right)d\tau dx\\
\leq &\ \eta\left\|\varphi_t\right\|_2^2+\frac{1}{2\eta}\int_0^L\left(\int_0^t \sqrt{h(t-\tau)}\sqrt{h(t-\tau)}\left(\psi_x(t)-\psi_x(\tau)\right)d\tau\right)^2 dx\\
&+\frac{\alpha^2}{2\eta}\int_0^L\left(\int_0^t g(t-\tau)\left(\psi_x(t)-\psi_x(\tau)\right)d\tau\right)^2 dx\\
\leq &\ \eta\left\|\varphi_t\right\|_2^2+\frac{\left(\int_0^t h(\tau)d\tau\right)}{2\eta}\int_0^L\left(h\Box\psi_x\right)dx+\frac{\alpha^2 C_\alpha}{2\eta}\int_0^L\left(h\Box\psi_x\right)dx\\
\leq &\ \eta\left\|\varphi_t\right\|_2^2+\frac{c}{\eta}\left(1+C_\alpha\right)\int_0^L\left(h\Box\psi_x\right)dx,\quad t\geq 0.
\end{aligned}$$

Collecting the previous estimates (3.24)-(3.26) in (3.23), (3.22) is established. □

Lemma 3.4 *Under the assumption (A1)-(A3), for any* η, $\eta_2>0$, *we have*

$$\begin{aligned}
I_4'(t) \leq &-\left[\left(b\psi_x(L,t)-\int_0^t g(t-\tau)\psi_x(L,\tau)d\tau\right)^2+\left(b\psi_x(0,t)-\int_0^t g(t-\tau)\psi_x(0,\tau)d\tau\right)^2\right] \qquad (3.27)\\
&+2\rho_2\left[b+\eta\left(1+g_0\right)\right]\left\|\psi_t\right\|_2^2+\left[\frac{\rho_2 g(t)}{2\eta}+\left(b-\int_0^t g(\tau)d\tau\right)^2\left(\frac{1}{\eta_2}+\frac{4}{L}\right)\right]\left\|\psi_x\right\|_2^2\\
&+2\eta_2 k^2\left\|\varphi_x+\psi\right\|_2^2-\rho_2 b\psi_t^2(L,t)+\left[\frac{c\rho_2}{\eta}\left(1+C_\alpha\right)+\frac{C_\alpha}{\eta_2}\right]\int_0^L\left(h\Box\psi_x\right)dx,\quad t\geq 0,
\end{aligned}$$

where $g_0=g(0)$ *since* $g(t)$ *is decreasing.*

Proof: Clearly, using equation (2.4), we have

$$\begin{aligned}
I_4'(t) = &-\left[\left(b\psi_x(L,t)-\int_0^t g(t-\tau)\psi_x(L,\tau)d\tau\right)^2+\left(b\psi_x(0,t)-\int_0^t g(t-\tau)\psi_x(0,\tau)d\tau\right)^2\right]\\
&\int_0^L m(x)\left(b\psi_x(t)-\int_0^t g(t-\tau)\psi_x(\tau)d\tau\right)_x\left(b\psi_x(t)-\int_0^t g(t-\tau)\psi_x(\tau)d\tau\right)dx\\
&-k\int_0^L m(x)\left(\varphi_x+\psi\right)\left(b\psi_x(t)-\int_0^t g(t-\tau)\psi_x(\tau)d\tau\right)dx\\
&+\rho_2\int_0^L m(x)\psi_t\left(b\psi_{xt}(t)-\int_0^t g'(t-\tau)\psi_x(\tau)d\tau-g(0)\psi_x(t)\right)dx,\quad t\geq 0,
\end{aligned}$$

or

$$\begin{aligned} I_4'(t) &= -\left[\left(b\psi_x(L,t)-\int_0^t g(t-\tau)\psi_x(L,\tau)d\tau\right)^2+\left(b\psi_x(0,t)-\int_0^t g(t-\tau)\psi_x(0,\tau)d\tau\right)^2\right] \\ &\quad -\rho_2 b\psi_t^2\left(L,t\right)+\frac{2}{L}\int_0^L\left(b\psi_x(t)-\int_0^t g(t-\tau)\psi_x(\tau)d\tau\right)^2dx+\rho_2\frac{2b}{L}\left\|\psi_t\right\|_2^2 \\ &\quad -k\int_0^L m(x)\left(\varphi_x+\psi\right)\left(b\psi_x(t)-\int_0^t g(t-\tau)\psi_x(\tau)d\tau\right)dx-\rho_2 g(t)\int_0^L m(x)\psi_t\psi_x dx \\ &\quad +\rho_2\int_0^L m(x)\psi_t\int_0^t g'(t-\tau)\left(\psi_x(\tau)-\psi_x(\tau)\right)d\tau dx,\quad t\geq 0. \end{aligned} \tag{3.28}$$

Now, the last term in the right-hand side of (3.28), Using (2.7), will be estimated as

$$\begin{aligned} & k\int_0^L m(x)\left(\varphi_x+\psi\right)\left(b\psi_x(t)-\int_0^t g(t-\tau)\psi_x(\tau)d\tau\right)dx \\ & \leq 2\eta_2 k^2\left\|\varphi_x+\psi\right\|_2^2+\frac{1}{2\eta_2}\int_0^L\left(b\psi_x(t)-\int_0^t g(t-\tau)\psi_x(\tau)d\tau\right)^2dx,\quad \eta_2>0,\quad t\geq 0, \end{aligned} \tag{3.29}$$

$$\rho_2 g(t)\int_0^L m(x)\psi_t\psi_x dx\leq 2\rho_2 g_0\eta\left\|\psi_t\right\|_2^2+\frac{\rho_2 g(t)}{2\eta}\left\|\psi_x\right\|_2^2,\quad \eta>0,\quad t\geq 0 \tag{3.30}$$

and

$$\begin{aligned} & \rho_2\int_0^L m(x)\psi_t\int_0^t g'(t-\tau)\left(\psi_x(\tau)-\psi_x(\tau)\right)d\tau dx \\ & \leq 2\rho_2\eta\left\|\psi_t\right\|_2^2+\frac{c\rho_2}{\eta}(1+C_\alpha)\int_0^L\left(h\square\psi_x\right)dx,\quad \eta>0,\quad t\geq 0. \end{aligned} \tag{3.31}$$

