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Necessary Results for Spectral Theory Associated with the Numerical Range on Hilbert Spaces

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ABSTRACT: This article examines aspects of the coupled numerical range for a linear relation and a linear operator on Hilbert spaces. First of all, we start by giving the new definition of this concept, and we study its properties. Additionally, necessary results for the spectral theory associated with the numerical range are discussed.

Key Words: Linear relations, numerical range, spectral sets, spectra.

Contents

1. Introduction

The concept of numerical range of an operator on Banach and Hilbert spaces were considered by many authors and studied in their work (e.g [2,7,8,9,10]). The classical definition of the numerical range of a closed linear operator in a Hilbert space is the range of the restriction to the unit sphere of the quadratic form associated with it. Among the most important properties of the numerical range proved by several authors are that it is convex and that its closure contains the spectrum of the operator.

Since we do not have the adjoint of non densely defined operators and the inverse of linear operators, we interest to study the theory of linear relation (or said to multivalued linear operator). In this case, each operator have an inverse and an adjoint. For more details, we refer the reader to [2,6].

Throughout this paper, H and K are infinite dimensional separable Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$. A linear relation $T : H \to K$ is a mapping from a subspace $\mathcal{D}(T) = \{x \in H : Tx \neq \emptyset\}$, the domain of T, into the collection of nonempty subsets of K such that

$$T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2),$$

for all nonzero scalars α_1, α_2 and $x_1, x_2 \in \mathcal{D}(T)$. Denote by $LR(H, K)$ a class of all linear relations from H to K. If T maps the points in its domain to singletons, then T is said to be an operator.

The graph, the null, the range space and the multivalued part of $T \in LR(H, K)$ are, respectively, denoted by

$$\begin{array}{rcl} G(T) & = & \{(x, y) \in H \times K : x \in \mathcal{D}(T)\}, \\ N(T) & = & \{x \in \mathcal{D}(T) : (x, 0) \in G(T)\}, \\ R(T) & = & \{y : (x, y) \in G(T)\}, \text{ and} \\ T(0) & = & \{y : (0, y) \in G(T)\}. \end{array}$$

The inverse and the adjoint relation of T are denoted by T^{-1} and T^* respectively, which are defined by

$$G(T^{-1}) = \{(y, x) : (x, y) \in G(T)\},$$

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and

$$G(T^*) = \Big\{(x,y) \in K \times H : \langle y, z\rangle = \langle x, t\rangle, \ \text{ for all } \ (z,t) \in G(T)\Big\}. \tag{1.1}$$

A linear relation $T \in LR(H)$ is said to be a self-adjoint relation if $T = T^*$.
For $T \in LR(K,H)$ and $S \in LR(H,K)$ where $R(T) \cap \mathcal{D}(S) \neq \emptyset$, we define ST by

$$G(ST) = \{(x,y) \in K \times K : (x,z) \in G(T) \text{ and } (z,y) \in G(S) \text{ for some } z \in H\}. \tag{1.2}$$

Denote by $BR(H,K)$ (respectively $CR(H,K)$) a class of all bounded (respectively closed) linear relations from H into K, by $\mathcal{L}(H,K)$ (respectively $\mathcal{C}(H,K)$) a class of all bounded (respectively closed) linear operators from H into K. If $H = K$, then $LR(H,H) = LR(H)$, $BR(H,H) = BR(H)$, $CR(H,H) = CR(H)$, $\mathcal{L}(H,H) = \mathcal{L}(H)$ and $\mathcal{C}(H,H) = \mathcal{C}(H)$. The closed and the open disc of $\mathbb{C}$ centred at λ_0 and with radius r denoted respectively by: $\overline{\mathbb{D}}(\lambda_0, r)$ and $\mathbb{D}(\lambda_0, r)$. If $\lambda_0 = 0$, then $\mathbb{D}(0,r) = \mathbb{D}_r$ and $\overline{\mathbb{D}}(0,r) = \overline{\mathbb{D}}_r$.

In this paper, we introduce the new concept of the numerical range of a linear relation and a linear operator on Hilbert spaces and study its properties. After that, we establish some relationship between the various concepts of spectral theory and the numerical range.

We organize our paper in the following way: Section 2, the numerical range of a linear relation and a linear operator on Hilbert spaces is introduced and studied. After that, some results of the spectral theory associated with the numerical range is established.

2. Main Results

Let $T \in LR(H,K)$ and let $A : \mathcal{D}(A) \subseteq H \longrightarrow K$ be a linear operator on Hilbert spaces H and K. To study the properties of the numerical range, coupled numerical range for a linear relation T and a linear operator A, we are need to define this notion, which is done in the following definition.

Definition 2.1 *Let H and K be Hilbert spaces. Let $T \in LR(H,K)$ and $A : \mathcal{D}(A) \subseteq H \longrightarrow K$ be a linear operator. The numerical range of T associated with A is defined as the set*

$$W(T,A) = \Big\{ \langle y, Ax\rangle : x \in \mathcal{D}(A) \cap \mathcal{D}(T), y \in Tx \ \text{and} \ \|Ax\| = 1\Big\}. \qquad \diamondsuit$$

Proposition 2.1 *Let H and K be Hilbert spaces.*

(i) If $T \in LR(H)$ and $A = I$, then

$$W(T,A) = W(T) = \Big\{ \langle y, x\rangle : (x,y) \in G(T) \ \text{and} \ \|x\| = 1\Big\},$$

where I is the identity operator on H.

(ii) If $T \in LR(H,K)$ and $A : \mathcal{D}(A) \subseteq H \longrightarrow K$ is a linear operator, then

$$W(T,A) = W(TA^{-1}).$$

(iii) If $T \in CR(H,K)$ and $A \in \mathcal{C}(H,K)$ are everywhere defined, then

$$W(T,A) \subset \overline{\mathbb{D}}_{\|TA^{-1}\|}. \qquad \diamondsuit$$

Proof: (i) For $A = I$, we have

$$\begin{aligned} W(T,I) &= \Big\{ \langle y, Ix\rangle : x \in \mathcal{D}(I) \cap \mathcal{D}(T), y \in Tx \text{ and } \|Ix\| = 1\Big\} \\ &= \Big\{ \langle y, x\rangle : x \in \mathcal{D}(T), y \in Tx \text{ and } \|x\| = 1\Big\} \\ &= \Big\{ \langle y, x\rangle : (x,y) \in G(T) \text{ and } \|x\| = 1\Big\} \\ &= W(T). \end{aligned}$$

