Solving Bilevel Quasimonotone Variational Inequality Problem In Hilbert Spaces

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ABSTRACT: In this paper, we propose and study a Bilevel quasimonotone Variational Inequality Problem (BVIP) in the framework of Hilbert space. We introduce a new modified inertial iterative technique with self-adaptive step size for approximating a solution of the BVIP. In addition, we established a strong convergence result of the proposed iterative technique with adaptive step-size conditions without prior knowledge of Lipschitz’s constant of the cost operators as well as the strongly monotonicity coefficient under some standard mild assumptions. Finally, we provide some numerical experiments to demonstrate the efficiency of our proposed methods in comparison with some recently announced results in the literature.

Key Words: Variational inequality problem, inertial technique, quasimonotone, Hilbert space.

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1. Introduction

Let $H$ be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$, $C$ a nonempty closed convex subset of $H$ and $F : H \to H$ be a nonlinear operator. The classical Variational Inequality Problem (VIP) is formulated as:

$$\text{Find } x \in C \text{ such that } \langle Fx, y - x \rangle \geq 0 \quad \forall \ y \in C. \quad (1.1)$$

The notion of VIP was introduced independently by Stampacchia [36] and Fichera [13,14] for modeling problems arising from mechanics and for solving Signorini problem. It is well-known that many problems in economics, mathematical sciences, and mathematical physics can be formulated as VIP. We denote the solution set of a VIP by $\Omega$. Due to the fruitful applications of the VIP, many researchers in this area have developed different iterative techniques to solve VIP $(1.1)$. In particular, Goldsten in [20] introduced an iterative technique defined as follows:

$$\begin{cases}
  x_1 \in C, \\
  x_{n+1} = P_C(x_n - \lambda Fx_n),
\end{cases} \quad (1.2)$$

for all $n \in \mathbb{N}$, where $\lambda \in (0, \frac{2\alpha}{L^2})$, $F$ is $\alpha$-strongly monotone and $L$-Lipschitz continuous and $P_C$ is a metric projection defined from $H$ onto $C$. The author established that the iterative method $(1.2)$ converges to the solution set of VIP $(1.1)$. However, it was observed that if $F$ monotone and $L$-Lipschitzian continuous, the iterative method $(1.2)$ may not converge to the solution set of VIP $(1.1)$, see [22] and the reference therein for details. In addition, computing the value of $\lambda$ may be very difficult or impossible. In the light

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of these draw back, Korpelevich in [24] introduced and studied the Extragradient Method (EM) iterative technique defined as follows:

\[
\begin{align*}
    x_1 &\in C, \\
    y_n &= P_C(x_n - \lambda_n Fx_n), \\
    x_{n+1} &= P_C(x_n - \lambda_n Fy_n),
\end{align*}
\]

(1.3)

for all \( n \geq 1 \), where \( \lambda_n \in (0, \frac{1}{L}) \), \( F \) is monotone and \( L \)-Lipschitz continuous and \( P_C \) is a metric projection defined from \( H \) onto \( C \). This method was able to provide an affirmative answer to the question of weakening the cost operator, however, the computation of \( \lambda_n \) remains a challenge. More so, another set back of this technique (1.3) is that it requires two projections onto the feasible set \( C \) per iteration, which is costly when \( C \) is not a simple structure. Since the inception of EM, many authors have introduced, modified and studied different EM in which the cost operator \( F \) is monotone and pseudomonotonicity.

For example, He et al. [23], Apostol et al. [2], He et al. [22], Ceng et al. [4], Ceng et al [5], Nadezhkina and Takahashi [27] and many others. In addition, the notion of VIP (1.1) has also been extended and generalized by many authors. For example, Mainge in [28] introduced and studied the notion of Bilevel Variational Inequality Problems (BVIP). The BVIP is defined as follows:

Find \( x^* \in \Omega \) such that \( \langle Gx^*, x - x^* \rangle \geq 0 \ \forall \ x \in \Omega \).  

(1.4)

where \( G : H \to H \) is \( L \)-Lipschitz continuous and \( \gamma \)-strongly monotone. It is easy to see that the BVIP (1.4) is a problem that is made up of the VIP (1.1) as a constraint. He proposed the following extragradient technique:

\[
\begin{align*}
    u_0 &\in C \\
    v_n &= P_C(u_n - \lambda_n Fu_n) \\
    t_n &= P_C(u_n - \lambda_n Fu_n) \\
    u_{n+1} &= t_n - \alpha_n G t_n,
\end{align*}
\]

(1.5)

where \( \{\lambda_n\} \subset [a, b] \subset \left(0, \frac{1}{L}\right) \) and \( \alpha_n \subset (0, 1) \) such that \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=1}^{\infty} \alpha_n = +\infty \). It was established that the sequence generated by \( \{x_n\} \) converges strongly to a unique solution of problem BVIP (1.4). It is easy to see that the iterative technique (1.5) has at least two setbacks, for example \( \{\lambda_n\} \subset [a, b] \subset \left(0, \frac{1}{L}\right) \), and the double metric projection \((P_C)\). In order to overcome this setbacks, researchers have introduced the Tseng type iterative technique, the projection contraction iterative technique and the subgradient extragradient iterative technique that are self adaptive, see [35,37,38,39,42] and the reference therein for details. In particular, Tan et al., [38] introduced and studied the following iterative technique:

\[
\begin{align*}
    x_0, x_1 &\in C, \\
    w_n &= x_n + \theta_n (x_n - x_{n-1}), \\
    y_n &= P_C(w_n - \lambda_n Fw_n), \\
    T_n &= \{x \in H : \langle w_n - \lambda_n Fw_n - y_n, x - y_n \rangle \leq 0\}, \\
    z_n &= P_{T_n}(w_n - \lambda_n Fy_n), \\
    x_{n+1} &= z_n - \alpha_n \gamma G z_n,
\end{align*}
\]

