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On CSI$-\xi^{\perp}-$Riemannian submersions from Sasakian manifolds

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ABSTRACT: In present paper, we study the Clairaut semi-invariant $\xi^{\perp}$-Riemannian submersions ($CSI-\xi^{\perp}$-Riemannian submersions, in short) from Sasakian manifolds onto Riemannian manifolds. We investigate fundamental results pertaining to the geometry of introduced submersions. We also work out on integrability conditions and totally geodesicness of distributions defined in such submersions. Finally, we construct a non-trivial example of $CSI-\xi^{\perp}$-Riemannian submersion from 5-dimensional Sasakian manifold onto Riemannian manifold.

Key Words: Riemannian map, Clairaut semi-invariant Riemannian map, Sasakian manifold.

Contents

1. Introduction

The theory of Riemannian submersions has the origin in the study of smooth maps between Riemannian manifolds. O' Neill [17] and Grey [10] independently studied the notion of Riemannian submersions. Later, Watson [29] studied almost Hermitian submersions and showed that horizontal and vertical distributions are invariant with respect to the almost complex structure. Since then this notion has been envolving in different directions of physics, robotics, mechanics etc. It is still an active and interesting field of research. Some important applications of Riemannian submersions are: Kaluza-Klein theory [9], Supergravity and superstring theories [11], in robotics [4], etc. The concept of anti-invariant submersion [23] and semi-invariant submersion ([13], [25]) was firstly defined by Sahin [23]. Further, Different kinds of Riemannian submersions have been studied, such as: slant submersions [24], semi-slant submersions [18], hemi-slant Riemannian submersions [19], quasi-bi-slant submersions [20], quasi-hemi-slant Riemannian submersions [21] (for details see [22]) etc. In 2013, Lee [15] studied anti invariant $\xi^{\perp}-$Riemannian submersions. As a generalization of anti-invariant $\xi^{\perp}-$Riemannian Submersions, Akyol, Sari and Aksoy [1] introduced semi-invariant $\xi^{\perp}-$ Riemannian Submersions as well as semi-slant $\xi^{\perp}-$Riemannian Submersions [2].

Firstly, Bishop [7] introduced and studied the concept of Clairaut submersion as: a submersion $\pi : M \rightarrow N$ is said to be a Clairaut submersion if there is a function $r : M \rightarrow R^{+}$ such that for every geodesic, making an angle θ with the horizontal subspaces, $r sin\theta$ is constant. Afterwards, this notion has been studied in Lorentzian spaces, timelike and spacelike spaces [14], static spacetimes [27], [28]. Moreover, Clairaut submersions have been further generalized in ([3], [5]). Lee et al. [14] investigated new conditions for anti-invariant Riemannian submersions to be Clairaut when the total manifolds are Kahlerian. In 2017, Sahin introduced Clairaut Riemannian map [26] and studied it's geometric properties. Recently, Yadav and Meena [30] studied Clairaut anti-invariant Riemannian maps from Kahler manifolds, Kumar et al. studied Clairaut semi-invariant Riemannian maps in [12] and Li and other Clairaut semi-invariant Riemannian maps from Cosymplectic manifolds in [16].

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In the present paper, we are interested in studying the idea of $CSI-\xi^{\perp}$-Riemannian submersions from Sasakian manifolds onto Riemannian manifolds. The article is organized as follows: In section 2, we gather some concepts, which are needed in the following parts. In section 3, we define the $CSI-\xi^{\perp}$-Riemannian submersions from Sasakian manifolds onto Riemannian manifolds and investigate differential geometric properties of such submersions. In section 4, we present an example of the $CSI-\xi^{\perp}$-Riemannian submersion from Sasakian manifold onto Riemannian manifold.

2. Preliminaries

A $(2m+1)$-dimensional differentiable manifold M_1 which admits a $(1,1)$ tensor field ϕ, a contravariant vector field ξ, a 1− form η such that

$$\phi^2 = -I + \eta \otimes \xi, \phi \circ \xi = 0, \eta \circ \xi = 0, \tag{2.1}$$

$$\eta(\xi) = 1, \tag{2.2}$$

where I denote the identity tensor. The manifold M_1 with an almost contact structure (ϕ, ξ, η) is called an almost contact manifold [8].

If there exists a Riemannian metric g_1 on an almost contact manifold M_1 satisfying the following conditions

$$g_1(\phi W_1, \phi W_2) = g_1(W_1, W_2) - \eta(W_1)\eta(W_2), g_1(\phi W_1, W_2) = -g_1(W_1, \phi W_2), \tag{2.3}$$

$$g_1(W_1, \xi) = \eta(W_1), \tag{2.4}$$

where W_1, W_2 are the vector fields on M_1, then structure (ϕ, ξ, η, g_1) is called almost contact metric structure and the manifold M_1 is called an almost contact metric manifold. An almost contact manifold M_1 with almost contact metric structure (ϕ, ξ, η, g_1) is denoted by $(M_1, \phi, \xi, \eta, g_1)$. Further, an almost contact structure (ϕ, ξ, η) is said to be normal if $N + d\eta \otimes \xi = 0$, where N is the Nijenhuis tensor [31] of ϕ. The fundamental 2-form Φ is defined by $\Phi(W_1, W_2) = g_1(W_1, \phi W_2)$.

An almost contact metric manifold $(M_1, \phi, \xi, \eta, g_1)$ is said to be Sasakian manifold [31] if it satisfies the following condition

$$(\nabla_{W_1}\phi)W_2 = g_1(W_1, W_2)\xi - \eta(W_2)W_1, \tag{2.5}$$

where ∇ represents the operator of covariant differentiation with respect to the Riemannian metric g_1 and W_1, W_2 vector fields on M_1.

For a Sasakian manifold M_1, we have

$$\nabla_{W_1}\xi = -\phi W_1 \tag{2.6}$$

for any vector field W_1 on M_1.