Recalling that

$$\begin{aligned} & \int_0^L\left(b\psi_x(t)-\int_0^t g(t-\tau)\psi_x(\tau)d\tau\right)^2dx \\ &= \int_0^L\left[\int_0^t g(t-\tau)\left(\psi_x(t)-\psi_x(\tau)\right)d\tau+\left(b-\int_0^t g(\tau)d\tau\right)\psi_x(t)\right]^2dx \\ &\leq 2\left(b-\int_0^t g(\tau)d\tau\right)^2\left\|\psi_x\right\|_2^2+2\int_0^L\left(\int_0^t g(t-\tau)\left(\psi_x(t)-\psi_x(\tau)\right)d\tau\right)^2dx \\ &\leq 2\left(b-\int_0^t g(\tau)d\tau\right)^2\left\|\psi_x\right\|_2^2+2C_\alpha\int_0^L\left(h\square\psi_x\right)dx,\quad t\geq 0. \end{aligned} \tag{3.32}$$

Collecting the previous estimates (3.29)-(3.32) in (3.23), (3.27) is established. □

Lemma 3.5 *The time derivative of* $I_5(t)$ *satisfies the estimate*

$$\begin{aligned} I_5'(t) \quad &\leq \quad -k\left[\varphi_x^2(L,t)+\varphi_x^2(0,t)\right]-\rho_1\left[\varphi_t^2(L,t)+\varphi_t^2(0,t)\right] \\ &\quad +\frac{2\rho_1}{L}\left\|\varphi_t\right\|_2^2+Lk\left\|\psi_x\right\|_2^2+\frac{3k}{L}\left\|\varphi_x\right\|_2^2, \quad t\geq 0. \end{aligned} \tag{3.33}$$

Proof: Clearly,

$$\begin{aligned} I_5'(t) \quad &= \quad \rho_1\int_0^L m(x)\varphi_{xt}\varphi_t dx+k\int_0^L m(x)\varphi_x\varphi_{xx}dx+k\int_0^L m(x)\varphi_x\psi_x dx \\ &= \quad \frac{\rho_1}{2}\left[m(x)\varphi_t^2\right]_0^L-\frac{\rho_1}{2}\int_0^L m_x(x)\varphi_t^2dx+\frac{k}{2}\left[m(x)\varphi_x^2\right]_0^L \\ &\quad -\frac{k}{2}\int_0^L m_x(x)\varphi_x^2dx+k\int_0^L m(x)\varphi_x\psi_x dx, \end{aligned}$$

Therefore, by using (2.7) with $\eta=\frac{L}{2}$, we obtain

$$\begin{aligned} I_5'(t) \quad &\leq \quad -k\left[\varphi_x^2(L,t)+\varphi_x^2(0,t)\right]-\rho_1\left[\varphi_t^2(L,t)+\varphi_t^2(0,t)\right] \\ &\quad +\frac{2\rho_1}{L}\left\|\varphi_t\right\|_2^2+Lk\left\|\psi_x\right\|_2^2+\frac{3k}{L}\left\|\varphi_x\right\|_2^2, \quad t\geq 0. \end{aligned}$$

This completes the proof. □

Lemma 3.6 *The functional* $I_6(t)$ *defined by*

$$I_6(t):=\int_0^L\int_0^t f(t-\tau)\psi_x^2(\tau)d\tau dx, \quad t\geq 0,$$

where $f(t)=\int_t^\infty g(\tau)d\tau$, *satisfies*

$$I_6'(t)\leq -\frac{1}{2}\int_0^L\left(g\square\psi_x\right)dx+3l\left\|\psi_x\right\|_2^2, \quad t\geq 0. \tag{3.34}$$

Proof: By Young's inequality and the fact $f'(t)=-g(t)$, we see that

$$\begin{aligned} I_6'(t) &= f(0)\int_0^L\psi_x^2(t)dx-\int_0^L\int_0^t g(t-\tau)\psi_x^2(\tau)d\tau dx \\ &= -\int_0^L\int_0^t g(t-\tau)\left(\psi_x(\tau)-\psi_x(t)\right)^2 d\tau dx+f(t)\left\|\psi_x\right\|_2^2 \\ &\quad -2\int_0^L\psi_x(t)\int_0^t g(t-\tau)\left(\psi_x(\tau)-\psi_x(t)\right)d\tau dx, \quad t\geq 0. \end{aligned}$$

But

$$\begin{aligned}
&-2\int_0^L \psi_x(t)\int_0^t g(t-\tau)\left(\psi_x(\tau)-\psi_x(t)\right)d\tau dx\\
&\leq 2l\left\|\psi_x\right\|_2^2+\frac{\int_0^t g(\tau)d\tau}{2l}\int_0^L\int_0^t g(t-\tau)\left(\psi_x(\tau)-\psi_x(t)\right)^2 d\tau dx, \quad t\geq 0.
\end{aligned}$$

Then, as $f(t)\leq f(0)=l$ and $\int_0^t g(\tau)d\tau\leq l$, we get (3.34). □

Using the previous lemmas we now give the proof of our main result.