(1.6)

for all \( n \in \mathbb{N} \), and

\[
\begin{align*}
    x_0, x_1 &\in C, \\
    w_n &= x_n + \theta_n (x_n - x_{n-1}), \\
    y_n &= P_C(w_n - \lambda_n Fw_n), \\
    z_n &= y_n - \lambda_n (F_t w_n - F y_n), \\
    x_{n+1} &= z_n - \alpha_n \gamma G z_n,
\end{align*}
\]

(1.7)
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for all $n \in \mathbb{N}$, where $F$ is $L$-Lipschitz continuous, pseudomonotone and sequentially weakly continuous, $G$ is $\alpha$-strongly monotone and $L_1$-Lipschitz continuous. They established that the iterative techniques (1.2) and (1.7) converge strongly to a unique solution of the BVIP (1.4) using some standard assumptions. The question is still wide open, if some (all) of the mentioned iterative techniques can further be improved. One of the ways in which these iterative techniques have been improved over the years is the introduction of the inertial extrapolation technique. In 1964, Polyak in [32] introduced an inertial extrapolation as an acceleration technique process for solving the smooth convex minimization problem. Since then, this technique has been employed by research to improve their iterative techniques. The inertial technique requires the first two initial terms of the iterative technique and the next iterate is defined by making use of the previous two iterates. Since inception of the inertial extrapolation, many authors have modified, extended and generalized the technique, see [15,16,35] and the references therein. However, it has always been a question of interest, if the extrapolation technique can further be improved. The importance of the BVIP cannot be overemphasized. The BVIP (1.4) has been applied to different areas of mathematical science, engineering, physics and so on. For example, the BVIP (1.4) has applications in equilibrium constraints, bilevel convex programming models, minimum-norm problems with the solution set of variational inequalities, bilevel linear programming, image restoration and many more see ([1,12,18,19,26,40,41]) and the references therein. Due to these applications, many authors have introduced different iterative techniques for solving the BVIP in the framework of Hilbert spaces (see, [1,15,16,17,28,29,39] and the references therein). It is well-known that the underlying cost operators have crucial roles to play in real applications of these iterative methods. In the light of this, introducing an iterative technique with weaker cost operators and better rate of convergence is highly sorted after. Having considered the above discussed literatures and the references therein, it is natural to ask the following question:

1. Can we construct an efficient inertial type iterative technique that does require the knowledge of the Lipschitz constant during implementation of the algorithm?

2. Can we construct an iterative technique for a BVIP (1.4) in which the cost operators are quasimonotone and $\alpha$-strongly monotone in the framework of infinite dimensional Hilbert spaces and obtain strong convergence?

3. Can we further explore the iterative techniques (1.7) and (1.5), in which the cost operator is quasimonotone and strongly monotone and the knowledge of the Lipschitz constant needed not to be known during implementation?

The purpose of this work is to provide and affirmative answer to the above questions by introducing a modified inertial iterative technique with self-adaptive step size for approximating the solution of quasimonotone BVIP (1.4). In addition, we use a modified inertial technique to accelerate the rate of convergence of our proposed methods. In addition, our numerical experiments justify that our method is better than other methods in the literature for solving the BVIP (1.4).

The rest of this paper is organized as follows: In Section 2, we recall some useful definitions and results that are relevant for our study. In Section 3, we present our proposed method. In Section 4, we establish strong convergence of our method and in Section 5, we present some numerical experiments to show the efficiency and applicability of our method in the framework of infinite dimensional Hilbert spaces. Lastly in Section 6, we give the conclusion of the paper.

2. Preliminaries

In this section, we begin by recalling some known and useful results which are needed in the sequel. Let $H$ be a real Hilbert space. The set of fixed points of a nonlinear mapping $T : H \to H$ will be denoted by $F(T)$, that is $F(T) = \{ x \in H : Tx = x \}$. We denotes strong and weak convergence by “→” and “⇀”, respectively. For any $x, y \in H$ and $\alpha \in [0, 1]$, it is well-known that

$$
\| x - y \|^2 = \| x \|^2 - 2\langle x, y \rangle + \| y \|^2. \tag{2.1}
$$
\[\|x + y\|^2 = \|x\|^2 + 2\langle x, y \rangle + \|y\|^2.\]  
(2.2)

\[\|x - y\|^2 \leq \|x\|^2 + 2\langle y, x - y \rangle.\]  
(2.3)

\[\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2.\]  
(2.4)

**Definition 2.1.** Let \( T : H \rightarrow H \) be an operator. Then \( T \) is called

(a) **L-Lipschitz continuous** if there exists \( L > 0 \) such that
\[\|Tx - Ty\| \leq L\|x - y\|,\]
for all \( x, y \in H \). If \( L = 1 \), then \( T \) is called nonexpansive. If \( y \in F(T) \), and
\[\|Tx - y\| \leq \|x - y\|,\]
for all \( x \in H \). Then \( T \) is called quasinonexpansive.