(ii) Suppose that $\lambda \in W(T,A)$. Then there exist $x \in \mathcal{D}(A) \cap \mathcal{D}(T)$ and $y \in Tx$ satisfying

$$\lambda = \langle y, Ax \rangle, \quad \|Ax\| = 1. \tag{2.1}$$

The fact that $(Ax, x) \in G(A^{-1})$ and $(x, y) \in G(T)$ implies from (1.2) that $(Ax, y) \in G(TA^{-1})$. This implies from (2.1) and (i) that $\lambda \in W(TA^{-1})$. Hence,

$$W(T,A) \subseteq W(TA^{-1}).$$

Conversely, let us assume that $\lambda \in W(TA^{-1})$. Then there exist $(x, y) \in G(TA^{-1})$ satisfying

$$\lambda = \langle y, x \rangle, \quad \|x\| = 1. \tag{2.2}$$

It follows from (1.2) that there exists $z \in \mathcal{D}(A) \cap \mathcal{D}(T)$ such that $(x, z) \in G(A^{-1})$ and $(z, y) \in G(T)$. This implies that $x = Az$ and $(z, y) \in G(T)$. Hence, there exist $z \in \mathcal{D}(A) \cap \mathcal{D}(T)$ and $(z, y) \in G(T)$ satisfies from (2.2) that

$$\lambda = \langle y, Az \rangle, \quad \|Az\| = 1.$$

This is equivalent to saying that $\lambda \in W(T,A)$.

(iii) Let $\lambda \in W(T,A)$. From (ii), we get $\lambda \in W(TA^{-1})$. Hence, there exists $(x, y) \in G(TA^{-1})$ satisfying $\lambda = \langle y, x \rangle$, where $\|x\| = 1$. This yields from Cauchy-Schwarz inequality that

$$\begin{aligned} |\lambda| &= |\langle y, x \rangle| \\ &\leq \|y\|\|x\| \\ &\leq \|y\| \quad (\text{as } \|x\| = 1). \end{aligned} \tag{2.3}$$

Since $y \in TA^{-1}x$, by using [6, Proposition II.1.4 (d)] , we infer that

$$\|TA^{-1}x\| = \inf_{y \in TA^{-1}x} \|y\|.$$

Using the definition of the lower bound, for any given $\eta > 0$, we obtain that

$$\|y\| < \|TA^{-1}x\| + \eta. \tag{2.4}$$

Since $A^{-1}(0) = N(A) = \mathcal{D}(T)$, by [6, Corollary II.3.13 (2)], we obtain $\|TA^{-1}\| \leq \|T\| \, \|A^{-1}\|$. Based on the hypothesis $\mathcal{D}(T)$ is closed and $T \in CR(H)$, we infer that $\|T\| < \infty$. The fact that $\gamma(A) > 0$ implies from [6, Theorem II.2.5] that $\|A^{-1}\| < \infty$. Hence, $\|TA^{-1}\| < \infty$. For any given $\eta > 0$, we deduce from (2.3) and (2.4) that

$$\begin{aligned} |\lambda| &\leq \|TA^{-1}x\| + \eta \\ &\leq \|TA^{-1}\| + \eta. \end{aligned}$$

By arbitrariness of η, we conclude that $|\lambda| \leq \|TA^{-1}\|$. This equivalent to saying that $W(T,A) \subset \overline{\mathbb{D}}_{\|TA^{-1}\|}$. □

Remark 2.1 *Let $T \in LR(H,K)$ and $A : \mathcal{D}(A) \subseteq H \longrightarrow K$ be a linear operator such that $N(T) \nsubseteq N(A)$. Then,*

$$0 \in W(T,A).$$

Indeed, since $N(T) \nsubseteq N(A)$, $AT^{-1}(0) \neq 0$, which yields that TA^{-1} is not injective. This implies that there exist $x \in \mathcal{D}(TA^{-1})$ such that $\|x\| = 1$ and $0 \in TA^{-1}x$. Thus, $(x, 0) \in G(TA^{-1})$ with $\|x\| = 1$. Hence, $\langle 0, x \rangle \in W(TA^{-1})$. Consequently, $0 \in W(TA^{-1})$. Finally, the use of Proposition 2.1 (ii) allows us to conclude that $0 \in W(T,A)$.

Proposition 2.2 *Let $T, S \in LR(H,K)$, $A : \mathcal{D}(A) \subseteq H \longrightarrow K$ be a linear operator and $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ such that $|\alpha_3| = 1$. Then,*

(i) $W(\alpha_1 T, A) = \alpha_1 W(T, A)$.

(ii) $W(\alpha_1 S + \alpha_2 T, A) \subset \alpha_1 W(S, A) + \alpha_2 W(T, A)$.

(iii) $W(T, \alpha_3 A) = \overline{\alpha}_3 W(T, A)$.

Proof: *(i)* If $\alpha_1 = 0$, then

$$W(0T, A) = \{\langle 0, Ax\rangle : x \in \mathcal{D}(A) \text{ and } \|Ax\| = 1\} = \{0\}.$$

This implies that $W(\alpha_1 T, A) = \alpha_1 W(T, A)$.

Suppose that $\alpha_1 \in \mathbb{C}\backslash\{0\}$ and $\lambda \in W(\alpha_1 T, A)$. Then, there exists $x \in \mathcal{D}(A) \cap \mathcal{D}(\alpha_1 T)$ and $y \in \alpha_1 Tx$ satisfying

$$\lambda = \langle y, Ax\rangle \quad \text{with } \|Ax\| = 1. \tag{2.5}$$

The fact that $(x, y) \in G(\lambda_1 T)$ implies from [6, Subsection I.4.1 (6)] that $\alpha_1^{-1} y \in Tx$. This yields that $\left\langle \alpha_1^{-1} y, Ax\right\rangle \in W(T, A)$. It follows from (2.5) that

$$\lambda = \alpha_1 \left\langle \alpha_1^{-1} y, Ax\right\rangle \subset \alpha_1 W(T, A).$$

Hence, $W(\alpha_1 T, A) \subset \alpha_1 W(T, A)$. Conversely, assume that $\alpha_1 \in \mathbb{C}\backslash\{0\}$ and $\lambda \in \alpha_1 W(T, A)$. Then, $\alpha_1^{-1}\lambda \in W(T, A)$. This implies that there exists $x \in \mathcal{D}(A) \cap \mathcal{D}(T)$ and $y \in Tx$ satisfying

$$\alpha_1^{-1}\lambda = \langle y, Ax\rangle \text{ with } \|Ax\| = 1.$$

Therefore,

$$\begin{aligned} \lambda &= \alpha_1 \alpha_1^{-1} \lambda \\ &= \alpha_1 \langle y, Ax\rangle \\ &= \langle \alpha_1 y, Ax\rangle. \end{aligned} \tag{2.6}$$

The use of the hypothesis $(x, y) \in G(T)$ leads to deduce from [6, Subsection I.4.1 (6)] that $(x, \alpha_1 y) \in G(\lambda_1 T)$. Hence, by the fact that $x \in \mathcal{D}(A) \cap \mathcal{D}(T)$, we conclude from (2.6) that $\lambda \in W(\alpha_1 T, A)$.