Lemma 3.7 *The functional L satisfies,*

$$\begin{aligned}
L'(t) \leq\ & -\sigma_1\varphi_t^2(0,t)-\sigma_2\varphi_t^2(L,t)-\sigma_3\varphi^2(0,t)-\sigma_4\left\|\varphi_t\right\|_2^2 \\
& -\sigma_5\left\|\psi_t\right\|_2^2-\sigma_6\psi_t^2(L,t)-\sigma_7\left\|\varphi_x+\psi\right\|_2^2-\sigma_8\left\|\psi_x\right\|_2^2 \\
& +\frac{\alpha}{2}\int_0^L\left(g\Box\psi_x\right)dx, \quad t\geq 0,
\end{aligned} \tag{3.35}$$

where

$$\begin{aligned}
\sigma_1 &= K+\frac{\lambda_3}{6}\left(\frac{L\rho_1}{6}+m-\frac{1}{4\eta}\right)-\frac{1}{4\eta},\\
\sigma_2 &= \frac{\lambda_3}{6}\left(m_E+\frac{L\rho_1}{6}\right),\\
\sigma_3 &= \frac{\lambda_3}{6}\left(1-K^2\eta\right)-2\eta,\\
\sigma_4 &= \lambda_3\rho_1\left[\frac{1}{9}-\frac{\eta}{k}\left(2+g_0\right)\right],\\
\sigma_5 &= \left\{\lambda_2\left(g_*-\eta\right)-\lambda_3\left[\frac{5}{6}+\frac{18}{Lk}\left[b+\eta\left(1+g_0\right)\right]\right]\right\}\rho_2,\\
\sigma_6 &= \left[\lambda_2\left(g_*-\eta\right)-\lambda_3\left(\frac{5}{6}-\frac{9\rho_2 b}{LkJ}\right)\right]J,\\
\sigma_7 &= \left[\lambda_3\left(\frac{2}{3}-\frac{18}{k}\eta_2\right)-\frac{2\lambda_2\eta}{k}\right]k,\\
\sigma_8 &= \frac{g_0}{2}-B_4,
\end{aligned}$$

and

$$\begin{aligned}
B_4 = \ & \lambda_2\eta\left(b-g_*\right)^2+\lambda_3\left\{\frac{\rho_1 g_0}{4k\eta}+\frac{\eta+b-g_*}{6}\right.\\
& \left.+\frac{9}{Lk}\left[\frac{1}{\eta_2}\left(b-g_*\right)^2+\frac{\rho_2 g_0}{2\eta}\right]+\frac{kL^2}{36}\right\}
\end{aligned}$$

Proof: We differentiate L defined in equation (3.1), substitute equation (2.12) into equation (2.11), take into account lamma 3.1-lamma 3.5 and the following estimate

$$\varphi_t(0,t)\varphi(0,t)\leq\frac{1}{4\eta}\varphi_t^2(0,t)+\eta\varphi^2(0,t), \quad \eta>0, \tag{3.36}$$

recalling that $g'(t) = \alpha g(t) - h(t)$, we obtain, for $t \geq t_* > 0$

$$\begin{aligned}
L'(t) \leq\ & -\left(K + \lambda_1 m + \lambda_5 \rho_1 - \frac{\lambda_1}{4\eta} - \frac{1}{4\eta}\right) \varphi_t^2(0,t) - \left(\lambda_1 m_E + \lambda_5 \rho_1\right) \varphi_t^2(L,t) \\
& - \left[\lambda_1\left(1 - K^2\eta\right) - 2\eta\right] \varphi^2(0,t) - \left\{\lambda_1 - \left[\frac{\lambda_3 \eta}{k}(1+g_0) + \frac{2\lambda_5}{L}\right]\right\} \rho_1 \|\varphi_t\|_2^2 \\
& - \left\{\lambda_1 + \lambda_2(g_* - \eta) - \left[\lambda_3 + 2\lambda_4\left[b + \eta(1+g_0)\right]\right]\right\} \rho_2 \|\psi_t\|_2^2 \\
& - \left[\lambda_1 + \lambda_2(g_* - \eta) + \frac{\lambda_4 \rho_2 b}{J} - \lambda_3\right] J \psi_t^2(L,t) + \lambda_5 \frac{3k}{L} \|\varphi_x\|_2^2 \\
& - \left[\lambda_3 - \left(\lambda_1 + \lambda_2 \frac{2\eta}{k} + 2\lambda_4 k \eta_2\right)\right] k \|\varphi_x + \psi\|_2^2 \\
& - \left(\frac{g_0}{2} - B_2\right) \|\psi_x\|_2^2 - (k\lambda_5 - \lambda_3 \eta_1)\left[\varphi_x^2(L,t) + \varphi_x^2(0,t)\right] \\
& - \left(\lambda_4 - \lambda_3 \frac{1}{4\eta_1}\right) \left[\left(b\psi_x(L,t) - \int_0^t g(t-\tau)\psi_x(L,t)\, d\tau\right)^2\right. \\
& \left. + \left(b\psi_x(0,t) - \int_0^t g(t-\tau)\psi_x(0,t)\, d\tau\right)^2\right] \\
& + \frac{\alpha}{2} \int_0^L (g \Box \psi_x)\, dx - \left(\frac{1}{2} - B_3\right) \int_0^L (h \Box \psi_x)\, dx, \quad t \geq 0,
\end{aligned}$$

where

$$B_2 = \lambda_1(\eta + b - g_*) + \lambda_2 (b - g_*)^2 \eta + \frac{\lambda_3 \rho_1 g_0}{4k\eta} + \lambda_4 \left[\frac{1}{\eta_2}(b - g_*)^2 + \frac{\rho_2 g_0}{2\eta}\right] + kL\lambda_5$$

and

$$\begin{aligned}
B_3 =\ & \lambda_1 \frac{c}{3\eta} + \lambda_2 \left[C_\alpha \left(1 + \frac{L^2}{4\eta} + \frac{1}{4\eta} + \frac{\alpha^2 L^2}{2\eta} + \frac{cL}{2\eta}\right) + \frac{cL}{2\eta}(1+L)\right] \\
& + \lambda_3 \frac{c}{\eta}(1 + C_\alpha) + \lambda_4 \left[\frac{c}{\eta}\rho_2(1 + C_\alpha) + \frac{C_\alpha}{\eta_2}\right].
\end{aligned}$$