(b) **monotone** if
\[\langle Tx - Ty, x - y \rangle \geq 0, \quad \forall x, y \in H;\]

(c) **pseudomonotone** if
\[\langle Tx, y - x \rangle \geq 0 \Rightarrow \langle Ty, y - x \rangle \geq 0, \quad \forall x, y \in H;\]

(d) **\( \alpha \)-strongly monotone** if there exists \( \alpha > 0 \), such that
\[\langle Tx - Ty, x - y \rangle \geq \alpha\|x - y\|^2, \quad \forall x, y \in H;\]

(e) **quasimonotone**
\[\langle Tx, x - y \rangle > 0 \Rightarrow \langle Ty, x - y \rangle \geq 0 \forall x, y \in H;\]

(f) **sequentially weakly continuous** if for each sequence \( \{x_n\} \), we obtain \( \{x_n\} \) converges weakly to \( x \) implies that \( Tx_n \) converges weakly to \( Tx \).

**Remark 2.1.** It is well-known that \( \alpha \)-strongly monotone \( \Rightarrow \) monotone \( \Rightarrow \) pseudomonotone \( \Rightarrow \) quasi-monotone. However, the converses are not generally true.

Let \( C \) be a nonempty, closed and convex subset of \( H \). For any \( u \in H \), there exists a unique point \( P_C u \in C \) such that
\[\|u - P_C u\| \leq \|u - y\| \forall y \in C.\]

The operator \( P_C \) is called the metric projection of \( H \) onto \( C \). It is well-known that \( P_C \) is a nonexpansive mapping and that \( P_C \) satisfies
\[\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2,\]  
(2.5)

for all \( x, y \in H \). Furthermore, \( P_C \) is characterized by the property
\[\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2\]
and
\[\langle x - P_C x, y - P_C x \rangle \leq 0,\]  
(2.6)

for all \( x \in H \) and \( y \in C \).
Lemma 2.2. \cite{21.43} Let $C$ be a nonempty, closed and convex subset of a Hilbert space $H$ and $A : H \to H$ ba $L$-Lipschitzian and quasimonotone operator. Suppose that $y \in C$ and for some $p \in C$, we have $\langle Ay, p - y \rangle \geq 0$, then at least on of the following hold

$$\langle Ap, p - y \rangle \geq 0 \text{ or } \langle Ay, q - y \rangle \leq 0$$

for all $q \in C$.

Lemma 2.3. \cite{34} Let $\{a_n\}$ be a sequence of positive real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $(0, 1)$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{d_n\}$ be a sequence of real numbers. Suppose that $a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n d_n, n \geq 1$.

If $\limsup_{k \to \infty} d_{n_k} \leq 0$ for all subsequences $\{a_{n_k}\}$ of $\{a_n\}$ satisfying the condition

$$\liminf_{k \to \infty} \{a_{n_k+1} - a_{n_k}\} \geq 0,$$

then, $\lim_{k \to \infty} a_n = 0$.

Lemma 2.4. \cite{1} Let $C$ be nonempty closed convex subset of a real Hilbert space $H$. For any $x \in H$ and $z \in C$, we have $z = P_C x$ if and only if $\langle x - z, y - z \rangle \leq 0 \forall y \in C$.

Lemma 2.5. \cite{1} Let $H$ be a Hilbert space and $F : H \to H$ be a $\tau$-strongly monotone and $L$-Lipschitz continuous operator on $H$. Let $\alpha \in (0, 1)$ and $\gamma \in (0, \frac{2\tau}{L^2})$. Then for any nonexpansive operator $T : H \to H$, we can associate $T^\gamma : H \to H$ defined by $T^\gamma x = (I - \alpha\gamma F)Tx$ for all $x \in H$. Then, $T^\gamma$ is a contraction. That is

$$\|T^\gamma x - T^\gamma y\| \leq (1 - \alpha\nu)\|x - y\|$$

for all $x, y \in H$, where $\nu = 1 - \sqrt{1 - \gamma(2\tau - \gamma L^2)} \in (0, 1)$.

3. Proposed Algorithm

In this section, we present our proposed method for solving a bilevel quasimonotone variational inequality problem.

Assumption 1.

Condition A. Suppose

1. The feasible sets $C$ is nonempty set, closed and convex subsets of the real Hilbert space $H$.
2. $\{S_n\}$ is a sequence of nonexpansive mapping on $H$.
3. $F : H \to H$ is quasimonotone, sequentially weakly continuous an Lipschitz continuous with Lipschitz constant $L > 0$.
4. $G : H \to H$ is $\tau$-strongly monotone and Lipschitz continuous with Lipschitz constant $L_1 > 0$.
5. The solution set $\Phi := \{x \in \Omega : \langle Gx^*, x - x^* \rangle \geq 0 \forall x \in \Omega\}$ is nonempty, where $\Omega$ is the solution of the classical VIP (1.1).

Condition B.

1. $\beta_n \in (0, 1), \lim_{n \to \infty} \beta_n = 0$ and $\sum_{n=0}^{\infty} \beta_n = \infty$.
2. $\alpha \in (0, \frac{2\tau}{L^2}), \lambda_0 > 0, \nu \in (0, 1)$ and choose the nonnegative real sequence $\{p_n\}$ such that $\sum_{n=1}^{\infty} p_n < \infty$.