(ii) If $\alpha_1 = \alpha_2 = 0$, it is clear that $W(\alpha_1 S + \alpha_2 T, A) \subset \alpha_1 W(S, A) + \alpha_2 W(T, A)$. Now, assume that $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$ and $\lambda \in W(\alpha_1 S + \alpha_2 T, A)$. Then, there exist $x \in \mathcal{D}(\alpha_1 S + \alpha_2 T) \cap \mathcal{D}(A) = \mathcal{D}(S) \cap \mathcal{D}(T) \cap \mathcal{D}(A)$ and $y \in (\alpha_1 S + \alpha_2 T)x$ satisfying

$$\lambda = \langle y, x\rangle, \quad \text{with } \|Ax\| = 1. \tag{2.7}$$

The fact that $(x, y) \in G(\alpha_1 S + \alpha_2 T)$ implies from [6, Subsection I.4.1 (5)] that there exists $z \in K$ such that

$$(x, z) \in G(\alpha_1 S) \quad \text{and } (x, y - z) \in G(\alpha_2 T).$$

This implies from [6, Subsection I.4.1 (6)] that $\left(x, \alpha_1^{-1} z\right) \in G(S)$ and $\left(x, \alpha_2^{-1}(y - z)\right) \in G(T)$. Since $x \in \mathcal{D}(S) \cap \mathcal{D}(T) \cap \mathcal{D}(A)$ and $\|Ax\| = 1$, we can conclude that

$$\left\langle \alpha_1^{-1} z, x\right\rangle \in W(S, A) \quad \text{and } \left\langle \alpha_2^{-1}(y - z), x\right\rangle \in W(T, A).$$

Thus, the use of (2.7) makes us deduce that

$$\begin{aligned} \lambda &= \langle y - z + z, x\rangle \\ &= \langle y - z, x\rangle + \langle z, x\rangle \\ &= \alpha_1 \left\langle \alpha_1^{-1} z, x\right\rangle + \alpha_2 \left\langle \alpha_2^{-1}(y - z), x\right\rangle \\ &\subset \alpha_1 W(S, A) + \alpha_2 W(T, A). \end{aligned}$$

(iii) Let $\lambda \in W(T, \alpha_3 A)$. Then, there exist $(x, y) \in G(T)$, $x \in \mathcal{D}(\alpha_3 A) = \mathcal{D}(A)$ such that $\|\alpha_3 Ax\| = 1$ and $\lambda = \langle y, \alpha_3 Ax\rangle$. Based on the assumption $|\alpha_3| = 1$, we infer that

$$1 = \|\alpha_3 Ax\| = |\alpha_3|\ \|Ax\| = \|Ax\|.$$

This implies that $\langle y, Ax\rangle \in W(T, A)$. Moreover, we have $\lambda = \overline{\alpha}_3\langle y, Ax\rangle$. Hence, $\lambda \in \overline{\alpha}_3 W(T, A)$. Conversely, let $\lambda \in \overline{\alpha}_3 W(T, A)$. Then, $(\overline{\alpha_3})^{-1}\lambda \in W(T, A)$. This implies that there exist $(x, y) \in G(T)$ and $x \in \mathcal{D}(A)$ satisfying

$$(\overline{\alpha_3})^{-1}\lambda = \langle y, Ax\rangle, \quad \text{with } \|Ax\| = 1.$$

Consequently, $\lambda = \overline{\alpha}_3\langle y, Ax\rangle = \langle y, \alpha_3(Ax)\rangle = \langle y, (\alpha_3 A)x\rangle$. Since $x \in \mathcal{D}(A) = \mathcal{D}(\alpha_3 A)$ and $\|\alpha_3 Ax\| = |\alpha_3|\ \|Ax\| = 1$, $\lambda \in W(T, \alpha_3 A)$. □

Proposition 2.3 *Let $T \in LR(H)$ and $A : \mathcal{D}(A) \subseteq H \longrightarrow H$ be a linear operator such that $\mathcal{D}(A) \subset R(T)$ and $G(T) \subset \mathcal{S}(G(A))$, where the operator $\mathcal{S}$ is defined by*

$$\begin{array}{rccc} \mathcal{S}: & H \times H & \longrightarrow & H \times H \\ & (x, y) & \longmapsto & (y, x). \end{array}$$

If T and A are self-adjoint, then $W(T, A)$ is a subset of the real axis.

Proof: Let us assume that $\lambda \in W(T, A)$. By referring to Proposition 2.1 *(ii)*, we get $\lambda \in W(TA^{-1})$. Then, there exist $(x, y) \in G(TA^{-1})$ satisfying

$$\lambda = \langle y, x\rangle, \quad \text{with } \|x\| = 1.$$

The fact that $(x, y) \in G(TA^{-1})$ implies from [6, Subsection I.1.3 (3)] that there exists $z \in \mathcal{D}(A) \cap \mathcal{D}(T)$ such that $(x, z) \in G(A^{-1})$ and $(z, y) \in G(T)$. Based on the hypotheses $G(T) \subset \mathcal{S}(G(A)) = G(A^{-1})$, $R(A^{-1}) = \mathcal{D}(A) \subset R(T)$ and $N(A^{-1}) = A(0) = 0 \in T^{-1}(0) = N(T)$, we infer from [6, Exercise I.2.14 (a)] that

$$G(T) = G(A^{-1}). \tag{2.8}$$

Since T and A are self-adjoint, we deduce from (2.8) and [6, Proposition III.1.3] that

$$G(T^*) = G(T) = G\left(A^{-1}\right) = G\left(\left(A^*\right)^{-1}\right) = G\left(\left(A^{-1}\right)^*\right).$$

This implies that $(x, z) \in G(T^*)$ and $(z, y) \in G\left(\left(A^{-1}\right)^*\right)$. Again from [6, Subsection I.1.3 (3)], we get that $(x, y) \in G\left(\left(A^{-1}\right)^* T^*\right)$. The use of [6, Theorem III.1.6] makes us to conclude that

$$(x, y) \in G\left(\left(TA^{-1}\right)^*\right).$$

It follows from the definition of the adjoint relation on Hilbert space (see (1.1)) that

$$\langle x, y\rangle = \langle y, x\rangle, \text{ for all } (x, y) \in G\left(TA^{-1}\right).$$

Therefore, $\lambda = \langle y, x\rangle = \langle x, y\rangle = \overline{\langle y, x\rangle} = \overline{\lambda}$. This is equivalent to saying that $\lambda \in \mathbb{R}$. □

Proposition 2.4 *If $T \in LR(H, K)$ and $A : \mathcal{D}(A) \subseteq H \longrightarrow K$ is a linear operator, then $W(T, A)$ is a convex set.*

Proof: Let us assume that λ_1 and λ_2 are two points in $W(T, A)$. We divide this proof into two cases.