Example 2.1 [8] Let R^{2s+1} with Cartesian coordinates (x_i, y_i, z) $(i = 1, 2.....s)$ and its usual contact form

$$\eta = \frac{1}{2}(dz - \sum_{i=1}^{s} y_i dx_i).$$

The characteristic vector field ξ is given by $2\frac{\partial}{\partial z}$ and its Riemannian metric $g_{R^{2s+1}}$ and tensor field ϕ are given by

$$g_{R^{2s+1}} = (\eta \otimes \eta) + \frac{1}{4}\sum_{i=1}^{s}(dx_i)^2 + (dy_i)^2),$$

$$\phi(\sum_{i=1}^{s}(X_i\frac{\partial}{\partial x_i} + Y_i\frac{\partial}{\partial y_i}) + Z\frac{\partial}{\partial z}) = \sum_{i=1}^{s}(Y_i\frac{\partial}{\partial x_i} - X_i\frac{\partial}{\partial y_i}) + \sum_{i=1}^{k} Y_i y^i \frac{\partial}{\partial z}, .$$

This gives a contact metric structure on R^{2s+1}. The vector fields $E_i = 2\frac{\partial}{\partial y_i}$, $E_{k+i} = 2(\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial z})$ and ξ form a $\phi-$basis for the contact metric structure. On the other hand, it can be shown that $(R^{2s+1}, \phi, \xi, \eta, g_1)$ is a Sasakian manifold.

Define O'Neill's tensors [17] $\mathcal{T}$ and $\mathcal{A}$ by

$$\mathcal{A}_{Z_1}Z_2 = \mathcal{H}\nabla_{\mathcal{H}Z_1}\mathcal{V}Z_2 + \mathcal{V}\nabla_{\mathcal{H}Z_1}\mathcal{H}Z_2, \tag{2.7}$$

$$\mathcal{T}_{Z_1}Z_2 = \mathcal{H}\nabla_{\mathcal{V}Z_1}\mathcal{V}Z_2 + \mathcal{V}\nabla_{\mathcal{V}Z_1}\mathcal{H}Z_2 \tag{2.8}$$

for any vector fields Z_1, Z_2 on M_1, where ∇ is the Levi-Civita connection of g_1. It is easy to see that $\mathcal{T}_{Z_1}$ and $\mathcal{A}_{Z_1}$ are skew-symmetric operators on the tangent bundle of M_1 reversing the vertical and the horizontal distributions.

From equations (2.7) and (2.8), we have

$$\nabla_{Y_1}Y_2 = \mathcal{T}_{Y_1}Y_2 + \mathcal{V}\nabla_{Y_1}Y_2, \tag{2.9}$$

$$\nabla_{Y_1}W_1 = \mathcal{T}_{Y_1}W_1 + \mathcal{H}\nabla_{Y_1}W_1, \tag{2.10}$$

$$\nabla_{W_1}Y_1 = \mathcal{A}_{W_1}Y_1 + \mathcal{V}\nabla_{W_1}Y_1, \tag{2.11}$$

$$\nabla_{W_1}W_2 = \mathcal{H}\nabla_{W_1}W_2 + \mathcal{A}_{W_1}W_2 \tag{2.12}$$

for all $Y_1, Y_2 \in \Gamma(\ker \pi_*)$ and $W_1, W_2 \in \Gamma(\ker \pi_*)^{\perp}$, where $\mathcal{H}\nabla_{Y_1}W_1 = \mathcal{A}_{W_1}Y_1$, if W_1 is basic. It is not difficult to observe that $\mathcal{T}$ acts on the fibers as the second fundamental form, while $\mathcal{A}$ acts on the horizontal distribution and measures the obstruction to the integrability of this distribution.

The differentiable map π between two Riemannian manifolds is totally geodesic if

$$(\nabla\pi_*)(U_1, U_2) = 0, \text{ for all } U_1, U_2 \in \Gamma(TM_1).$$

A totally geodesic map is that it maps every geodesic in the total space into a geodesic in the base space in proportion to arc lengths. A Riemannian submersion is a Riemannian submersion with totally umbilical fibers [6] if

$$\mathcal{T}_{Y_1}Y_2 = g_1(Y_1, Y_2)H, \tag{2.13}$$

for all $Y_1, Y_2 \in \Gamma(\ker \pi_*)$, where H is the mean curvature vector field of fibers.

Let $\pi : (M_1, g_1) \to (M_2, g_2)$ is a smooth map between Riemannian manifolds. Then the differential map π_* of π can be observed a section of the bundle $Hom(TM_1, \pi^{-1}TM_2) \to M_1$, where $\pi^{-1}TM_2$ is the bundle which has fibers $(\pi^{-1}TM_2) = TM_2$ has a connection ∇ induced from the Riemannian connection ∇^{M_1} and the pullback connection. Then the second fundamental form of π is given by

$$(\nabla\pi_*)(W_1, W_2) = \nabla^{\pi}_{W_1}\pi_*(W_2) - \pi_*(\nabla^{M_1}_{W_1}W_2), \tag{2.14}$$

for vector field $W_1, W_2 \in \Gamma(TM_1)$, where ∇^{π} is the pullback connection. We know that the second fundamental form is symmetric.

Lemma 2.1 *[6] Let (M_1, g_1) and (M_2, g_2) are two Riemannian manifolds. If $\pi : M_1 \to M_2$ Riemannian submersion between Riemannian manifolds, then for any horizontal vector fields U_1, U_2 and vertical vector fields V_1, V_2, we have*

(i) $(\nabla\pi_*)(U_1, U_2) = 0$,
(ii) $(\nabla\pi_*)(V_1, V_2) = -\pi_*(\mathcal{T}_{V_1}V_2) = -\pi_*(\nabla^{M_1}_{V_1}V_2)$,
(iii) $(\nabla\pi_*)(U_1, V_1) = -\pi_*(\nabla^{M_1}_{U_1}V_1) = -\pi_*((\mathcal{A}_{U_1}V_1)$.