For small $\eta < g_*$, we choose $\lambda_1 = \frac{\lambda_3}{6}$, $\lambda_4 = \frac{\lambda_3}{4\eta_1}$, $\lambda_5 = \frac{\lambda_3 \eta_1}{k}$, $\eta_1 = \frac{Lk}{36}$ and the fact that

$$\|\varphi_x\|_2^2 \leq 2\|\varphi_x + \psi\|_2^2 + 2L^2 \|\psi_x\|_2^2,$$

imply that

$$\begin{aligned} L'(t) \leq & -\left[K+\frac{\lambda_3}{6}\left(\frac{L\rho_1}{6}+m-\frac{1}{4\eta}\right)-\frac{1}{4\eta}\right]\varphi_t^2(0,t)-\frac{\lambda_3}{6}\left(m_E+\frac{L\rho_1}{6}\right)\varphi_t^2(L,t) \\ & -\left[\frac{\lambda_3}{6}\left(1-K^2\eta\right)-2\eta\right]\varphi^2(0,t)-\lambda_3\rho_1\left[\frac{1}{9}-\frac{\eta}{k}(2+g_0)\right]\|\varphi_t\|_2^2 \\ & -\left\{\lambda_2(g_*-\eta)-\lambda_3\left[\frac{5}{6}+\frac{18}{Lk}\left[b+\eta(1+g_0)\right]\right]\right\}\rho_2\|\psi_t\|_2^2 \\ & -\left[\lambda_2(g_*-\eta)-\lambda_3\left(\frac{5}{6}-\frac{9\rho_2 b}{LkJ}\right)\right]J\psi_t^2(L,t) \\ & -\left[\lambda_3\left(\frac{2}{3}-\frac{18}{k}\eta_2\right)-\frac{2\lambda_2\eta}{k}\right]k\|\varphi_x+\psi\|_2^2 \\ & -\left(\frac{g_0}{2}-B_4\right)\|\psi_x\|_2^2-\left(\frac{1}{2}-B_5\right)\int_0^L(h\Box\psi_x)\,dx+\frac{\alpha}{2}\int_0^L(g\Box\psi_x)\,dx, \quad t\geq 0, \end{aligned}$$

where

$$B_4=\lambda_2\eta(b-g_*)^2+\lambda_3\left\{\frac{\rho_1 g_0}{4k\eta}+\frac{\eta+b-g_*}{6}+\frac{9}{Lk}\left[\frac{1}{\eta_2}(b-g_*)^2+\frac{\rho_2 g_0}{2\eta}\right]+\frac{kL^2}{36}\right\}$$

and

$$\begin{aligned} B_5 = & \lambda_2\left[C_\alpha\left(1+\frac{L^2}{4\eta}+\frac{1}{4\eta}+\frac{\alpha^2L^2}{2\eta}+\frac{cL}{2\eta}\right)+\frac{cL}{2\eta}(1+L)\right] \\ & +\lambda_3\left\{\frac{c}{\eta}(1+C_\alpha)+\frac{c}{18\eta}+\frac{9}{Lk}\left[\frac{c}{\eta}\rho_2(1+C_\alpha)+\frac{C_\alpha}{\eta_2}\right]\right\}. \end{aligned}$$

At this point, we choose λ_2, λ_3 η and η_2 small enough and K large enough, so that

$$\begin{cases} K+\frac{\lambda_3}{6}\left(\frac{L\rho_1}{6}+m-\frac{1}{4\eta}\right)-\frac{1}{4\eta}>0 \\ \frac{\lambda_3}{6}\left(1-K^2\eta\right)-2\eta>0 \\ \frac{1}{9}-\frac{\eta}{k}(2+g_0)>0 \\ \lambda_2(g_*-\eta)-\lambda_3\left[\frac{5}{6}+\frac{18}{Lk}\left[b+\eta(1+g_0)\right]\right]>0 \\ \lambda_2(g_*-\eta)-\lambda_3\left(\frac{5}{6}-\frac{9\rho_2 b}{LkJ}\right)>0 \\ \lambda_3\left(\frac{2}{3}-\frac{18}{k}\eta_2\right)-\frac{2\lambda_2\eta}{k}>0 \\ \frac{g_0}{2}-B_4>0 \\ \frac{1}{2}-B_5>0. \end{cases}$$

So we arrive at

$$\begin{aligned} L'(t) \leq & -\left[K+\frac{\lambda_3}{6}\left(\frac{L\rho_1}{6}+m-\frac{1}{4\eta}\right)-\frac{1}{4\eta}\right]\varphi_t^2(0,t)-\frac{\lambda_3}{6}\left(m_E+\frac{L\rho_1}{6}\right)\varphi_t^2(L,t) \\ & -\left[\frac{\lambda_3}{6}\left(1-K^2\eta\right)-2\eta\right]\varphi^2(0,t)-\lambda_3\rho_1\left[\frac{1}{9}-\frac{\eta}{k}(2+g_0)\right]\|\varphi_t\|_2^2 \\ & -\left\{\lambda_2(g_*-\eta)-\lambda_3\left[\frac{5}{6}+\frac{18}{Lk}\left[b+\eta(1+g_0)\right]\right]\right\}\rho_2\|\psi_t\|_2^2 \\ & -\left[\lambda_2(g_*-\eta)-\lambda_3\left(\frac{5}{6}-\frac{9\rho_2 b}{LkJ}\right)\right]J\psi_t^2(L,t) \\ & -\left[\lambda_3\left(\frac{2}{3}-\frac{18}{k}\eta_2\right)-\frac{2\lambda_2\eta}{k}\right]k\|\varphi_x+\psi\|_2^2 \\ & -\left(\frac{g_0}{2}-B_4\right)\|\psi_x\|_2^2+\frac{\alpha}{2}\int_0^L(g\Box\psi_x)\,dx, \quad t\geq 0. \end{aligned}$$

This completes the proof. □

The main result of the present work is to establish the general decay rate of the energy, which is given by the following theorem.

Theorem 3.1 *Assume that (A1)-(A3) holds. Then there exist positive constants $k_1 \leq 1$ and k_2 such that the energy functional satisfies*

$$E(t) \leq k_2 H_1^{-1}\left(k_1 \int_{g^{-1}(r)}^{t} \zeta(\tau) d\tau\right), \tag{3.37}$$

where $H_1(t) = \int_t^r \frac{1}{\tau H'(\tau)} d\tau$.