First case. If $\lambda_1 = \lambda_2$, then we can conclude that

$$t\lambda_1 + (1 - t)\lambda_1 = \lambda_1 \in W(T, A), \quad \text{for all } t \in [0, 1].$$

This is equivalent to saying that $W(T,A)$ is a convex set.

Second case. If $\lambda_1 \neq \lambda_2 \in W(T,A)$, then there exist x_1, $x_2 \in \mathcal{D}(A) \cap \mathcal{D}(T)$, $y_1 \in Tx_1$ and $y_2 \in Tx_2$ such that $\|Ax_1\| = \|Ax_2\| = 1$ satisfying

$$\lambda_1 = \langle y_1, Ax_1 \rangle \quad \text{and } \lambda_2 = \langle y_2, Ax_2 \rangle \,.$$

• Suppose that x_1 and x_2 are linearly dependent, i.e., $x_2 = \alpha x_1$, for some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. This implies that $(x_1, y_1) \in G(T)$ and $(\alpha x_1, y_2) \in G(T)$. The fact that $G(T)$ is a vector subspace of $H \times K$ implies that $(\alpha x_1, \alpha y_1) \in G(T)$, which yields that $(0, \alpha y_1 - y_2) \in G(T)$. Hence, for all $\beta \in \mathbb{C}$, we have $(0, \beta(\alpha y_1 - y_2)) \in G(T)$. Thus,

$$(x_1, \beta(\alpha y_1 - y_2) + y_1) \in G(T).$$

Based on the hypothesis $(Ax_1, x_1) \in G(A^{-1})$, we infer from [6, Subsection I.1.3 (3)] that

$$(Ax_1, \beta(\alpha y_1 - y_2) + y_1) \in G(TA^{-1}), \quad \text{with } \|Ax_1\| = 1.$$

This implies that $\langle \beta(\alpha y_1 - y_2) + y_1, Ax_1 \rangle \in W(TA^{-1})$, which yields from Proposition 2.1 (ii) that

$$\langle \beta(\alpha y_1 - y_2) + y_1, Ax_1 \rangle \in W(T,A), \quad \text{for all } \beta \in \mathbb{C}.$$

As a result,

$$\mathbb{C} = \{\langle \beta(\alpha y_1 - y_2) + y_1, Ax_1 \rangle : \beta \in \mathbb{C}\} \subset W(T,A) \subset \mathbb{C}.$$

Therefore, $W(T,A) = \mathbb{C}$ is a convex set.

• Suppose that x_1 and x_2 are linearly independent. If λ_1 and λ_2 are distinct points in $W(T,A)$, then there exist $x_1, x_2 \in \mathcal{D}(A) \cap \mathcal{D}(T)$, $y_1 \in Tx_1$ and $y_2 \in Tx_2$ satisfying

$$\lambda_1 = \langle y_1, Ax_1 \rangle \quad \text{and } \lambda_2 = \langle y_2, Ax_2 \rangle \,, \quad \text{with } \|Ax_1\| = \|Ax_2\| = 1.$$

We must show that whenever λ is a point of the line segment joining λ_1 and λ_2, there exists an element $x_0 \in \mathcal{D}(A) \cap \mathcal{D}(T)$ and $y_0 \in Tx_0$, for which

$$\langle y_0, Ax_0 \rangle = \lambda, \quad \text{with } \|Ax_0\| = 1.$$

We find x_0 as a linear combination of x_1 and x_2. For this purpose and based on the hypotheses $\mathcal{D}(A) \cap \mathcal{D}(T)$ and $G(T)$ are vector subspaces of H and $H \times K$, respectively, we can conclude, for all $\alpha_1, \alpha_2 \in \mathbb{C}$, that

$$(\alpha_1 x_1 + \alpha_2 x_2, \alpha_1 y_1 + \alpha_2 y_2) \in G(T) \text{ and } (\alpha_1 x_1 + \alpha_2 x_2, A(\alpha_1 x_1 + \alpha_2 x_2)) \in G(A).$$

Hence, we can consider the binary forms $B_i : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$ where $i = 1, 2$ defined by

$$\begin{aligned} B_1(\alpha_1, \alpha_2) &= \langle \alpha_1 y_1 + \alpha_2 y_2, A(\alpha_1 x_1 + \alpha_2 x_2) \rangle \\ B_2(\alpha_1, \alpha_2) &= \langle A(\alpha_1 x_1 + \alpha_2 x_2), A(\alpha_1 x_1 + \alpha_2 x_2) \rangle \,. \end{aligned}$$

It follows from the fact that $\|Ax_1\| = \|Ax_2\| = 1$ that

$$\begin{aligned} B_1(\alpha_1, \alpha_2) &= \alpha_1 \overline{\alpha_1} \langle y_1, Ax_1 \rangle + \alpha_1 \overline{\alpha_2} \langle y_1, Ax_2 \rangle + \alpha_2 \overline{\alpha_1} \langle y_2, Ax_1 \rangle + \alpha_2 \overline{\alpha_2} \langle y_2, Ax_2 \rangle \\ &= |\alpha_1|^2 \lambda_1 + \alpha_1 \overline{\alpha_2} \langle y_1, Ax_2 \rangle + \alpha_2 \overline{\alpha_1} \langle y_2, Ax_1 \rangle + |\alpha_2|^2 \lambda_2, \end{aligned}$$

and

$$\begin{aligned} B_2(\alpha_1, \alpha_2) &= \alpha_1 \overline{\alpha_1} \langle Ax_1, Ax_1 \rangle + \alpha_1 \overline{\alpha_2} \langle Ax_1, Ax_2 \rangle + \alpha_2 \overline{\alpha_1} \langle Ax_2, Ax_1 \rangle + \alpha_2 \overline{\alpha_2} \langle Ax_2, Ax_2 \rangle \\ &= |\alpha_1|^2 \|Ax_1\|^2 + \alpha_1 \overline{\alpha_2} \langle Ax_1, Ax_2 \rangle + \alpha_2 \overline{\alpha_1} \langle Ax_2, Ax_1 \rangle + |\alpha_2|^2 \|Ax_2\|^2 \\ &= |\alpha_1|^2 + \alpha_1 \overline{\alpha_2} \langle Ax_1, Ax_2 \rangle + \alpha_2 \overline{\alpha_1} \langle Ax_2, Ax_1 \rangle + |\alpha_2|^2 \\ &= |\alpha_1|^2 + |\alpha_2|^2 + 2\Re\Big(\alpha_2 \overline{\alpha_1} \langle Ax_2, Ax_1 \rangle \Big). \end{aligned} \tag{2.9}$$