Now, we recall following definitions for later use:

Definition 2.1 *[22] Let π be a Riemannian submersion from an almost Hermitian manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2). Then, we say that π is an invariant Riemannian submersion if the vertical distribution is invariant with respect to the complex structure J, i.e.,*

$$J(\ker \pi_*) = \ker \pi_*.$$

Definition 2.2 *[15] Let $\pi : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ be a Riemannian submersion from an almost contact metric manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (M_2, g_2). Suppose that there exists a Riemannian submersion π such that ξ is normal to $(\ker \pi_*)$ and $(\ker \pi_*)$ is anti-invariant with respect to ϕ i.e., $\phi(\ker \pi_*) \subset (\ker \pi_*)^\perp$. Then we say that π is an anti-invariant $\xi^\perp$-Riemannian submersion.*

Definition 2.3 *[1] Let $\pi : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ be a Riemannian submersion from an almost contact metric manifold onto a Riemannian manifold manifold. A Riemannian submersion π is called a semi-invariant $\xi^\perp$-Riemannian submersion if there is a distribution $D_1 \subset (\ker \pi_*)$ such that*

$$(\ker \pi_*) = D_1 \oplus D_2, \phi(D_1) = D_1, \phi(D_2) \subset (\ker \pi_*)^\perp,$$

where D_2 is orthogonal complementary to D_1 in $(\ker \pi_*)$.

3. CSI$-\xi^\perp-$Riemannian submersions from a Sasakian manifold

Firstly, Bishop [7] introduces the notion of Clairaut submersion in the following way:

Definition 3.1 *[7] A Riemannian submersion $\pi : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ is called a Clairaut submersion if there exists a positive function r on M_1, such that, for any geodesic α on M_1, the function $(r \circ \alpha) \sin \theta$ is constant, where, for any t, $\theta(t)$ is the angle between $\dot{\alpha}$ and the horizontal space at $\alpha(t)$.*

He also gave the following necessary and sufficient condition for a Riemannian submersion to be a Clairaut submersion as:

Theorem 3.1 *[7] Let $\pi : (M_1, g_1) \to (M_2, g_2)$ be a Riemannian submersion with connected fibers. Then, π is a Clairaut Riemannian submersion with $r = e^f$ if and only if each fiber is totally umbilical and has the mean curvature vector field $H = -\nabla f$, where ∇f is the gradient of the function f with respect to g_1.*

Definition 3.2 *A semi-invariant $\xi^\perp$-Riemannian submersion from a Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (M_2, g_2) is called Clairaut semi-invariant $\xi^\perp$-Riemannian submersion if it satisfies the condition of Clairaut Riemannian submersion.*

Now, let π be a $CSI - \xi^\perp$-Riemannian submersion from a Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (M_2, g_2). Using definition (3.2), we have

$$(\ker \pi_*) = D_1 \oplus D_2.$$

Thus for any $X_1 \in (\ker \pi_*)$, we put

$$X_1 = PX_1 + QX_1,$$

where $PX_1 \in \Gamma(D_1)$ and $QX_1 \in \Gamma(D_2)$.

In addition, for $V_1 \in (\ker \pi_*)$, we get

$$\phi V_1 = \psi V_1 + \omega V_1, \tag{3.1}$$

where $\psi V_1 \in \Gamma(D_1)$ and $\omega V_1 \in \Gamma \phi(D_2)$.

The horizontal distribution $\Gamma(\ker \pi_*)^\perp$ is decomposed as

$$\Gamma(\ker \pi_*)^\perp = \phi(D_2) \oplus \mu.$$

Here μ is an invariant distribution of ϕ and contains ξ.

Also for $V_2 \in \Gamma(\ker \pi_*)^\perp$, we have

$$\phi V_2 = BV_2 + CV_2, \tag{3.2}$$

where $BV_2 \in \Gamma(D_2)$ and $CV_2 \in \Gamma(\mu)$.

Lemma 3.1 *Let π be a semi-invariant $\xi^{\perp}$-Riemannian submersion from a Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (M_2, g_2). Then, we get*

$$\mathcal{V}\nabla_{Y_1}\psi Y_2 + \mathcal{T}_{Y_1}\omega Y_2 = B\mathcal{T}_{Y_1}Y_2 + \psi\mathcal{V}\nabla_{Y_1}Y_2, \tag{3.3}$$

$$\mathcal{T}_{Y_1}\psi Y_2 + \mathcal{H}\nabla_{Y_1}\omega Y_2 = C\mathcal{T}_{Y_1}Y_2 + \omega\mathcal{V}\nabla_{Y_1}Y_2 + g_1(Y_1, Y_2)\xi, \tag{3.4}$$

$$\mathcal{V}\nabla_{V_1}BV_2 + \mathcal{A}_{V_1}CV_2 = B\mathcal{H}\nabla_{V_1}V_2 + \psi\mathcal{A}_{V_1}V_2, \tag{3.5}$$

$$\mathcal{A}_{V_1}BV_2 + \mathcal{H}\nabla_{V_1}CV_2 + \eta(V_2)V_1 = C\mathcal{H}\nabla_{V_1}V_2 + \omega\mathcal{A}_{V_1}V_2 + g_1(V_1, V_2)\xi, \tag{3.6}$$

$$\mathcal{V}\nabla_{Y_1}BV_1 + \mathcal{T}_{Y_1}CV_1 + \eta(V_1)Y_1 = \psi\mathcal{T}_{Y_1}V_1 + B\mathcal{H}\nabla_{Y_1}V_1, \tag{3.7}$$

$$\mathcal{T}_{Y_1}BV_1 + \mathcal{H}\nabla_{Y_1}CV_1 = \omega\mathcal{T}_{Y_1}V_1 + C\mathcal{H}\nabla_{Y_1}V_1, \tag{3.8}$$

$$\mathcal{V}\nabla_{V_1}\psi Y_1 + \mathcal{A}_{V_1}\omega Y_1 = B\mathcal{A}_{V_1}Y_1 + \psi\mathcal{V}\nabla_{V_1}Y_1, \tag{3.9}$$

$$\mathcal{A}_{V_1}\psi Y_1 + \mathcal{H}\nabla_{V_1}\omega Y_1 = C\mathcal{A}_{V_1}Y_1 + \omega\mathcal{V}\nabla_{V_1}Y_1, \tag{3.10}$$

where $Y_1, Y_2 \in \Gamma(\ker\pi_*)$ and $V_1, V_2 \in \Gamma(\ker\pi_*)^{\perp}$.