Here, H_1 is strictly decreasing and convex on $]0, r]$, with $\lim\limits_{t \to 0} H_1(t) = +\infty$.

Proof: We start using (2.11) and (2.14) to conclude that, for any $t \geq t_1 > 0$

$$\begin{aligned} &\int_0^{t_1} g(\tau) \int_0^L (\psi_x(t) - \psi_x(t-\tau))^2 \, dx d\tau - 3\varphi_t(0,t)\varphi(0,t) \\ &\leq -\frac{g(0)}{a} \int_0^{t_1} g'(\tau) \int_0^L (\psi_x(t) - \psi_x(t-\tau))^2 \, dx d\tau - 3\varphi_t(0,t)\varphi(0,t) \\ &\leq -c\left(E(t) + \varphi^2(0,t)\right)'. \end{aligned} \tag{3.38}$$

Now, use addition and subtraction $6\varphi_t(0,t)\varphi(0,t)$ in the right-hand side of (3.35) and using (3.36), we obtian

$$\begin{aligned} L'(t) \leq \; & -\left(\sigma_1 - \frac{3}{2\eta}\right)\varphi_t^2(0,t) - \sigma_2\varphi_t^2(L,t) - (\sigma_3 - 6\eta)\varphi^2(0,t) \\ & -\sigma_4 \|\varphi_t\|_2^2 - \sigma_5 \|\psi_t\|_2^2 - \sigma_6\psi_t^2(L,t) - \sigma_7 \|\varphi_x + \psi\|_2^2 \\ & -\sigma_8 \|\psi_x\|_2^2 + \frac{\alpha}{2}\int_0^L (g\Box\psi_x)\, dx - 6\varphi_t(0,t)\varphi(0,t), \quad t \geq 0, \end{aligned}$$

imply that

$$\begin{aligned} L'(t) \leq \; & -\left[K + \frac{\lambda_3}{6}\left(\frac{L\rho_1}{6} + m - \frac{1}{4\eta}\right) - \frac{7}{4\eta}\right]\varphi_t^2(0,t) \\ & -\sigma_2\varphi_t^2(L,t) - \left[\frac{\lambda_3}{6}\left(1 - K^2\eta\right) - 7\eta\right]\varphi^2(0,t) \\ & -\sigma_4 \|\varphi_t\|_2^2 - \sigma_5 \|\psi_t\|_2^2 - \sigma_6\psi_t^2(L,t) - \sigma_7 \|\varphi_x + \psi\|_2^2 \\ & -\sigma_8 \|\psi_x\|_2^2 + \frac{\alpha}{2}\int_0^L (g\Box\psi_x)\, dx - 6\varphi_t(0,t)\varphi(0,t), \quad t \geq 0. \end{aligned}$$

At this point, we choose η small enough and K large enough so that

$$K + \lambda_1 m + \lambda_3 \frac{L\eta\rho_1}{2k} - \frac{\lambda_1}{4\eta} - \frac{7}{4\eta} > 0$$

and

$$\lambda_1\left(1 - K^2\eta\right) - 7\eta > 0.$$

So we arrive at

$$\begin{aligned}
L'(t) &\leq -m\left(E\left(t\right)+\varphi^{2}(0,t)\right)+c\left(\int\limits_{0}^{L}\left(g\Box\psi_{x}\right)dx-6\varphi_{t}\left(0,t\right)\varphi\left(0,t\right)\right)\\
&\leq -m\left(E\left(t\right)+\varphi^{2}(0,t)\right)-c\left(E\left(t\right)+\varphi^{2}(0,t)\right)'\\
&\quad+c\left(\int\limits_{t_1}^{t}g(\tau)\int\limits_{0}^{L}\left(\psi_{x}(t)-\psi_{x}(t-\tau)\right)^{2}dxd\tau-3\varphi_{t}\left(0,t\right)\varphi\left(0,t\right)\right),\quad t\geq 0,
\end{aligned}$$

then take $F(t)=L(t)+c\left(E\left(t\right)+\varphi^{2}(0,t)\right)$, which is clearly equivalent to $\left(E\left(t\right)+\varphi^{2}(0,t)\right)$, to get, for some constant $m>0$,

$$F'(t)\leq -m\left(E\left(t\right)+\varphi^{2}(0,t)\right)+c\left(\int\limits_{t_1}^{t}g(\tau)\int\limits_{0}^{L}\left(\psi_{x}(t)-\psi_{x}(t-\tau)\right)^{2}dxd\tau-3\varphi_{t}\left(0,t\right)\varphi\left(0,t\right)\right),\quad t\geq 0,\tag{3.39}$$

so,

$$F'(t)\leq -m\left(E\left(t\right)+\varphi^{2}(0,t)\right)(t)-c\left(E\left(t\right)+\varphi^{2}(0,t)\right)'(t).\tag{3.40}$$

We consider the following two cases:

H(t) is linear Multiplying (3.40) by $\zeta(t)$, we obtain

$$\zeta(t)F'(t)\leq -m\zeta(t)\left(E\left(t\right)+\varphi^{2}(0,t)\right)(t)-c\zeta(t)\left(E\left(t\right)+\varphi^{2}(0,t)\right)'(t),$$

which gives, as ζ is nonincreasing,

$$\left[\zeta(t)F(t)+c\left(E\left(t\right)+\varphi^{2}(0,t)\right)\right]'\leq -m\zeta(t)\left(E\left(t\right)+\varphi^{2}(0,t)\right),\quad \forall t\geq t_1>0.$$

Hence, using the fact that $\zeta(t)F(t)+c\left(E\left(t\right)+\varphi^{2}(0,t)\right)\sim\left(E\left(t\right)+\varphi^{2}(0,t)\right)(t)$, we easily obtain

$$\left(E\left(t\right)+\varphi^{2}(0,t)\right)\leq c'e^{-m\int_{t_1}^{t}\zeta(\tau)d\tau}.$$