We present the following iterative algorithm.
Algorithm 1. Initialization Step:

**Step 1:** Choose \(x_0, x_1 \in H\), given the iterates \(x_{n-1}\) and \(x_n\) for all \(n \in \mathbb{N}\), choose \(\theta_n\) such that \(0 \leq \theta_n \leq \bar{\theta}\), where

\[
\bar{\theta}_n = \begin{cases} 
\min \left\{ \frac{n-1}{n+\beta-1}, \frac{\epsilon_n}{||x_n-x_{n-1}||} \right\}, & \text{if } x_n \neq x_{n-1} \\
\frac{n-1}{n+\beta-1}, & \text{otherwise,}
\end{cases}
\]  

(3.1)

with \(\theta\) been a positive constant and \(\{\epsilon_n\}\) is a positive sequence such that \(\epsilon_n = o(\beta_n)\).

**Step 2.** Set

\[
w_n = x_n + \theta_n(S_nx_n - S_nx_{n-1}).
\]

Then, compute

\[
y_n = P_C(w_n - \lambda_n Fw_n),
\]

(3.2)

\[
z_n = y_n + \lambda_n(Fw_n - Fy_n),
\]

(3.3)

\[
\lambda_{n+1} = \begin{cases} 
\min \left\{ \frac{n}{||w_n-y_n||}, \lambda_n + p_n \right\}, & \text{if } Fw_n \neq Fy_n \\
\lambda_n + p_n, & \text{otherwise.}
\end{cases}
\]

(3.4)

**Step 4.** Compute

\[
x_{n+1} = z_n - \beta_n \alpha Gz_n
\]

(3.5)

4. Convergence Analysis

**Lemma 4.1.** The step-size \(\lambda_{n+1}\) in Algorithm 1 is well defined.

**Proof** The proof that \(\lambda_{n+1}\) is well defined follows similar approach as in Lemma 3.1 of [25], thus we omit it.

**Lemma 4.2.** Let \(\{x_n\}\) be a sequence generated by Algorithm 1 under Assumption 3. Then, \(\{x_n\}\) is bounded.

**Proof** Let \(p \in \Phi\). Since \(\lim_{n \to \infty} \frac{\theta_n}{\beta_n} ||x_n - x_{n-1}|| = 0\), there exists \(N_1 > 0\) such that \(\frac{\theta_n}{\beta_n} ||x_n - x_{n-1}|| \leq N_1\), for all \(n \in \mathbb{N}\). Then using Algorithm 1, we have

\[
||w_n - p|| = ||x_n + \theta_n(S_nx_n - S_nx_{n-1}) - p|| \\
\leq ||x_n - p|| + \theta_n ||S_nx_n - S_nx_{n-1}|| \\
\leq ||x_n - p|| + \beta_n \frac{\theta_n}{\beta_n} ||x_n - x_{n-1}|| \\
\leq ||x_n - p|| + \beta_n N_1.
\]

(4.1)

Also, using Algorithm 1 and (2.2)

\[
||z_n - p||^2 = ||y_n + \lambda_n(Fw_n - Fy_n) - p||^2 \\
= ||y_n - \lambda_n(Fy_n - Fw_n) - p||^2 \\
= ||y_n - p||^2 + \lambda_n^2 ||Fy_n - Fw_n||^2 - 2\lambda_n \langle Fy_n - Fw_n, y_n - p \rangle \\
= ||y_n - w_n + w_n - p||^2 + \lambda_n^2 ||Fy_n - Fw_n||^2 - 2\lambda_n \langle Fy_n - Fw_n, y_n - p \rangle \\
= ||w_n - p||^2 + ||y_n - w_n||^2 + 2\langle y_n - w_n, w_n - p \rangle + \lambda_n^2 ||Fy_n - Fw_n||^2 \\
- 2\lambda_n \langle Fy_n - Fw_n, y_n - p \rangle \\
= ||w_n - p||^2 + ||y_n - w_n||^2 - 2\langle y_n - w_n, y_n - w_n \rangle + 2\langle y_n - w_n, y_n - p \rangle \\
+ \lambda_n^2 ||Fy_n - Fw_n||^2 - 2\lambda_n \langle Fy_n - Fw_n, y_n - p \rangle.
\]

(4.2)
Since $y_n = P_C(w_n - \lambda_n Aw_n)$ and $p \in \Phi \subseteq C$, and by Lemma 2.4, we have
\[ \langle y_n - w_n + \lambda_n F w_n, y_n - p \rangle \leq 0. \]

It then follows that $\langle y_n - w_n, y_n - p \rangle \leq -\lambda_n \langle F w_n, y_n - p \rangle$. Using the fact that $y_n \in C$ and $p \in \Phi$, we have $\langle F y_n, y_n - p \rangle \geq 0$. Using the above facts and (3.4), we have (4.2) becomes
\[
\| z_n - p \|^2 \leq \| w_n - p \|^2 - \| y_n - w_n \|^2 + 2\lambda_n \langle F w_n, y_n - p \rangle + \lambda^2_n \| F y_n - F w_n \|^2 \\
- 2\lambda_n \langle F y_n - F w_n, y_n - p \rangle \\
= \| w_n - p \|^2 - \| y_n - w_n \|^2 + \lambda^2_n \| F y_n - F w_n \|^2 - 2\lambda_n \langle F y_n, y_n - p \rangle \\
\leq \| w_n - p \|^2 - \| y_n - w_n \|^2 + \lambda^2_n \| F y_n - F w_n \|^2 \\
\leq \| w_n - p \|^2 - (1 - \frac{\lambda^2_n}{\lambda_n + 1}) \| y_n - w_n \|^2 \\
\leq \| w_n - p \|^2.
\]

It follows that
\[ \| z_n - p \| \leq \| w_n - p \|. \]