Proof: Using equations (2.5), (2.9)−(2.12), (3.1) and (3.2), we get all equations of Lemma 3.1. □

Lemma 3.2 *Let π be a semi-invariant $\xi^{\perp}$-Riemannian submersion from a Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (M_2, g_2). If $\alpha : I_2 \subset R \to M_1$ is a regular curve and $Y_1(t)$ and $Y_2(t)$ are the vertical and horizontal components of the tangent vector field $\dot{\alpha} = E$ of $\alpha(t)$, respectively, then α is a geodesic if and only if along α the following equations hold:*

$$\mathcal{V}\nabla_{\dot{\alpha}}\psi Y_1 + \mathcal{V}\nabla_{\dot{\alpha}}BY_2 + (\mathcal{T}_{Y_1} + \mathcal{A}_{Y_2})\omega Y_1 + (\mathcal{T}_{Y_1} + \mathcal{A}_{Y_2})CY_2 + \eta(Y_2)Y_1 = 0, \tag{3.11}$$

$$\mathcal{H}\nabla_{\dot{\alpha}}\omega Y_1 + \mathcal{H}\nabla_{\dot{\alpha}}CY_2 + (\mathcal{T}_{Y_1} + \mathcal{A}_{Y_2})\psi Y_1 + (\mathcal{T}_{Y_1} + \mathcal{A}_{Y_2})BY_2 - g_1(\dot{\alpha}, \dot{\alpha})\xi + \eta(Y_2)Y_2 = 0. \tag{3.12}$$

Proof: Let $\alpha : I_2 \to M_1$ be a regular curve on M_1. Since $Y_1(t)$ and $Y_2(t)$ are the vertical and horizontal parts of the tangent vector field $\dot{\alpha}(t)$, i.e., $\dot{\alpha}(t) = Y_1(t) + Y_2(t)$. From equations (2.1), (2.5), (2.9)-(2.12), (3.1) and (3.2), we get

$$\begin{aligned}
& \phi\nabla_{\dot{\alpha}}\dot{\alpha} \\
= \; & \nabla_{\dot{\alpha}}\phi\dot{\alpha} - (\nabla_{\dot{\alpha}}\phi)\dot{\alpha}, \\
= \; & \nabla_{Y_1}\psi Y_1 + \nabla_{Y_1}\omega Y_1 + \nabla_{Y_1}BY_2 + \nabla_{Y_1}CY_2 + \nabla_{Y_2}\psi Y_1 + \nabla_{Y_2}\omega Y_1 + \\
& \nabla_{Y_2}BY_2 + \nabla_{Y_2}CY_2 - g_1(\dot{\alpha}, \dot{\alpha})\xi + \eta(Y_2)Y_1 + \eta(Y_2)Y_2, \\
= \; & \mathcal{T}_{Y_1}\psi Y_1 + \mathcal{V}\nabla_{Y_1}\psi Y_1 + \mathcal{T}_{Y_1}\omega Y_1 + \mathcal{H}\nabla_{Y_1}\omega Y_1 + \mathcal{T}_{Y_1}BY_2 + \mathcal{V}\nabla_{Y_1}BY_2 + \\
& \mathcal{T}_{Y_1}CY_2 + \mathcal{H}\nabla_{Y_1}CY_2 + \mathcal{A}_{Y_2}\psi Y_1 + \mathcal{V}\nabla_{Y_2}\psi Y_1 + \mathcal{H}\nabla_{Y_2}\omega Y_1 + \mathcal{A}_{Y_2}\omega Y_1 + \\
& \mathcal{A}_{Y_2}BY_2 + \mathcal{V}\nabla_{Y_2}BY_2 + \mathcal{H}\nabla_{Y_2}CY_2 + \mathcal{A}_{Y_2}CY_2 - g_1(\dot{\alpha}, \dot{\alpha})\xi + \eta(Y_2)Y_1 + \eta(Y_2)Y_2.
\end{aligned}$$

Taking the vertical and horizontal components in above equation, we have

$$\mathcal{V}\phi\nabla_{\dot{\alpha}}\dot{\alpha} = \mathcal{V}\nabla_{\dot{\alpha}}\psi Y_1 + \mathcal{V}\nabla_{\dot{\alpha}}BY_2 + (\mathcal{T}_{Y_1} + \mathcal{A}_{Y_2})\omega Y_1 + (\mathcal{T}_{Y_1} + \mathcal{A}_{Y_2})CY_2 + \eta(Y_2)Y_1$$

$$\mathcal{H}\phi\nabla_{\dot{\alpha}}\dot{\alpha} = \mathcal{H}\nabla_{\dot{\alpha}}\omega Y_1 + \mathcal{H}\nabla_{\dot{\alpha}}CY_2 + (\mathcal{T}_{Y_1} + \mathcal{A}_{Y_2})\psi Y_1 + (\mathcal{T}_{Y_1} + \mathcal{A}_{Y_2})BY_2 - g_1(\dot{\alpha}, \dot{\alpha})\xi + \eta(Y_2)Y_2,$$

Now, α is a geodesic on M_1 if and only if $\mathcal{V}\phi\nabla_{\dot{\alpha}}\dot{\alpha} = 0$ and $\mathcal{H}\phi\nabla_{\dot{\alpha}}\dot{\alpha} = 0$, which completes the proof. □

Theorem 3.2 *Let π be a semi-invariant $\xi^{\perp}$-Riemannian submersion from a Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (M_2, g_2). Then π is a $CSI - \xi^{\perp}$-Riemannian submersion with $r = e^f$ if and only if*

$$\begin{aligned} & g_1(\nabla f, V_2)||V_1||^2 \\ = \; & g_1(\mathcal{V}\nabla_{\dot{\alpha}} BV_2, \psi V_1) + g_1(\mathcal{H}\nabla_{\dot{\alpha}} CV_2, \omega V_1) + g_1((\mathcal{T}_{V_1} + \mathcal{A}_{V_2})CV_2, \psi V_1) + \\ & g_1((\mathcal{T}_{V_1} + \mathcal{A}_{V_2})BV_2, \omega V_1) + \eta(V_2) g_1(V_1, \psi V_1) + \eta(V_2) g_1(V_2, \omega V_1)). \end{aligned}$$

where $\alpha : I_2 \to M_1$ is a geodesic on M_1 and V_1, V_2 are vertical and horizontal components of $\dot{\alpha}(t)$.