We must show that B_1 assumes every values on the line segment joining λ_1 and λ_2 while $B_2 = 1$. First, we introduce a binary form $B(\alpha_1, \alpha_2)$ as follows:

$$B(\alpha_1, \alpha_2) = \frac{B_1(\alpha_1, \alpha_2) - \lambda_2 B_2(\alpha_1, \alpha_2)}{\lambda_1 - \lambda_2}.$$

A simple calculation shows that

$$B(\alpha_1, \alpha_2) = |\alpha_1|^2 + \overline{\alpha_1}\alpha_2 b_{12} + \alpha_1 \overline{\alpha_2} b_{21}, \tag{2.10}$$

where

$$b_{12} = \left(\frac{\langle y_2, Ax_1 \rangle - \lambda_2 \langle Ax_1, Ax_2 \rangle}{\lambda_1 - \lambda_2} \right)$$

and

$$b_{21} = \left(\frac{\langle y_1, Ax_2 \rangle - \lambda_2 \langle Ax_1, Ax_2 \rangle}{\lambda_1 - \lambda_2} \right).$$

In order that B_1 should have the requisite behavior, it is necessary and sufficient that while $B_2 = 1$, the form B should take on every real value from 0 to 1 inclusive. We shall now exhibit values of α_1 and α_2 which bring about the desired result. Choose $\xi \in \mathbb{C}$ as follows:

$$\xi = \begin{cases} \pm 1 & \text{if } b_{21} = \overline{b_{12}} \\ \pm \dfrac{\overline{b_{12}} - b_{21}}{|\overline{b_{12}} - b_{21}|} & \text{if } b_{21} \neq \overline{b_{12}}. \end{cases}$$

So that $\Re(\xi \langle Ax_2, Ax_1 \rangle) \geq 0$. Now, we will replace α_1 and α_2 respectively by f and ξg, where $f, g \in \mathbb{R}$. On the one hand, we obtain from (2.10) that

$$\begin{aligned} B(f, \xi g) &= f^2 + \overline{\xi} f g b_{21} + \xi f g b_{12} \\ &= f^2 + f g (\overline{\xi} b_{21} + \xi b_{12}). \end{aligned} \tag{2.11}$$

At this point, we need to calculate the value of $\theta = \overline{\xi} b_{21} + \xi b_{12}$. Then, we will consider two cases:

- If $b_{21} = \overline{b_{12}}$, then $\overline{\xi} = \xi = \pm 1$. It follows from (2.11) that

$$B(f, \xi g) = f^2 \pm f g (\overline{b_{12}} + b_{12}).$$

Hence, $\theta = \overline{b_{12}} + b_{12} \in \mathbb{R}$, which yields that $\theta = \overline{\theta}$.

- If $b_{21} \neq \overline{b_{12}}$, then $\xi = \pm \dfrac{\overline{b_{12}} - b_{21}}{|\overline{b_{12}} - b_{21}|}$ and $\overline{\xi} = \pm \dfrac{b_{12} - \overline{b_{21}}}{|\overline{b_{12}} - b_{21}|}$. Hence,

$$\begin{aligned} \theta &= \frac{b_{21}(b_{12} - \overline{b_{21}}) + b_{12}(\overline{b_{12}} - b_{21})}{|\overline{b_{12}} - b_{21}|} \\ &= \frac{b_{21} b_{12} - b_{21}\overline{b_{21}} + b_{12}\overline{b_{12}} - b_{12} b_{21}}{|\overline{b_{12}} - b_{21}|} \\ &= \frac{|b_{12}|^2 - |b_{21}|^2}{|\overline{b_{12}} - b_{21}|} \in \mathbb{R}. \end{aligned}$$

Therefore, the real number θ is defined as follows:

$$\theta = \begin{cases} \pm(b_{21} + b_{12}) & \text{if } b_{21} = \overline{b_{12}} \\ \pm \dfrac{|b_{12}|^2 - |b_{21}|^2}{|\overline{b_{12}} - b_{21}|} & \text{if } b_{21} \neq \overline{b_{12}}. \end{cases}$$

On the other hand, we get from (2.9) that

$$\begin{aligned} B_2(f,\xi g) &= f^2+|\xi|^2g^2+2\Re\Big(\xi fg\langle Ax_2,Ax_1\rangle\Big) \\ &= f^2+g^2+2fg\;\Re\Big(\xi\langle Ax_2,Ax_1\rangle\Big),\quad (\text{as } |\xi|=1). \end{aligned}$$

Suppose that $\delta=\Re\Big(\xi\langle Ax_2,Ax_1\rangle\Big)$. We observe that $0\le\delta\le 1$. Indeed, by Cauchy-Schwarz inequality, we infer that

$$\begin{aligned} |\delta|=\Big|\Re\Big(\xi\langle Ax_2,Ax_1\rangle\Big)\Big| &\le |\xi\langle Ax_2,Ax_1\rangle| \\ &\le |\xi|\,|\langle Ax_2,Ax_1\rangle| \\ &\le |\xi|\,\|Ax_2\|\,\|Ax_1\| \\ &\le 1\ \ (\text{as } |\xi|=\|Ax_2\|=\|Ax_1\|=1). \end{aligned}$$

Solving for $B_2(f,\xi g)=1$, we get $g=-\delta f+\sqrt{1-(1-\delta^2)f^2}$, which is a real valued for $f\in[-1,1]$. Since g is a function of f, we can let $\widehat{B}(f)=B(f,\xi g)$, which yields from (2.11) that

$$\begin{aligned} \widehat{B}(f) &= f^2+\theta f\left(-\delta f+\sqrt{1-(1-\delta^2)f^2}\right) \\ &= f^2-\theta\delta f^2+\theta f\sqrt{1-(1-\delta^2)f^2} \\ &= (1-\theta\delta)f^2+\theta f\sqrt{1-(1-\delta^2)f^2}. \end{aligned}$$

For $f\in[0,1]$, $\widehat{B}(f)$ is evidently a continuous real valued function with $\widehat{B}(0)=0$ and $\widehat{B}(1)=1$, so that $\widehat{B}(f)$ assumes all values between 0 and 1. This result shows that $W(T,A)$ is a convex set. $\square$

To discuss the results of the classical spectral theory associated with the numerical range, we need to give the following definitions.