In addition, using Algorithm 1, Lemma 2.5 and the fact that $\gamma = 1 - \sqrt{1 - \alpha(2\tau - \alpha L^2)} \in (0, 1)$, we have
\[
\| x_{n+1} - p \| = \| z_n - \beta_n \alpha G z_n - p \| \\
= \| (1 - \beta_n \alpha G) z_n - (1 - \beta_n \alpha G)p - \beta_n \alpha Gp \| \\
\leq (1 - \gamma \beta_n) \| z_n - p \| + \beta_n \alpha \| Gp \| \\
\leq (1 - \gamma \beta_n) \| w_n - p \| + \beta_n \alpha \| Gp \| \\
\leq (1 - \gamma \beta_n) \| x_n - p \| + \beta_n \| G N_1 \| + \beta_n \alpha \| F_2 p \| \\
\leq (1 - \gamma \beta_n) \| x_n - p \| + \gamma \beta_n \left[ \frac{\| N_1 \| + \alpha \| Gp \|}{\gamma} \right] \\
\leq \max \{\| x_n - p \|, \frac{\| N_1 \| + \alpha \| Gp \|}{\gamma} \} \\
\vdots \\
\leq \max \{\| x_1 - p \|, \frac{\| N_1 \| + \alpha \| Gp \|}{\gamma} \}.
\]

Thus, we have that $\{x_n\}$ is bounded.

**Lemma 4.3.** Let $\{x_n\}$ be a sequence generated by Algorithm 1 under Assumption 3 and suppose that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to $x^* \in H$ and $\lim_{k \to \infty} \| w_{n_k} - y_{n_k} \| = 0$. Then, $x^* \in \Omega$.

**Proof** Since $y_{n_k} = P_C(w_{n_k} - \lambda_{n_k} F w_{n_k})$, then from the characteristic of the metric projection, we have
\[ \langle w_{n_k} - \lambda_{n_k} F w_{n_k} - y_{n_k}, x - y_{n_k} \rangle \leq 0 \\forall \ \ x \in C, \]
which implies that
\[ \langle w_{n_k} - y_{n_k}, x - y_{n_k} \rangle - \lambda_{n_k} \langle F w_{n_k}, x - y_{n_k} \rangle \leq 0, \]
which implies that
\[ \langle w_{n_k} - y_{n_k}, x - y_{n_k} \rangle \leq \lambda_{n_k} \langle F w_{n_k}, x - y_{n_k} \rangle = \lambda_{n_k} \langle F w_{n_k}, w_{n_k} - y_{n_k} \rangle + \lambda_{n_k} \langle F w_{n_k}, x - w_{n_k} \rangle. \]
Since $\lambda_{n_k} > 0$, we have
\[
\frac{1}{\lambda_{n_k}}(w_{n_k} - y_{n_k}, x - y_{n_k}) + \langle Fw_{n_k}, y_{n_k} - w_{n_k} \rangle \leq \langle Fw_{n_k}, x - w_{n_k} \rangle.
\] (4.13)

Using our hypothesis and the fact that $\lim_{k \to \infty} \lambda_{n_k} > 0$, we have
\[
0 \leq \liminf_{k \to \infty} \langle Fw_{n_k}, x - w_{n_k} \rangle \leq \limsup_{k \to \infty} \langle Fw_{n_k}, x - w_{n_k} \rangle.
\] (4.14)

Now, observe that
\[
\langle Fy_{n_k}, x - y_{n_k} \rangle = \langle Fy_{n_k}, x - w_{n_k} \rangle + \langle Fy_{n_k}, w_{n_k} - y_{n_k} \rangle
\]
\[
= \langle Fy_{n_k} - Fw_{n_k}, x - w_{n_k} \rangle + \langle Fw_{n_k}, x - w_{n_k} \rangle + \langle Fy_{n_k}, w_{n_k} - y_{n_k} \rangle.
\] (4.15)

Since $F$ is Lipschitz continuous on $H$ and our hypothesis, we have
\[
\lim_{k \to \infty} \|Fw_{n_k} - Fy_{n_k}\| \leq L \lim_{k \to \infty} \|w_{n_k} - y_{n_k}\| = 0.
\] (4.16)

Combining (4.14), (4.15) and (4.16), we have
\[
0 \leq \liminf_{k \to \infty} \langle Fy_{n_k}, x - y_{n_k} \rangle \leq \limsup_{k \to \infty} \langle Fy_{n_k}, x - y_{n_k} \rangle.
\] (4.17)

In what follows, we now establish that $x^* \in \Omega$. To start with, we consider the case in which $\limsup_{k \to \infty} \langle Fy_{n_k}, x - y_{n_k} \rangle > 0$ for all $x \in C$. Then there exists a subsequence $\{y_{n_{k_m}}\}$ of sequence $\{y_{n_k}\}$ such that $\limsup_{m \to \infty} \langle Fy_{n_{k_m}}, x - y_{n_{k_m}} \rangle > 0$ for all $x \in C$. It follows that we can find $N_0$ such that
\[
\langle Fy_{n_{k_m}}, x - y_{n_{k_m}} \rangle > 0 \ \forall m > N_0.
\] (4.18)

Since $A$ is quasimonotone, it follows that
\[
\langle Fx, x - y_{n_{k_m}} \rangle > 0 \ \forall m > N_0.
\] (4.19)

Now observe that
\[
\|w_{n_{k_m}} - x_{n_{k_m}}\| = \alpha_{n_{k_m}} \frac{\theta_{n_{k_m}}}{\alpha_{n_{k_m}}} \|S_{n_{k_m}} x_{n_{k_m}} - S_{n_{k_m}} x_{n_{k_m} - 1}\| \to 0 \text{ as } m \to \infty.
\] (4.20)

Since, the subsequence $\{x_{n_{k_m}}\}$ of $\{x_n\}$ is weakly convergent to a point $x^* \in H$. Hence, using the fact that $\lim_{n \to \infty} \|w_{n_{k_m}} - y_{n_{k_m}}\| = 0$, we have that $\{y_{n_{k_m}}\}$ also converges to $x^*$. Now passing the limit as $m \to \infty$ in (4.19), we have
\[
\lim_{m \to \infty} \langle Fx, x - y_{n_{k_m}} \rangle = \langle Fx, x - x^* \rangle > 0.
\] (4.21)

Hence, $x^* \in \Omega$.