Proof: Let $\alpha : I_2 \to M_1$ be a geodesic on M_1 with $V_1(t) = \mathcal{V}\dot{\alpha}(t)$ and $V_2(t) = \mathcal{H}\dot{\alpha}(t)$. Let $\theta(t)$ denote the angle in $[0, \pi]$ between $\dot{\alpha}(t)$ and $V_2(t)$. Assuming $\nu = ||\dot{\alpha}(t)||,^2$ then we get

$$g_1(V_1(t), V_1(t)) = \nu \sin^2 \theta(t), \tag{3.13}$$

$$g_1(V_2(t), V_2(t)) = \nu \cos^2 \theta(t). \tag{3.14}$$

Now, differentiating (3.13), we get

$$\frac{d}{dt} g_1(V_1(t), V_1(t)) = 2\nu \cos\theta(t) \sin\theta(t) \frac{d\theta}{dt}.$$

Using equation (2.3), we get

$$g_1(\phi \nabla_{\dot{\alpha}} V_1, \phi V_1) = \nu \cos\theta(t) \sin\theta(t) \frac{d\theta}{dt}. \tag{3.15}$$

Now, using equation (2.5), we get

$$\nabla_{\dot{\alpha}} \phi V_1 = \phi \nabla_{\dot{\alpha}} V_1 + g_1(\dot{\alpha}, V_1)\xi,$$

$$\begin{aligned} & g_1(\phi \nabla_{\dot{\alpha}} V_1, \phi V_1) \\ = \; & g_1(\nabla_{\dot{\alpha}} \phi V_1, \phi V_1), \\ = \; & g_1(\mathcal{V}\nabla_{\dot{\alpha}} \psi V_1, \psi V_1) + g_1(\mathcal{H}\nabla_{\dot{\alpha}} \omega V_1, \omega V_1) + g_1((\mathcal{A}_{V_2} + \mathcal{T}_{V_1})\psi V_1, \omega V_1) + \\ & g_1((\mathcal{A}_{V_2} + \mathcal{T}_{V_1})\omega V_1, \psi V_1). \end{aligned}$$

Using equations (3.11) and (3.12), in above equation, we get

$$\begin{aligned} & g_1(\phi \nabla_{\dot{\alpha}} V_1, \phi V_1) \\ = \; & -g_1(\mathcal{V}\nabla_{\dot{\alpha}} BV_2, \psi V_1) - g_1(\mathcal{H}\nabla_{\dot{\alpha}} CV_2, \omega V_1) - g_1((\mathcal{T}_{V_1} + \mathcal{A}_{V_2})CV_2, \psi V_1) - \\ & g_1((\mathcal{T}_{V_1} + \mathcal{A}_{V_2})BV_2, \omega V_1) - \eta(V_2) g_1(V_1, \psi V_1) - \eta(V_2) g_1(V_2, \omega V_1). \end{aligned} \tag{3.16}$$

From equations (3.15) and (3.16), we have

$$\begin{aligned} & \nu \cos\theta(t) \sin\theta(t) \frac{d\theta}{dt} \\ = \; & -g_1(\mathcal{V}\nabla_{\dot{\alpha}} BV_2, \psi V_1) - g_1(\mathcal{H}\nabla_{\dot{\alpha}} CV_2, \omega V_1) - g_1((\mathcal{T}_{V_1} + \mathcal{A}_{V_2})CV_2, \psi V_1) - \\ & g_1((\mathcal{T}_{V_1} + \mathcal{A}_{V_2})BV_2, \omega V_1) - \eta(V_2) g_1(V_1, \psi V_1) - \eta(V_2) g_1(V_2, \omega V_1). \end{aligned} \tag{3.17}$$

Moreover, π is a $CSI - \xi^{\perp}$-Riemannian submersion with $r = e^f$ if and only if $\frac{d}{dt}(e^{f \circ \alpha} \sin\theta) = 0$, i.e., $e^{f \circ \alpha}(\cos\theta \frac{d\theta}{dt} + \sin\theta \frac{df}{dt}) = 0$. By multiplying this with non-zero factor $\upsilon \sin\theta$, we have

$$\begin{aligned} -\upsilon \cos\theta \sin\theta \frac{d\theta}{dt} &= \upsilon \sin^2\theta \frac{df}{dt}, \\ \upsilon \cos\theta \sin\theta \frac{d\theta}{dt} &= -g_1(V_1, V_1) \frac{df}{dt}, \\ \upsilon \cos\theta \sin\theta \frac{d\theta}{dt} &= -g_1(\nabla f, \dot{\alpha})||V_1||^2, \\ \upsilon \cos\theta \sin\theta \frac{d\theta}{dt} &= -g_1(\nabla f, V_2)||V_1||^2. \end{aligned} \tag{3.18}$$

Thus, from equations (3.17) and (3.18), we have

$$\begin{aligned} & g_1(\nabla f, V_2)||V_1||^2 \\ = \; & g_1(\mathcal{V}\nabla_{\overset{\circ}{\alpha}} BV_2, \psi V_1) + g_1(\mathcal{H}\nabla_{\overset{\circ}{\alpha}} CV_2, \omega V_1) + g_1((\mathcal{T}_{V_1} + \mathcal{A}_{V_2})CV_2, \psi V_1) + \\ & g_1((\mathcal{T}_{V_1} + \mathcal{A}_{V_2})BV_2, \omega V_1) + \eta(V_2) g_1(V_1, \psi V_1) + \eta(V_2) g_1(V_2, \omega V_1). \end{aligned}$$

Hence the theorem 3.2 is proved. □

Corollary 3.1 *Let π be a semi-invariant $\xi^{\perp}$-Riemannian submersion from a Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (M_2, g_2). Then, we get*

$$g_1(\nabla f, \xi)||V_1||^2 = g_1(V_1, \psi V_1),$$

where V_1 is vertical.

Proof: One can easily prove it from theorem 3.2. □

Theorem 3.3 *Let π be a $CSI - \xi^{\perp}$-Riemannian submersion from a Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (M_2, g_2) with $r = e^f$. Then, we get*

$$\mathcal{A}_{\phi U_1} \phi V_1 = V_1(f) U_1 \tag{3.19}$$

for $V_1 \in \Gamma(\mu)$ and $U_1 \in \Gamma(D_2)$, such that ϕU_1 is basic.