H(t) is nonlinear First, we use (2.11) and (3.34) to deduce that

$$\mathcal{L}(t)=L(t)+\lambda_6 I_6(t),$$

is non-negative and satisfies, for suitable choice of λ_6 and for all $t\geq t_1$

$$\begin{aligned}
\mathcal{L}'(t) \quad \leq \quad & -\sigma_1\varphi_t^2\left(0,t\right)-\sigma_2\varphi_t^2\left(L,t\right)-\sigma_3\varphi^2\left(0,t\right)-\sigma_4\left\|\varphi_t\right\|_2^2\\
&-\sigma_5\left\|\psi_t\right\|_2^2-\sigma_6\psi_t^2\left(L,t\right)-\sigma_7\left\|\varphi_x+\psi\right\|_2^2\\
&-\left(\sigma_8-3\lambda_6 l\right)\left\|\psi_x\right\|_2^2-\frac{\lambda_6-\alpha}{2}\int\limits_{0}^{L}\left(g\Box\psi_x\right)dx.
\end{aligned}$$

Now, for small λ_6 and α, we have

$$\mathcal{L}'(t)\leq -b\left(E\left(t\right)+\varphi^{2}(0,t)\right),\quad t\geq 0,$$

where b is some positive constant. Therefore,

$$b\int_{t_1}^{t}\left(E\left(t\right)+\varphi^{2}(0,t)\right)(\tau)\,d\tau\leq\mathcal{L}(t_1)-\mathcal{L}\left(t\right)\leq\mathcal{L}(t_1).$$

This implies that

$$\int_0^\infty \left(E\left(t\right)+\varphi^2(0,t)\right)\left(\tau\right)d\tau<\infty. \tag{3.41}$$

Now, we define $I(t)$ by

$$I(t):=\mu\int_{t_1}^{t}\int_0^L\left(\psi_x(t)-\psi_x(t-\tau)\right)^2dxd\tau,$$

where (3.41) allows for a constant $0<\mu<1$ chosen so that, for all $t\geq t_1$

$$I(t)<1. \tag{3.42}$$

We also assume, without loss of generality that $I(t)>0$, for all $t\geq t_1$, otherwise (3.39) yields an exponential decay.
Also, we define $\lambda\left(t\right)$ by

$$\lambda\left(t\right):=-\int_{t_1}^{t}g'\left(\tau\right)\int_0^L\left(\psi_x(t)-\psi_x(t-\tau)\right)^2dxd\tau.$$

Using (2.11) and (3.36) with $\eta=\frac{1}{K}$, we get

$$\left(E\left(t\right)+\varphi^2(0,t)\right)'\leq\frac{1}{2}\int\limits_0^L(g'\Box\psi_x)dx-\frac{g(t)}{2}\left\|\psi_x\right\|_2^2-\frac{K}{4}\varphi_t^2\left(0,t\right)+\frac{3}{K}\varphi^2\left(0,t\right),\quad t\geq0. \tag{3.43}$$

Then, we select K large enough such that

$$-\frac{K}{4}\varphi_t^2\left(0,t\right)+\frac{3}{K}\varphi^2\left(0,t\right)<0.$$

Therefore,

$$\left(E\left(t\right)+\varphi^2(0,t)\right)'\leq\frac{1}{2}\int\limits_0^L(g'\Box\psi_x)dx\leq0, \tag{3.44}$$

implies that

$$\lambda\left(t\right)\leq-c\left(E\left(t\right)+\varphi^2(0,t)\right)'.$$

Since H is stictly convexe on $]0,r]$ and $H(0)=0$, then

$$H\left(\theta x\right)\leq\theta H\left(x\right),$$

provided $0<\theta<1$ and $x\in]0,r]$. The use of this fact, hypothesis (A1)-(A3), (3.42) and Jensen's inequality leads to

$$\begin{aligned}
\lambda(t) &= \frac{1}{\mu I(t)} \int_{t_1}^{t} I(t)\left(-g'(\tau)\right) \int_0^L \mu \left(\psi_x(t)-\psi_x(t-\tau)\right)^2 dx d\tau \\
&\geq \frac{1}{\mu I(t)} \int_{t_1}^{t} I(t)\zeta(\tau) H(g(\tau)) \int_0^L \mu \left(\psi_x(t)-\psi_x(t-\tau)\right)^2 dx d\tau \\
&\geq \frac{\zeta(t)}{\mu I(t)} \int_{t_1}^{t} H(I(t)g(\tau)) \int_0^L \mu \left(\psi_x(t)-\psi_x(t-\tau)\right)^2 dx d\tau \\
&\geq \frac{\zeta(t)}{\mu} H\left(\frac{1}{I(t)} \int_{t_1}^{t} I(t)g(\tau) \int_0^L \mu \left(\psi_x(t)-\psi_x(t-\tau)\right)^2 dx d\tau\right) \\
&= \frac{\zeta(t)}{\mu} H\left(\mu \int_{t_1}^{t} g(\tau) \int_0^L \left(\psi_x(t)-\psi_x(t-\tau)\right)^2 dx d\tau\right) \\
&= \frac{\zeta(t)}{\mu} \overline{H}\left(\mu \int_{t_1}^{t} g(\tau) \int_0^L \left(\psi_x(t)-\psi_x(t-\tau)\right)^2 dx d\tau\right),
\end{aligned}$$

where $\overline{H}$ is in extension of H such that $\overline{H}$ is strictly increasing and strictly convex C^2 function on $]0,\infty[$.
This implies that

$$\int_{t_1}^{t} g(\tau) \int_0^L \left(\psi_x(t)-\psi_x(t-\tau)\right)^2 dx d\tau \leq \frac{1}{\mu} \overline{H}^{-1}\left(\frac{\mu\lambda(t)}{\zeta(t)}\right), \tag{3.45}$$

and now, use addition and subtraction $3\varphi_t(0,t)\varphi(0,t)$ in the right-hand side of (3.35) and using (3.36) then we choose η small enough and K large enough, we obtian

$$\begin{aligned}
L'(t) \quad &\leq \quad -m\left(E(t)+\varphi^2(0,t)\right) \\
&\quad\quad +\frac{\alpha}{2}\int_0^L (g\square\psi_x)\, dx - 3\varphi_t(0,t)\varphi(0,t), \quad t \geq 0.
\end{aligned}$$