The resolvent set $\rho(T,A)$ of T and A is defined as

$$\rho(T,A)=\{\lambda\in\mathbb{C}:\lambda A-T\ \text{ is injective, open and has dense range }\}.$$

The spectrum $\sigma(T,A)$ of T and A is defined as

$$\sigma(T,A)=\mathbb{C}\backslash\rho(T,A).$$

The point spectrum $\sigma_p(T,A)$ of T and A is defined by

$$\sigma_p(T,A)=\{\lambda\in\mathbb{C}:N(\lambda A-T)\neq\{0\}\}.$$

The approximate point spectrum $\sigma_{ap}(T,A)$ of T and A is defined as

$$\sigma_{ap}(T,A)=\Big\{\lambda\in\mathbb{C}:\exists(x_n)\in\mathcal{D}(\lambda A-T)\text{ with }\|x_n\|=1\text{ and }\lim_{n\to+\infty}\|(\lambda A-T)x_n\|=0\Big\}.$$

The compression spectrum $\sigma_0(T,A)$ of T and A is defined as

$$\sigma_0(T,A)=\{\lambda\in\mathbb{C}:\overline{R(\lambda A-T)}\subsetneq K\}.$$

If $A=I$, then we recover the usual definition of the above spectra of linear relations, $\sigma(T,I)=\sigma(T)$ and $\sigma_i(T,I)=\sigma_i(T)$ for $i=p,ap,0$.

Remark 2.2 *Let $T\in CR(H,K)$ and $A\in\mathcal{L}(H,K)$. Then,*

$$\rho(T,A)=\left\{\lambda\in\mathbb{C}:(\lambda A-T)^{-1}\in\mathcal{L}(K,H)\right\}.$$

Indeed, it is clear that $\{\lambda \in \mathbb{C} : (\lambda A - T)^{-1} \in \mathcal{L}(K, H)\} \subset \rho(T, A)$. *Then, it is sufficient to show that*

$$\rho(T, A) \subset \{\lambda \in \mathbb{C} : (\lambda A - T)^{-1} \in \mathcal{L}(K, H)\}.$$

let $\lambda \in \rho(T, A)$. *Then,* $\lambda A - T$ *is injective, open and has dense range. This implies from* [6, Proposition II.3.2 (a)] *that* $(\lambda A - T)^{-1}$ *is a continuous operator. It remains to prove that* $\mathcal{D}\left((\lambda A - T)^{-1}\right) = K$. *Since* $T \in CR(H, K)$ *and* $\lambda A \in \mathcal{L}(H, K)$ *for all* $\lambda \in \mathbb{C}$, *by using* [6, Exercice II.5.16], *we infer that* $\lambda A - T \in CR(H, K)$. *It follows from the openness of* $\lambda A - T$ *and* [6, Theorem III.4.2 (b)] *that* $R(\lambda A - T)$ *is closed. Hence,*

$$\mathcal{D}\left((\lambda A - T)^{-1}\right) = R(\lambda A - T) = \overline{R\,(\lambda A - T)} = K. \qquad \diamondsuit$$

Proposition 2.5 *Let* $T \in LR(H, K)$, A *be an operator from* H *to* K *and* $\lambda \in \mathbb{C}$. *Then, the following statements are equivalent*

(i) $\lambda \notin \sigma_{ap}(T, A)$.

(ii) There exists a constant $c > 0$ *such that* $\|(\lambda A - T)x\| \geq c\,\|x\|$ *for all* $x \in \mathcal{D}(T) \cap \mathcal{D}(A)$.

(iii) $\lambda A - T$ *is injective and open.*

Proof: This proof is analogous to proof of [3, Proposition 2.1]. □

Theorem 2.1 *Let* $T \in LR(H, K)$ *and* $A : \mathcal{D}(A) \subseteq H \longrightarrow K$ *be a linear operator.*

(i) If $N(A) \cap \mathcal{D}(T) = \{0\}$, *then* $\sigma_p(T, A) \subset W(T, A)$.

(ii) $\sigma_{ap}(T, A) \subset \overline{W(T, A)}$.

(iii) If A *is surjective and* $\mathcal{D}(A) \subset \mathcal{D}(T)$, *then* $\sigma_0(T, A) \subset W(T, A)$.

Proof: (i) Let $\lambda \in \sigma_p(T, A)$. Then, the fact that $\mathcal{D}(\lambda A - T) = \mathcal{D}(A) \cap \mathcal{D}(T)$ implies that there exists $x \in \Big(\mathcal{D}(A) \cap \mathcal{D}(T)\Big)\backslash\{0\}$ such that $0 \in (\lambda A - T)x$. Since $N(A) \cap \mathcal{D}(T) = \{0\}$, then we get $x \notin N(A)$. This yields that $Ax \neq 0$. By taking $y = \dfrac{x}{\|Ax\|}$, we obtain $y \in \mathcal{D}(A) \cap \mathcal{D}(T)$ and $\|Ay\| = 1$. We know that $G(T)$ is a vector subspace of $H \times K$, we get $(y, 0) = \left(\dfrac{x}{\|Ax\|}, 0\right) \in G(\lambda A - T)$. It follows from [6, Subsection I.4.1 (5) and (6)] that there exists $z \in K$ such that $(y, z) \in G(\lambda A)$ and $(y, z) \in G(T)$. Hence,

$$\langle z, Ay\rangle = \langle \lambda Ay, Ay\rangle = \lambda\|Ay\|^2 = \lambda.$$

This implies from the above that $\lambda \in W(T, A)$.

(ii) Let us assume that $\lambda \in \sigma_{ap}(T, A)$. Then, there exists $(x_n) \subset \mathcal{D}(A) \cap \mathcal{D}(T)$ with $\|Ax_n\| = 1$ and $\lim\limits_{n\to+\infty} \|(\lambda A - T)x_n\| = 0$. The fact that $(x_n) \subset \mathcal{D}(T)$ implies that there exists $(y_n) \subset K$ such that $(x_n, y_n) \in G(T)$. This implies that $\langle y_n, Ax_n\rangle \in W(T, A)$. On the one hand, we have

$$\begin{aligned} |\lambda - \langle y_n, Ax_n\rangle| &= |\lambda\,\langle Ax_n, Ax_n\rangle - \langle y_n, Ax_n\rangle| \\ &= |\langle \lambda Ax_n - y_n, Ax_n\rangle| \\ &\leq \|\lambda Ax_n - y_n\|\|Ax_n\| \text{ (by Cauchy-Schwarz inequality)} \\ &\leq \|\lambda Ax_n - y_n\| \quad (\text{as } \|Ax_n\| = 1). \end{aligned} \tag{2.12}$$

On the other hand, by using the fact that $(x_n, y_n) \in G(T)$ and $(x_n, \lambda Ax_n) \in G(\lambda A)$, we infer from [6, Subsection I.4.1 (5) and (6)] that $(x_n, \lambda Ax_n - y_n) \in G(\lambda A - T)$. Hence, for all $\varepsilon > 0$, we have

$$\|\lambda Ax_n - y_n\| \leq \|(\lambda A - T)x_n\| + \varepsilon.$$

It follows from (2.12) that

$$|\langle y_n, Ax_n\rangle - \lambda| \leq \|(\lambda A - T)x_n\| + \varepsilon.$$

By arbitrariness of ε, we deduce that $|\langle y_n, Ax_n\rangle - \lambda| \to 0$ as $n \to +\infty$. This yields that $\langle y_n, Ax_n\rangle \to \lambda$ as $n \to +\infty$. This equivalent to saying that $\lambda \in \overline{W(T,A)}$.