Proof: Let π be $CSI - \xi^{\perp}$-Riemannian submersion from a Sasakian manifold onto a Riemannian manifold. For $Z_1, Z_2 \in \Gamma(D_2)$, using equation (2.13) and Theorem 3.1, we get

$$\mathcal{T}_{Z_1} Z_2 = -g_1(Z_1, Z_2) gradf. \tag{3.20}$$

Taking inner product in equation (3.20) with ϕU_1, we have

$$g_1(\mathcal{T}_{Z_1} Z_2, \phi U_1) = -g_1(Z_1, Z_2) g_1(gradf, \phi U_1), \tag{3.21}$$

for all $U_1 \in \Gamma(D_2)$.

From equations (2.3), (2.4), (2.5) and (2.9), we obtain

$$g_1(\nabla_{Z_1} \phi Z_2, U_1) = g_1(Z_1, Z_2) g_1(gradf, \phi U_1).$$

Since ∇ is metric connection, using equations (2.9) and (2.13) in above equation, we get

$$g_1(Z_1, U_1) g_1(gradf, \phi Z_2) = g_1(Z_1, Z_2) g_1(gradf, \phi U_1). \tag{3.22}$$

Taking $U_1 = Z_2$ and interchanging the role of Z_1 and Z_2, we obtain

$$g_1(Z_2, Z_2) g_1(gradf, \phi Z_1) = g_1(Z_1, Z_2) g_1(gradf, \phi Z_2). \tag{3.23}$$

Using equation (3.22) with $U_1 = Z_1$ in (3.23), we have

$$g_1(gradf, \phi Z_1) = \frac{(g_1(Z_1, Z_2))^2}{||Z_1||^2 ||Z_2||^2} g_1(gradf, \phi Z_1). \tag{3.24}$$

If $gradf \in \Gamma(\phi(D_2))$, then equation (3.24) and the condition of equality in the Schwarz inequality implies that either f is constant on $\phi(D_2)$ or the fibers are one dimensional.

On the other hand, using equations (2.3) and (2.5), we get

$$g_1(\phi \nabla_{Z_1} U_1, \phi V_1) = g_1(\nabla_{Z_1} \phi U_1, \phi V_1)$$

for $V_1 \in \Gamma(\mu)$ and $V_1 \neq \xi$. Now, using equation (2.3), we obtain

$$g_1(\nabla_{Z_1}\phi U_1, \phi V_1) = g_1(\nabla_{Z_1}U_1, V_1).$$

Using equations (2.9) and (2.13) in above equation, we get

$$g_1(\nabla_{Z_1}\phi U_1, \phi V_1) = -g(Z_1, U_1)g_1(gradf, V_1).$$

Since ϕU_1 is basic and using the fact that $\mathcal{H}\nabla_{Z_1}\phi U_1 = \mathcal{A}_{\phi U_1}Z_1$, we get

$$\begin{aligned} g_1(\nabla_{Z_1}\phi U_1, \phi V_1) &= -g(Z_1, U_1)g_1(gradf, V_1), \\ g_1(\mathcal{A}_{\phi U_1}Z_1, \phi V_1) &= -g(Z_1, U_1)g_1(gradf, V_1), \\ g_1(\mathcal{A}_{\phi U_1}\phi V_1, Z_1) &= g(Z_1, U_1)g_1(gradf, V_1) \\ g_1(\mathcal{A}_{\phi U_1}\phi V_1, Z_1) &= g(Z_1, U_1)g_1(\nabla f, V_1). \end{aligned}$$

Since $\mathcal{A}_{\phi U_1}\phi V_1$ and U_1 are vertical and ∇f is horizontal, we obtain equation (3.19). □

Lemma 3.3 *Let π be a $CSI-\xi^{\perp}$-Riemannian submersion from a Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (M_2, g_2) with $r = e^f$ and $\dim(D_2) > 1$. Then, $\overset{\pi}{\nabla}_{X_1}\pi_*(\phi Z_1) = X_1(f)\pi_*(\phi Z_1)$, for all $Z_1 \in \Gamma(D_2)$ and $X_1 \in \Gamma(\ker \pi_*)^{\perp}$.*

Proof: Let π be a $CSI - \xi^{\perp}$-Riemannian submersion from a Sasakian manifold onto a Riemannian manifold. From Theorem 3.1, fibers are totally umbilical with mean curvature vector field $H = -gradf$, then we have

$$\begin{aligned} -g_1(\nabla_{Z_1}X_1, Z_2) &= g_1(\nabla_{Z_1}Z_2, X_1), \\ -g_1(\nabla_{Z_1}X_1, Z_2) &= -g_1(Z_1, Z_2)g_1(gradf, X_1), \end{aligned}$$

for all $Z_1, Z_2 \in \Gamma(D_2)$ and $X_1 \in \Gamma(\ker \pi_*)^{\perp}$.

Using equation (2.3) in above equation, we get

$$g_1(\nabla_{V_1}\phi Z_1, \phi Z_2) = g_1(\phi Z_1, \phi Z_2)g_1(gradf, X_1). \tag{3.25}$$

Since π is semi-invariant $\xi^{\perp}$-Riemannian submersion, we have

$$g_2(\pi_*(\nabla_{V_1}\phi Z_1), \pi_*(\phi Z_2)) = g_2(\pi_*(\phi Z_1), \pi_*(\phi Z_2))g_1(gradf, X_1). \tag{3.26}$$

From (2.14) in (3.26), we obtain

$$g_2(\overset{\pi}{\nabla}_{X_1}\pi_*(\phi Z_1), \pi_*(\phi Z_2)) = g_2(\pi_*(\phi Z_1), \pi_*(\phi Z_2))g_1(gradf, X_1), \tag{3.27}$$

which implies $\overset{\pi}{\nabla}_{X_1}\pi_*(\phi Z_1) = X_1(f)\pi_*(\phi Z_1)$, for all $Z_1 \in \Gamma(D_2)$ and $X_1 \in \Gamma(\ker \pi_*)^{\perp}$. □

Theorem 3.4 *Let π be a $CSI - \xi^{\perp}$-Riemannian submersion with $r = e^f$ from a Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (M_2, g_2). If $\mathcal{T}$ is not equal to zero identically, then the invariant distribution D_1 cannot defined a totally geodesic foliation on N_1.*