Secondly, we consider the case in which $\limsup_{k \to \infty} \langle Fy_{n_k}, x - y_{n_k} \rangle = 0$ for $x \in C$. Let $\{\delta_k\}$ be a non-increasing positive sequence defined by
\[
\delta_k = |\langle Fy_{n_k}, x - y_{n_k} \rangle| + \frac{1}{k + 1}.
\] (4.22)

By our hypothesis, it is easy to see that
\[
\lim_{k \to \infty} \delta_k = \lim_{k \to \infty} \langle Fy_{n_k}, x - y_{n_k} \rangle + \lim_{k \to \infty} \frac{1}{k + 1} = 0.
\] (4.23)
By our hypothesis and (4.22), we have
\[ \langle F y_{n_k}, x - y_{n_k} \rangle + \delta_k > 0 \tag{4.24} \]
for each \( k \geq 1 \), since \( \{y_{n_k}\} \subset C \), it implies that \( \{F y_{n_k}\} \) is strictly non-zero and \( \liminf_{k \to \infty} \|F y_{n_k}\| = N_0 > 0 \). We therefore deduce that
\[ \|F y_{n_k}\| > \frac{N_0}{2} \tag{4.25} \]
In addition, let \( \{\epsilon_{n_k}\} \) be a sequence defined by \( \epsilon_{n_k} = \frac{F y_{n_k}}{\|F y_{n_k}\|^2} \). It implies that
\[ \langle F y_{n_k}, \epsilon_{n_k} \rangle = 1. \tag{4.26} \]
Combining (4.24) and (4.26), we have
\[ \langle F y_{n_k}, x + \delta_k \epsilon_{n_k} - y_{n_k} \rangle > 0. \tag{4.27} \]
By quasimonotonicity of the operator \( F \) on \( H \), we get that
\[ \langle F(x + \delta_k \epsilon_{n_k}), x + \delta_k \epsilon_{n_k} - y_{n_k} \rangle \geq 0. \tag{4.28} \]
Now, observe that
\[ \langle Fx, x + \delta_k \epsilon_{n_k} - y_{n_k} \rangle = \langle Fx - F(x + \delta_k \epsilon_{n_k}), x + \delta_k \epsilon_{n_k} - y_{n_k} \rangle \tag{4.29} \]
\[ = \langle Fx - F(x + \delta_k \epsilon_{n_k}), x + \delta_k \epsilon_{n_k} - y_{n_k} \rangle + \langle F(x + \delta_k \epsilon_{n_k}), x + \delta_k \epsilon_{n_k} - y_{n_k} \rangle \tag{4.30} \]
Combining (4.28), (4.29) and applying the well known Cauchy Schwartz inequality, we have
\[ \langle Fx, x + \delta_k \epsilon_{n_k} - y_{n_k} \rangle \geq \langle Fx - F(x + \delta_k \epsilon_{n_k}), x + \delta_k \epsilon_{n_k} - y_n \rangle \tag{4.31} \]
\[ \geq -\|Fx - F(x + \delta_k \epsilon_{n_k})\| \|x + \delta_k \epsilon_{n_k} - y_n\|. \tag{4.32} \]
Since \( F \) is Lipschitz continuous, we have
\[ \langle Fx, x + \delta_k \epsilon_{n_k} - y_{n_k} \rangle + L\|\delta_k \epsilon_{n_k}\| \|x + \delta_k \epsilon_{n_k} - y_{n_k}\| \geq 0 \tag{4.33} \]
Combining (4.25) and (4.33) and using the definition of \( \{\epsilon_{n_k}\} \), we have
\[ \langle Fx, x + \delta_k \epsilon_{n_k} - y_{n_k} \rangle + \frac{2L}{N_0} \delta_k \|x + \delta_k \epsilon_{n_k} - y_{n_k}\| \geq 0. \tag{4.34} \]
Since, the subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) is weakly convergent to a point \( x^* \in H \). Hence, using the fact that \( \lim_{n \to \infty} \|w_{n_k} - y_{n_k}\| = 0 \), we have that \( \{y_{n_k}\} \) also converges to \( x^* \). Taking limit as \( k \to \infty \), since \( \delta_k \to 0 \), we have
\[ \lim_{k \to \infty} \left[ \langle Fx, x + \delta_k \epsilon_{n_k} - y_{n_k} \rangle + \frac{2L}{N_0} \delta_k \|x + \delta_k \epsilon_{n_k} - y_{n_k}\| \right] = \langle Fx, x - x^* \rangle > 0. \tag{4.35} \]
Hence \( x^* \in \Omega \).