(iii) Let $\lambda \in \sigma_0(T,A)$. Then $\overline{R(\lambda A - T)} \subsetneq K$. Since every closed subspace of a Hilbert space is topologically complemented by its orthogonal complement, we can conclude that

$$K = \overline{R(\lambda A - T)} \oplus R(\lambda A - T)^{\perp}.$$

This implies that there exists $z \in K$ such that $\|z\| = 1$ and $z \in R(\lambda A - T)^{\perp}$. The fact that A is surjective implies that there exists $x \in \mathcal{D}(A)$ such that $Ax = z$ and $\|Ax\| = 1$. Based on the hypothesis $\mathcal{D}(A) \subset \mathcal{D}(T)$, we infer that there exists $y \in K$ such that $(x,y) \in G(T)$. Hence, by referring to [6, Subsection I.4.1 (5) and (6)], we get $(x, \lambda Ax - y) \in G(\lambda A - T)$, which yields that $\lambda Ax - y \in R(\lambda A - T)$. This implies from the fact that $Ax \in R(\lambda A - T)^{\perp}$ that

$$\begin{aligned} 0 &= \langle \lambda Ax - y, Ax\rangle \\ &= \lambda\langle Ax, Ax\rangle - \langle y, Ax\rangle \\ &= \lambda - \langle y, Ax\rangle \quad (\text{as } \|Ax\| = 1). \end{aligned}$$

Therefore, $\lambda = \langle y, Ax\rangle$. This is equivalent to saying that $\lambda \in W(T,A)$. □

Lemma 2.1 *Let $T \in LR(H,K)$ and let $A \in \mathcal{L}(H,K)$ be non null operator.*

(i) $\sigma_{ap}(T,A)$ is closed.

(ii) If T is closed, then $\partial\sigma(T,A) \subset \sigma_{ap}(T,A)$.

Proof: (i) Our purpose is to prove that $\mathbb{C}\backslash\sigma_{ap}(T,A)$ is open set. In order to show it, let us assume that $\lambda \in \mathbb{C}\backslash\sigma_{ap}(T,A)$. Then, by Proposition 2.5, we infer that there exists $c > 0$ such that $\|(\lambda A - T)x\| \geq c\,\|x\|$ for all $x \in \mathcal{D}(T)$. Then, for $\mu \in \mathbb{C}$,

$$\begin{aligned} c\,\|x\| &\leq \|(\lambda - \mu)\,Ax + (\mu A - T)\,x\| \\ &\leq |\lambda - \mu|\ \|Ax\| + \|(\mu A - T)\,x\| \\ &\leq |\lambda - \mu|\ \|A\|\ \|x\| + \|(\mu A - T)\,x\| \quad (\text{as } A \in \mathcal{L}(H,K)). \end{aligned}$$

This implies that

$$\big(c - |\lambda - \mu|\ \|A\|\big)\ \|x\| \leq \|(\mu A - T)\,x\|\,.$$

Assume that $|\lambda - \mu| < c\,\|A\|^{-1}$. This enables us to conclude from Proposition 2.5 that $\mu \in \mathbb{C}\backslash\sigma_{ap}(T,A)$. Hence, $\mathbb{D}\left(\lambda, c\,\|A\|^{-1}\right) \subset \mathbb{C}\backslash\sigma_{ap}(T,A)$. This is equivalent to saying that $\mathbb{C}\backslash\sigma_{ap}(T,A)$ is open.

(ii) We assume that $\partial\sigma(T,A) \subset \mathbb{C}\backslash\sigma_{ap}(T,A)$. Then, $\partial\sigma(T,A) \cap \mathbb{C}\backslash\sigma_{ap}(T,A) \neq \emptyset$, which yields that there exist

$$\lambda \in \partial\sigma(T,A) \cap \mathbb{C}\backslash\sigma_{ap}(T,A).$$

By referring to (i), we have $\mathbb{C}\backslash\sigma_{ap}(T,A)$ is open. Then, we can consider a connected Ω of $\mathbb{C}\backslash\sigma_{ap}(T,A)$ containing λ. The fact that Ω is open and $\lambda \in \Omega$ implies that Ω is a neighborhood of λ. Consequently, $\lambda \in \partial\sigma(T,A) \cap \Omega$. Thus, Ω contains points of $\rho(T,A)$. Hence,

$$\rho(T,A) \cap \Omega \neq \emptyset,$$

which yields that there exist $\mu \in \rho(T,A) \cap \Omega$. Since $T \in CR(H,K)$ and $\mu A \in \mathcal{L}(H,K)$, by using [6, Exercice II.5.16], we obtain $\mu A - T \in CR(H,K)$. This leads us to $(\mu A - T)^{-1} \in \mathcal{L}(K,H)$.

If $\lambda = \mu$, then $\lambda \in \rho(T,A)$, which is contradiction because $\lambda \in \partial\sigma(T,A)$.

We assume $\lambda \neq \mu$. The fact that $\lambda \in \mathbb{C}\backslash\sigma_{ap}(T,A)$ implies from Proposition 2.5 and [6, Proposition II.3.2 (a)] that $(\lambda A - T)^{-1}$ is a continuous operator. At this point, we shall prove that

$$\mathcal{D}\left((\lambda A - T)^{-1}\right) = K.$$

We assume that $\mathcal{D}\left((\lambda A-T)^{-1}\right) \subsetneq K$. Then, there exist $y \in K$ such that $y \notin \mathcal{D}\left((\lambda A-T)^{-1}\right) = R\left(\lambda A-T\right)$. This implies for all $x \in \mathcal{D}(\lambda A-T) = \mathcal{D}(T)$ that $(x,y) \notin G\left((\lambda-\mu)A+(\mu A-T)\right)$. It follows from [6, Subsection I.4.1 (5) and (6)] that $(x,Ax) \notin G\left(A\right)$ or $(x, y-(\lambda-\mu)Ax) \notin G\left(\mu A-T\right)$. But, $(x,Ax) \notin G\left(A\right)$ is impossible and $y-(\lambda-\mu)Ax \notin R\left(\mu A-T\right) = K$ is also impossible. This contradiction implies that $\mathcal{D}\left((\lambda A-T)^{-1}\right) = K$. Hence, $\lambda \in \rho(T,A)$, which is contradiction. □