Proof: For $V_1, V_2 \in \Gamma(D_1)$ and $Z_1 \in \Gamma(D_2)$, using equations (2.3), (2.9) and (2.13), we get

$$\begin{aligned} g_1(\nabla_{V_1}V_2, Z_1) &= g_1(\nabla_{V_1}\phi V_2, \phi Z_1), \\ &= g_1(\mathcal{T}_{V_1}\phi V_2, \phi Z_1), \\ &= -g_1(V_1, \phi V_2)g_1(gradf, \phi Z_1). \end{aligned}$$

Thus, the assertion can be seen from above equation and the fact that $gradf \in \phi(D_2)$. □

Theorem 3.5 *Let π be a $CSI-\xi^{\perp}$-Riemannian submersion with $r=e^{f}$ from a Sasakian manifold (M_1,ϕ,ξ,η,g_1) onto a Riemannian manifold (M_2,g_2). Then, the fibers of π are totally geodesic or the anti-invariant distribution D_2 one-dimensional.*

Proof: If the fibers of π are totally geodesic, it is obvious. For second one, since π is a Clairaut proper semi-invariant $\xi^{\perp}$-Riemannian submersion, then either $\dim(D_2)=1$ or $\dim(D_2)>1$. If $\dim(D_2)>1$, then we can choose $X_1,X_2\in\Gamma(D_2)$ such that $\{X_1,X_2\}$ is orthonormal. From equations (2.9), (3.1) and (3.2), we get

$$\begin{aligned}\mathcal{T}_{X_1}\phi X_2+\mathcal{H}\nabla_{X_1}\phi X_2 &= \nabla_{X_1}\phi X_2,\\ \mathcal{T}_{X_1}\phi X_2+\mathcal{H}\nabla_{X_1}\phi X_2 &= B\mathcal{T}_{X_1}X_2+C\mathcal{T}_{X_1}X_2+\psi\mathcal{V}\nabla_{X_1}X_2+\omega\mathcal{V}\nabla_{X_1}X_2.\end{aligned}$$

Taking inner product above equation with X_1, we obtain

$$g_1(\mathcal{T}_{X_1}\phi X_2,X_1)=g_1(B\mathcal{T}_{X_1}X_2,X_1)+g_1(\psi\mathcal{V}\nabla_{X_1}X_2,X_1). \tag{3.28}$$

From equation (2.3), (2.9) and (2.13), we have

$$g_1(\mathcal{T}_{X_1}X_1,\phi X_2)=-g_1(\mathcal{T}_{X_1}\phi X_2,X_1)=-g_1(gradf,\phi X_2)=g_1(\mathcal{T}_{X_1}X_2,\phi X_1). \tag{3.29}$$

From above equation, we obtain

$$\begin{aligned}g_1(gradf,\phi X_2) &= g_1(\mathcal{T}_{X_1}X_2,\phi X_1),\\ g_1(gradf,\phi X_2) &= g_1(X_1,X_2)g_1(gradf,\phi X_1),\\ g_1(gradf,\phi X_2) &= 0.\end{aligned}$$

So, we get

$$gradf\perp\phi(D_2).$$

Therefore, the dimension of D_2 must be one. □

4. Example

Example 4.1 Let (M_5,ϕ,ξ,η,g_5) denotes the manifold with its Sasakian structure given by

$$\eta=\frac{1}{2}(dz-\sum_{i=1}^{2}y_idx_i),\xi=2\frac{\partial}{\partial z},$$

$$g_{R^5}=(\eta\otimes\eta)+\frac{1}{4}\sum_{i=1}^{2}((dx_i)^2+(dy_i)^2),\lambda_1,\lambda_2,\lambda_3\in R,$$

$$\phi(\sum_{i=1}^{2}(X_i\frac{\partial}{\partial x_i}+Y_i\frac{\partial}{\partial y_i})+Z\frac{\partial}{\partial z})=\sum_{i=1}^{2}(Y_i\frac{\partial}{\partial x_i}-X_i\frac{\partial}{\partial y_i})+\sum_{i=1}^{k}Y_iy^i\frac{\partial}{\partial z},$$

where (x_1,x_2,y_1,y_2,z) denotes the Cartesian coordinates on R^5 and will be used throughout this section.

Note that, M_5 is a 5-dimensional Sasakian manifold given by the following:

$$M_5=\{(x_1,x_2,y_1,y_2,z)\in\mathbb{R}^5\mid x_1,y_1,z>0\}.$$

We consider the map $\pi:(M_5,\phi,\xi,\eta,g_5)\to(M_2,g_2)$ define by the following:

$$\pi(x_1,x_2,y_1,y_2,z)=(\sqrt{x_1^2+y_1^2},z).$$

Let $M_2 = \{(v_1, v_2), v_1, v_2 > 0\}$ be a 2-dimensional Riemannian manifold with $g_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the Riemannian metric on M_2. Then, the Jacobian matrix of π is as follows:

$$\begin{bmatrix} \frac{x_1}{\kappa} & 0 & \frac{y_1}{\kappa} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

where $\kappa = \sqrt{x_1^2 + y_1^2}$. Since the rank of this matrix is 2, the map π is a submersion. After some computations, we have

$$(\ker \pi_*) =< X_1 = \frac{y_1}{\kappa} E_1 - \frac{x_1}{\kappa} E_3, X_2 = E_2, X_3 = E_4 >,$$

$$D_1 =< X_2 = E_2, X_3 = E_4 >, D_2 =< X_1 = \frac{y_1}{\kappa} E_1 - \frac{x_1}{\kappa} E_3 >,$$

$$(\ker \pi_*)^\perp =< H_1 = \frac{x_1}{\kappa} E_1 + \frac{y_1}{\kappa} E_3, H_2 = \xi = 2\frac{\partial}{\partial z} >,$$

where $\{E_1 = (2\frac{\partial}{\partial x_1} + 2y_1\frac{\partial}{\partial z}), E_2 = (2\frac{\partial}{\partial x_2} + 2y_2\frac{\partial}{\partial z}), E_3 = 2\frac{\partial}{\partial y_1}, E_4 = 2\frac{\partial}{\partial y_2}, E_5 = 2\frac{\partial}{\partial z}\}$, $\{E_1^* = \frac{\partial}{\partial v_1}, E_2^* = \frac{\partial}{\partial v_2}\}$ are bases on T_qM_1 and $T_{\pi(q)}M_2$ respectively, for all $q \in M_1$.