**Theorem 4.4.** Let \( \{x_n\} \) be a sequence generated by Algorithm 1 under Assumption 3. Then, \( \{x_n\} \) converges strongly to \( p \in \Phi \).
Proof Let $p \in \Phi$, observe that
\[
\|w_n - p\|^2 = \|x_n + \theta_n(S_n x_n - S_n x_{n-1}) - p\|^2 \\
\quad = \|x_n - p\|^2 + 2\theta_n\langle x_n - p, S_n x_n - S_n x_{n-1} \rangle + \theta_n^2\|S_n x_n - S_n x_{n-1}\|^2 \\
\quad \leq \|x_n - p\|^2 + 2\theta_n\|x_n - p\|\|x_n - x_{n-1}\| + \theta_n^2\|x_n - x_{n-1}\|^2 \\
\quad = \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|\|2\|x_n - p\| + \theta_n\|x_n - x_{n-1}\| \\
\quad = \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|\{\|2\|x_n - p\| + \beta_n\theta_n\|x_n - x_{n-1}\|\} \\
\quad \leq \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|\{\|2\|x_n - p\| + \beta_n N_1\} \\
\quad \leq \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|N_2, \tag{4.36}
\]
for some $N_2 > 0$.
\[
\|x_{n+1} - p\|^2 = \|x_n - \beta_n G p - \beta_n x_n - p\|^2 \\
\quad = \|\|1 - \beta_n G\| x_n - (1 - \beta_n G) p - \beta_n x_n - p\|^2 \\
\quad \leq \|\|1 - \beta_n G\| x_n - (1 - \beta_n G) p\|^2 + 2\beta_n G p = x_{n+1} - p\| \\
\quad \leq \|\|1 - \gamma \beta_n\| x_n - p\|^2 + 2\beta_n\alpha G p = x_{n+1} - p\| \\
\quad \leq \|\|1 - \gamma \beta_n\| x_n - x_{n-1}\|N_2\| + 2\beta_n\alpha G p = x_{n+1} - p\| \\
\quad = 1 - \gamma \beta_n\| x_n - p\|^2 + \gamma \beta_n\| x_n - x_{n-1}\|N_2 + 2\alpha\gamma G p = x_{n+1} - p\| \\
\quad = 1 - \gamma \beta_n\| x_n - p\|^2 + \gamma \beta_n\Psi_n \tag{4.37}
\]
where $\Psi_n = \frac{\theta_n}{\beta_n\gamma}||x_n - x_{n-1}||N_2 + 2\alpha\gamma G p = x_{n+1} - p\|$. According to Lemma 2.3, to conclude our proof, it is sufficient to establish that $\limsup_{k \to \infty} \Psi_n \leq 0$ for every subsequence $\{\|x_n - p\|\}$ of $\{\|x_n\|\}$ satisfying the condition:
\[
\liminf_{k \to \infty} \{\|x_{n_k+1} - p\| - \|x_{n_k} - p\|\} \geq 0. \tag{4.38}
\]
To establish that $\limsup_{k \to \infty} \Psi_n \leq 0$, we suppose that for every subsequence $\{\|x_n - p\|\}$ of $\{\|x_n\|\}$ such that (4.38) holds. Then,
\[
\liminf_{k \to \infty} \{\|x_{n_k+1} - p\|^2 - \|x_{n_k} - p\|^2\} \\
\quad = \liminf_{k \to \infty} \{\|x_{n_k+1} - p\| - \|x_{n_k} - p\|\}(\|x_{n_k+1} - p\| + \|x_{n_k} - p\|) \geq 0. \tag{4.39}
\]
It is easy to see from (4.37),
\[
\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 + 2\beta_n\alpha G p = x_{n+1} - p\| \\
\quad \leq \|w_n - p\|^2 + (1 - \frac{\lambda^2 n^2}{\lambda_n^2})\|y_n - w_n\|^2 + 2\beta_n\alpha G p = x_{n+1} - p\| \\
\quad \leq \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|N_2 - (1 - \frac{\lambda^2 n^2}{\lambda_n^2})\|y_n - w_n\|^2 + 2\beta_n\alpha G p = x_{n+1} - p\|. \tag{4.40}
\]
which implies that
\[
\limsup_{k \to \infty} \left( 1 - \frac{\lambda^2}{\beta_{n_k}^2} \right) |y_{n_k} - w_{n_k}|^2 
\leq \limsup_{k \to \infty} \left[ |x_{n_k} - p|^2 + \frac{\theta_{n_k}}{\beta_{n_k}} |x_{n_k} - x_{n_k-1}| N_2 + 2\beta_{n_k} \langle Gp, p - x_{n_k+1} \rangle - |x_{n_k+1} - p|^2 \right] 
\leq - \liminf_{k \to \infty} |x_{n_k+1} - p|^2 - |x_{n_k} - p|^2 \leq 0.
\]
Thus, we have
\[
\lim_{k \to \infty} |y_{n_k} - w_{n_k}| = 0. \tag{4.41}
\]
It is easy to see that, as \( k \to \infty \), we have
\[
|w_{n_k} - x_{n_k}| = \theta_{n_k} |S_n x_{n_k} - S_n x_{n_k-1}| = \beta_{n_k} \frac{\theta_{n_k}}{\beta_{n_k}} |S_n x_{n_k} - S_n x_{n_k-1}| \to 0. \tag{4.42}
\]
\[
|z_{n_k} - w_{n_k}| \leq |y_{n_k} - w_{n_k}| + \frac{\lambda_{n_k} \nu}{\lambda_{n_k+1}} |y_{n_k} - w_{n_k}| \to 0 \text{ as } k \to \infty. \tag{4.43}
\]
\[
|y_{n_k} - x_{n_k}| \leq |y_{n_k} - w_{n_k}| + |w_{n_k} - x_{n_k}| \to 0 \text{ as } k \to \infty. \tag{4.44}
\]
\[
|z_{n_k} - x_{n_k}| \leq |z_{n_k} - w_{n_k}| + |w_{n_k} - x_{n_k}| \to 0 \text{ as } k \to \infty. \tag{4.45}
\]
\[
|x_{n_k+1} - z_{n_k}| \leq |z_{n_k} - \beta_{n_k} \alpha G z_{n_k} - z_{n_k}| = \beta_{n_k} |\alpha G z_{n_k}| \to 0 \text{ as } k \to \infty. \tag{4.46}
\]
\[
|x_{n_k+1} - x_{n_k}| \leq |x_{n_k+1} - z_{n_k}| + |z_{n_k} - x_{n_k}| \to 0 \text{ as } k \to \infty. \tag{4.47}
\]
Now, since \( \{x_{n_k}\} \) is bounded, then, there exists a subsequence \( \{x_{n_{k_j}}\} \) of \( \{x_{n_k}\} \) such that \( \{x_{n_{k_j}}\} \) converges weakly to \( x^* \in H \). In addition, using (4.41), and Lemma 4.3, we obtain that \( x^* \in \Phi \). Furthermore, since \( x_{n_{k_j}} \) converges weakly to \( x^* \), we obtain that
\[
\limsup_{k \to \infty} \langle Gp, p - x_{n_k} \rangle = \lim_{j \to \infty} \langle Gp, p - x_{n_{k_j}} \rangle = \langle Gp, p - x^* \rangle. \tag{4.48}
\]
Hence, since \( p \) is a unique solution of \( \Phi \), it follows that
\[
\limsup_{k \to \infty} \langle Gp, p - x_{n_k} \rangle = \langle Gp, p - x^* \rangle \leq 0, \tag{4.49}
\]
we have obtain from (4.49) and (4.47)
\[
\limsup_{k \to \infty} \langle Gp, p - x_{n_k+1} \rangle \leq 0. \tag{4.50}
\]
Using our assumption and (4.49), we have that
\[
\lim_{k \to \infty} \Psi_{n_k} = \lim_{k \to \infty} \left( \frac{\theta_{n_k}}{\beta_{n_k}} |x_{n_k} - x_{n_k-1}| N_2 + 2\beta_{n_k} \langle Gp, p - x_{n_k+1} \rangle \right) \leq 0. \text{ Thus, From Lemma 2.3, we have that } \lim_{n \to \infty} |x_n - p| = 0.
\]
5. Numerical Example