At this point, we use (3.38) so that

$$F'(t) \leq -m\left(E(t)+\varphi^2(0,t)\right) + c\int_{t_1}^{t} g(\tau) \int_0^L \left(\psi_x(t)-\psi_x(t-\tau)\right)^2 dx d\tau,$$

clearly, using (3.45), we have

$$F'(t) \leq -m\left(E(t)+\varphi^2(0,t)\right) + c\overline{H}^{-1}\left(\frac{\mu\lambda(t)}{\zeta(t)}\right), \quad t \geq t_1. \tag{3.46}$$

For $\varepsilon_0 < r$, using (3.44) and (3.46), and the fact that $\overline{H}' > 0$ and $\overline{H}'' > 0$, we find that the functional F_1, defined by

$$F_1(t) := \overline{H}'\left(\frac{\varepsilon_0}{E_*}\left(E(t)+\varphi^2(0,t)\right)\right) F(t) + E(t) + \varphi^2(0,t),$$

where

$$E_* = \left(E\left(t\right) + \varphi^2(0,t)\right)(0),$$

F_1 is equivalent to $E\left(t\right) + \varphi^2(0,t)$ and

$$\begin{aligned} F_1'(t) &= \frac{\varepsilon_0}{E_*}\left(E\left(t\right) + \varphi^2(0,t)\right)\overline{H}''\left(\frac{\varepsilon_0}{E_*}\left(E\left(t\right) + \varphi^2(0,t)\right)\right)F(t) \\ &\quad + \overline{H}'\left(\frac{\varepsilon_0}{E_*}\left(E\left(t\right) + \varphi^2(0,t)\right)\right)F'(t) + \left(E(t) + \varphi^2(0,t)\right)' \\ &\leq -m\left(E\left(t\right) + \varphi^2(0,t)\right)\overline{H}'\left(\frac{\varepsilon_0}{E_*}\left(E\left(t\right) + \varphi^2(0,t)\right)\right) \\ &\quad + c\overline{H}'\left(\frac{\varepsilon_0}{E_*}\left(E\left(t\right) + \varphi^2(0,t)\right)\right)\overline{H}^{-1}\left(\frac{\mu\lambda(t)}{\zeta\left(t\right)}\right) \\ &\quad + \left(E(t) + \varphi^2(0,t)\right)'. \end{aligned} \tag{3.47}$$

Let $\overline{H}^*$ be the convex conjugate of $\overline{H}$ in the sense of Young (see Arnold [6] [p. 61-62], Daoulatli, Lasiecka and Doundykov [22], Lasiecka and Doundykov [41], for more information), then

$$,\overline{H}^*(\tau) = \tau\left(\overline{H}'\right)^{-1}(\tau) - \overline{H}\left[\left(\overline{H}'\right)^{-1}(\tau)\right], \tag{3.48}$$

and $\overline{H}^*$ satisfies the following Young's inequality

$$AB \leq \overline{H}^*(A) + \overline{H}(B), \tag{3.49}$$

with $A = \overline{H}'\left(\frac{\varepsilon_0}{E_*}\left(E\left(t\right) + \varphi^2(0,t)\right)\right)$ and $B = \overline{H}^{-1}\left(\frac{\mu\lambda(t)}{\zeta(t)}\right)$, using (2.11) and (3.47)-(3.49), we arrive at

$$\begin{aligned} F_1'(t) &\leq -m\left(E\left(t\right) + \varphi^2(0,t)\right)\overline{H}'\left(\frac{\varepsilon_0}{E_*}\left(E\left(t\right) + \varphi^2(0,t)\right)\right) \\ &\quad + c\overline{H}^*\left(\overline{H}'\left(\frac{\varepsilon_0}{E_*}\left(E\left(t\right) + \varphi^2(0,t)\right)\right)\right) + c\frac{\mu\lambda(t)}{\zeta\left(t\right)} \\ &\quad + \left(E(t) + \varphi^2(0,t)\right)' \\ &\leq -m\left(E\left(t\right) + \varphi^2(0,t)\right)\overline{H}'\left(\frac{\varepsilon_0}{E_*}\left(E\left(t\right) + \varphi^2(0,t)\right)\right) \\ &\quad + c\frac{\varepsilon_0}{E_*}\left(E\left(t\right) + \varphi^2(0,t)\right)\overline{H}'\left(\frac{\varepsilon_0}{E_*}\left(E\left(t\right) + \varphi^2(0,t)\right)\right) \\ &\quad + c\frac{\mu\lambda(t)}{\zeta\left(t\right)} + \left(E(t) + \varphi^2(0,t)\right)'. \end{aligned}$$

Then, we multiply by $\zeta\left(t\right)$ and use the fact that, as

$$\begin{aligned} &\tfrac{\varepsilon_0}{E_*}\left(E\left(t\right) + \varphi^2(0,t)\right) < r, \\ &\overline{H}'\left(\tfrac{\varepsilon_0}{E_*}\left(E\left(t\right) + \varphi^2(0,t)\right)\right) = H'\left(\tfrac{\varepsilon_0}{E_*}\left(E\left(t\right) + \varphi^2(0,t)\right)\right), \end{aligned}$$

to get

$$\begin{aligned}\zeta\left(t\right)F_1'(t) &\le -m\zeta\left(t\right)\left(E\left(t\right)+\varphi^2(0,t)\right)H'\left(\frac{\varepsilon_0}{E_*}\left(E\left(t\right)+\varphi^2(0,t)\right)\right)\\&\quad+c\frac{\varepsilon_0}{E_*}\left(E\left(t\right)+\varphi^2(0,t)\right)\zeta(t)H'\left(\frac{\varepsilon_0}{E_*}\left(E\left(t\right)+\varphi^2(0,t)\right)\right)\\&\quad+c\mu\lambda(t)+\zeta\left(t\right)\left(E\left(t\right)+\varphi^2(0,t)\right)'\left(t\right)\\&\le -m\zeta\left(t\right)\left(E\left(t\right)+\varphi^2(0,t)\right)H'\left(\frac{\varepsilon_0}{E_*}\left(E\left(t\right)+\varphi^2(0,t)\right)\right)\\&\quad+c\frac{\varepsilon_0}{E_*}\left(E\left(t\right)+\varphi^2(0,t)\right)\zeta(t)H'\left(\frac{\varepsilon_0}{E_*}\left(E\left(t\right)+\varphi^2(0,t)\right)\right)\\&\quad-c\left(E\left(t\right)+\varphi^2(0,t)\right)'.\end{aligned}$$