Remark 2.3 *Let $T \in CR(H)$. Then, $\sigma_{ap}(T)$ is closed and $\partial\sigma(T) \subset \sigma_{ap}(T)$.*

As a direct consequence of Theorem 2.1 and Lemma 2.1, we infer the following result:

Corollary 2.1 *Let $T \in LR(H,K)$ and let $A \in \mathcal{L}(H,K)$ be non null operator.*

$$\partial\sigma(T,A) \subset \overline{W(T,A)}. \qquad \diamondsuit$$

Theorem 2.2 *Let $T \in LR(H,K)$ and $A \in \mathcal{C}(H,K)$. If A is injective and $\gamma(A)>0$, then*

$$\sigma(T,A) \subset \overline{W(T,A)}. \qquad \diamondsuit$$

Proof: Let $\lambda \in \sigma(T,A)$. To prove $\lambda \in \overline{W(T,A)}$, assume the contrary, i.e., let $\lambda \notin \overline{W(T,A)}$. Then, $r = d(\lambda, \overline{W(T,A)}) = d(\lambda, W(T,A)) > 0$. This implies that there exist $x \in \mathcal{D}(T)\cap\mathcal{D}(A)$, $y \in Tx$ and $\|Ax\| = 1$ such that

$$\begin{aligned} 0<r &\leq |\lambda - \langle y, Ax\rangle| \\ &= |\lambda\left\langle Ax, Ax\right\rangle - \langle y, Ax\rangle| \\ &= |\langle \lambda Ax - y, Ax\rangle| \\ &\leq \|\lambda Ax - y\|\|Ax\| \text{ (by Cauchy-Schwarz inequality)} \\ &\leq \|\lambda Ax - y\| \quad (\text{as } \|Ax\|=1). \end{aligned} \tag{2.13}$$

The use of [6, Subsection I.4.1 (5) and (6)] makes us to infer that $(x, \lambda Ax - y) \in G(\lambda A - T)$. Hence, for all $\varepsilon_0 > 0$, we have

$$\|\lambda Ax - y\| \leq \|(\lambda A - T)x\| + \varepsilon_0.$$

Choose $0<\varepsilon_0<r$, we get

$$0 < (r-\varepsilon_0)\|Ax\| \leq \|(\lambda A - T)x\|. \tag{2.14}$$

By using the hypotheses A is injective and $\gamma(A)>0$, we deduce from [6, Definition III.6.1] that there exists $c>0$ such that $\|Ax\| \geq c\|x\|$. This implies from (2.14) that

$$c(r-\varepsilon_0)\|x\| \leq \|(\lambda A - T)x\|,$$

where $c(r-\varepsilon)>0$. Again from [6, Definition III.6.1], we conclude that $\lambda A - T$ is injective and open. Now, we shall show that

$$\overline{R(\lambda A - T)} = K.$$

By referring to [6, Proposition III.1.4 (a)], it is sufficient to prove that

$$N((\lambda A - T)^*) = \{0\}.$$

Assume that $(z, \overline{\lambda}t) \in G(\overline{\lambda}A^*)$ and $(z,f) \in G(T^*)$, we infer from [6, Subsection I.4.1 (5)] that $(z, \overline{\lambda}t - f) \in G(\overline{\lambda}A^* - T^*)$. This implies from [6, Proposition II.1.5 (a)] that

$$(z, \overline{\lambda}t - f) \in G((\lambda A - T)^*).$$

Hence, for all $\varepsilon_1 > 0$, we have

$$\|\overline{\lambda}t - f\| \leq \|(\lambda A - T)^* z\| + \varepsilon_1. \tag{2.15}$$

The fact that $(x, \lambda Ax - y) \in G(T - \lambda A)$ implies that $\langle \overline{\lambda} t - f, x\rangle = \langle z, \lambda Ax - y\rangle$. It follows from (2.15) that

$$\begin{aligned} \varepsilon_1 + \|(\lambda A - T)^* z\| &\geq \|\overline{\lambda} t - f\| \|Ax\| \\ &\geq c\, \|\overline{\lambda} t - f\| \|x\| \\ &\geq c\, |\langle \overline{\lambda} t - f, x\rangle| \text{ (by Cauchy-Schwarz inequality)} \\ &= c\, |\langle z, \lambda Ax - y\rangle|. \end{aligned} \tag{2.16}$$

The generalized Riesz representation theorem asserts that one can denote that $\varphi_{\lambda Ax-y}(z) = \langle z, \lambda Ax - y\rangle$, where φ is a bounded linear functional defined on K and verifies the following property

$$\|\varphi_{\lambda Ax-y}\| = \|\lambda Ax - y\|. \tag{2.17}$$

For all $z \neq 0$, we conclude from (2.16) that

$$\begin{aligned} \varepsilon_1 + \|(\lambda A - T)^* z\| &\geq c\, |\varphi_{\lambda Ax-y}(z)| \\ &\geq c\, \frac{|\varphi_{\lambda Ax-y}(z)|}{\|z\|} \|z\|. \end{aligned} \tag{2.18}$$

Using the definition of the upper bound, for any given $\varepsilon_2 > 0$, we obtain from (2.16) that

$$\frac{|\varphi_{\lambda Ax-y}(z)|}{\|z\|} \geq \|\varphi_{\lambda Ax-y}\| - \varepsilon_2.$$

This implies that

$$\begin{aligned} \varepsilon_1 + \|(\lambda A - T)^* z\| &\geq c\, \|\varphi_{\lambda Ax-y}\| \|z\| - c\, \varepsilon_2 \|z\| \\ &\geq c\, \|\lambda Ax - y\| \|z\| - c\, \varepsilon_2 \|z\| \quad \text{(from (2.17))} \\ &\geq c\, (r - \varepsilon_2)\, \|z\| \quad \text{(from (2.13))}. \end{aligned} \tag{2.19}$$

In addition, if $z = 0$, we infer that (2.19) holds. Hence, by arbitrariness of ε_1, we deduce that $(\lambda A - T)^*$ is bounded below, which yields from [6, Definition III.6.1] that $(\lambda A - T)^*$ is injective. Thus, $\overline{R(\lambda A - T)} = K$. This is equivalent to saying that $\lambda \in \rho(T, A)$, which is a contradiction. Therefore, $\lambda \in \overline{W(T, A)}$. □

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