Now, we will find smooth function f on N_1 satisfying $\mathcal{T}_X X = g_1(X, X)\nabla f$, for all $X \in \Gamma(\ker \pi_*)$.

Using the Sasakian structure, we see that

$$\begin{aligned} [E_1, E_1] &= [E_2, E_2] = [E_3, E_3] = [E_4, E_4] = [E_5, E_5] = 0, \\ [E_1, E_2] &= 0, [E_1, E_3] = -2E_5, [E_1, E_4] = 0, [E_1, E_5] = 0, \\ [E_2, E_3] &= 0, [E_2, E_4] = -2E_5, [E_2, E_5] = 0, [E_3, E_4] = 0, \\ [E_3, E_5] &= 0, [E_4, E_5] = 0. \end{aligned} \tag{4.1}$$

The Riemannian connection ∇ of the metric g_1 is given by the Koszul′s formula which is

$$\begin{aligned} & 2g_1(\nabla_X Y, Z) \\ = \; & Xg_1(Y, Z) + Yg_1(Z, X) - Zg_1(X, Y) + g_1([X, Y], Z) - \\ & g_1([Y, Z], X) + g_1([Z, X], Y). \end{aligned} \tag{4.2}$$

$$\begin{aligned} \nabla_{E_1} E_1 &= \nabla_{E_2} E_2 = \nabla_{E_3} E_3 = \nabla_{E_4} E_4 = \nabla_{E_5} E_5 = 0, \\ \nabla_{E_1} E_3 &= -E_5, \nabla_{E_3} E_1 = E_5, \nabla_{E_1} E_2 = 0, \nabla_{E_2} E_1 = 0, \\ \nabla_{E_2} E_3 &= 0, \nabla_{E_3} E_2 = 0, \nabla_{E_2} E_4 = -E_5, \nabla_{E_4} E_2 = E_5, \\ \nabla_{E_1} E_4 &= 0, \nabla_{E_4} E_1 = 0, \nabla_{E_3} E_4 = 0, \nabla_{E_4} E_3 = 0. \end{aligned} \tag{4.3}$$

Thus, we have

$$\begin{aligned} \nabla_{X_1} X_1 &= \nabla_{\frac{y_1}{\kappa} E_1 - \frac{x_1}{\kappa} E_3} \frac{y_1}{\kappa} E_1 - \frac{x_1}{\kappa} E_3, \\ \nabla_{X_2} X_2 &= \nabla_{E_2} E_2 = 0, \nabla_{X_3} X_3 = \nabla_{E_4} E_4 = 0, \\ \nabla_{X_1} X_2 &= \nabla_{\frac{y_1}{\kappa} E_1 - \frac{x_1}{\kappa} E_3} E_2 = 0, \nabla_{X_2} X_1 = \nabla_{E_2} \frac{y_1}{\kappa} E_1 - \frac{x_1}{\kappa} E_3 = 0, \\ \nabla_{X_1} X_3 &= \nabla_{\frac{y_1}{\kappa} E_1 - \frac{x_1}{\kappa} E_3} E_4 = 0, \nabla_{X_3} X_1 = \nabla_{E_4} \frac{y_1}{\kappa} E_1 - \frac{x_1}{\kappa} E_3 = 0, \\ \nabla_{X_2} X_3 &= \nabla_{E_2} E_4 = -E_5, \nabla_{X_3} X_2 = \nabla_{E_4} E_2 = E_5. \end{aligned} \tag{4.4}$$

Using equation (4.4), we get

$$\nabla_X X = \nabla_{\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3} \lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3,$$

$$\begin{aligned}\nabla_X X &= \lambda_1^2\nabla_{X_1}X_1+\lambda_2^2\nabla_{X_2}X_2+\lambda_3^2\nabla_{X_3}X_3+\\ &\lambda_1\lambda_2\nabla_{X_1}X_2+\lambda_1\lambda_2\nabla_{X_2}X_1+\lambda_1\lambda_3\nabla_{X_1}X_3+\\ &\lambda_1\lambda_3\nabla_{X_3}X_1+\lambda_2\lambda_3\nabla_{X_2}X_3+\lambda_2\lambda_3\nabla_{X_3}X_2,\end{aligned}$$

$$\nabla_X X=\lambda_1^2\nabla_{\frac{y_1}{\kappa}E_1-\frac{x_1}{\kappa}E_3}\frac{y_1}{\kappa}E_1-\frac{x_1}{\kappa}E_3. \tag{4.5}$$

Using equations (2.13) and (4.5), we have

$$\begin{aligned}\mathcal{T}_X X &= -\lambda_1^2\frac{2}{\kappa^2}(x_1E_1+y_1E_3), \qquad (4.6)\\ &= -4\lambda_1^2\Big(\frac{x_1}{x_1^2+y_1^2}\frac{\partial}{\partial x_1}+\frac{x_1y_1}{x_1^2+y_1^2}\frac{\partial}{\partial z}+\frac{y_1}{x_1^2+y_1^2}\frac{\partial}{\partial y_1}\Big).\end{aligned}$$

Since $X=\lambda_1X_1+\lambda_2X_2+\lambda_3X_3$, so $g_1(\lambda_1X_1+\lambda_2X_2+\lambda_3X_3,\lambda_1X_1+\lambda_2X_2+\lambda_3X_3)=\lambda_1^2+\lambda_2^2+\lambda_3^2$. For any smooth function f on R^5, the gradient of f with respect to the metric g_1 is given by $\nabla f=\sum\limits_{i,j}g_1^{i,j}\frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_j}$. Hence $\nabla f=4\{(\frac{\partial f}{\partial x_1}+y_1\frac{\partial f}{\partial z})\frac{\partial}{\partial x_1}+\frac{\partial f}{\partial y_1}\frac{\partial}{\partial y_1}+(y_1\frac{\partial f}{\partial x_1}+(1+y_1^2)\frac{\partial f}{\partial z})\frac{\partial}{\partial z}\}$ for the function $f=\frac{\lambda_1^2}{2(\lambda_1^2+\lambda_2^2+\lambda_3^2)}\ln(x_1^2+y_1^2)$. Then it is easy to see that $\mathcal{T}_XX=-g_1(X,X)\nabla f$, thus by Theorem (3.1), π is $CSI-\xi^{\perp}$-Riemannian submersion.

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