Consequenly, with $F_2\left(t\right)=\zeta\left(t\right)F_1\left(t\right)+c\left(E\left(t\right)+\varphi^2(0,t)\right)$, which satisfies, for some α_1, $\alpha_2>0$

$$\alpha_1F_2(t)\le\left(E\left(t\right)+\varphi^2(0,t)\right)\le\alpha_2F_2(t), \tag{3.50}$$

and with a suitable choice of ε_0, we obtain, for some constant $k>0$

$$\begin{aligned}F_2'(t) &\le -k\zeta(t)\left(\frac{\left(E\left(t\right)+\varphi^2(0,t)\right)}{E_*}\right)H'\left(\frac{\varepsilon_0}{E_*}\left(E\left(t\right)+\varphi^2(0,t)\right)\right)\\&= -k\zeta(t)H_2\left(\frac{\left(E\left(t\right)+\varphi^2(0,t)\right)}{E_*}\right),\quad t\ge t_1>0,\end{aligned} \tag{3.51}$$

where $H_2(t)=tH'(\varepsilon_0t)$.
Since $H_2'(t)=H'(\varepsilon_0t)+\varepsilon_0tH''(\varepsilon_0t)$, then, using the strict convexity of H on $]0,r]$, we find that $H_2'(t)$, $H_2(t)>0$ on $]0,1]$. Thus, with

$$R(t)=\frac{\alpha_1F_2(t)}{E_*},$$

taking in account (3.50) and (3.51), we have

$$R(t)\sim E\left(t\right)+\varphi^2(0,t) \tag{3.52}$$

and, for some $k_1>0$,

$$R'(t)\le -k_1\zeta(t)H_2\left(R(t)\right),\quad t\ge t_1.$$

Then, the integration over $]t_1,t[$ yields

$$\int_{t_1}^{t}\frac{-R'(\tau)}{H_2\left(R\left(\tau\right)\right)}d\tau\ge k_1\int_{t_1}^{t}\zeta(\tau)d\tau\Rightarrow\int_{\varepsilon_0R(t)}^{\varepsilon_0R(t_1)}\frac{1}{\tau H'(\tau)}d\tau\ge k_1\int_{t_1}^{t}\zeta(\tau)d\tau,$$

so,

$$R(t)\le\frac{1}{\varepsilon_0}H_1^{-1}\left(k_1\int_{t_1}^{t}\zeta(\tau)d\tau\right), \tag{3.53}$$

where $H_1(t)=\int_t^r\frac{1}{\tau H'(\tau)}d\tau$. Here, we have used, based on the properties of H_1 is strictly decreasing function on $]0,r]$ and $\lim\limits_{t\to0}H_1(t)=+\infty$. A combination of (3.52) and (3.53), estimate (3.37) is established.

□

4. example

1. If $g(t) = ae^{-t^{\alpha}}$ pour $0 < \alpha < 1$ and a chosen so that g satisfies (2.1), then
$$g'(t) = -H(g(t)),$$
where $H(t) = \frac{\alpha^{t}}{\left[\ln\left(\frac{a}{t}\right)\right]^{\frac{1}{\alpha}-1}}$, then the functions H satisfies hypothesis (**A1**)-(**A3**) on the interval $(0, r]$ for any $0 < r < a$. Therefore, we can directly use corollary 2.4 in [57] to get $E(t) \leq k'e^{-kt^{\alpha}}$.

2. On the other hand, if $g(t)$ is another function that satisfies (**A1**) and the inequality $g'(t) \leq -H(g(t))$ for the same H, then we should use (inequality 3.37) where
$$\begin{aligned} H_1(t) &= \int_t^r \frac{1}{sH'(s)}ds \\ &= \int_t^r \frac{\left[\ln\frac{a}{s}\right]^{\frac{1}{\alpha}}}{s\left[1-\alpha+\alpha\ln\frac{a}{s}\right]}ds \\ &= \frac{1}{\alpha}\int_{\ln\frac{a}{r}}^{\ln\frac{a}{t}} u^{\frac{1}{\alpha}-1}\left[\frac{u}{\frac{1-\alpha}{\alpha}+u}\right]du \\ &\leq \left[\ln\frac{a}{t}\right]^{\frac{1}{\alpha}}. \end{aligned}$$
So
$$H_1^{-1}(t) \leq ae^{-t^{\alpha}},$$
wich implies
$$E(t) \leq ak_1e^{-k_2t^{\alpha}}.$$
Considering the function $g(t) = \frac{a}{(t+e)[\ln(t+e)]^{\beta}}$ where $\beta > 1$ and a chosen so that (**A1**) is satisfied. $g(t)$ satisfies the conditions of Corollary 2.5 in [57], so we have the energy decay rates announced in Corollary 2.4 in [57]. We have,
$$g'(t) = \frac{-\left[\ln(t+e)+\beta\right]}{(t+e)\ln(t+e)}g(t)$$
and
$$\begin{aligned} E(t) &\leq ke^{-k'\int_0^t \frac{\ln(t+e)+\beta}{(t+e)\ln(t+e)}ds} \\ &= \frac{k}{\left[(t+e)\left[\ln(t+e)\right]^{\beta}\right]^{k'}}. \end{aligned}$$

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