In this section, we will give some numerical examples which will show the applicability and the efficiency of our proposed iterative technique in comparison to Algorithm 1.7, and Algorithm 1.6.

**Example 5.1.** Let \( H = L_2([0,1]) \) be equipped with the inner product
\[
\langle x, y \rangle = \int_0^1 x(t)y(t)dt \quad \forall x, y \in L_2([0,1]) \quad \text{and} \quad \|x\|^2 := \int_0^1 |x(t)|^2dt \quad \forall x, y, \in L_2([0,1]).
\]

Let \( F, G : L_2([0,1]) \to L_2([0,1]) \) be defined by
\[
F(x)(t) = \max\{0, x(t)\}, \quad t \in [0,1],
\]
\[
G(x)(t) = \frac{x(t)}{2}, \quad t \in [0,1].
\]

It is easy to see that \( F \) is 1-Lipschitz continuous and monotone, and \( G \) \( \tau \)-strongly monotone. We used this example due to Remark ? so that we can compare. Let \( S_n : L_2([0,1]) \to L_2([0,1]) \) be defined by
\[
S_n x(t) = \sin x(t).
\]

Let \( C \) be defined by \( C = \{x \in L_2 : \langle a, x \rangle = b\} \) where \( a \neq 0 \) and \( b = 2 \). Thus, we have
\[
P_C(\bar{x}) = \max\left\{0, \frac{b - \langle a, \bar{x}\rangle}{\|a\|^2}\right\} a + \bar{x}.
\]

We choose \( \beta_n = \frac{2}{2^{1/n}+5}, \theta_n = \theta, \alpha = 0.2, \nu = 0.3, \lambda_0 = \frac{1}{3}, \lambda_{n+1} = \frac{100}{(n+1)}, \epsilon_n = \frac{\alpha}{n^2}, \) for all \( n \in \mathbb{N} \). Also if we consider \( \epsilon = \|x_n - x_n^{(i)}\| \leq 10^{-5} \) as the stopping criterion and choose the following as starting points:

- **Case I:** \( x_0(t) = 2t^2 + t + 2, \quad x_1(t) = t; \)
- **Case II:** \( x_0(t) = 2t^2 + e^{2t} + 1, \quad x_1(t) = 3t^3 + 3; \)
- **Case III:** \( x_0(t) = t + 2, \quad x_1(t) = \cos(t); \)

**Table 1:** Computation result for Example ??.

<table>
<thead>
<tr>
<th></th>
<th>Algorithm 1</th>
<th>Algorithm 1.7</th>
<th>Algorithm 1.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case I</td>
<td>25</td>
<td>28</td>
<td>26</td>
</tr>
<tr>
<td>CPU time (sec)</td>
<td>0.0231</td>
<td>0.0345</td>
<td>0.0245</td>
</tr>
<tr>
<td>Case II</td>
<td>25</td>
<td>30</td>
<td>27</td>
</tr>
<tr>
<td>CPU time (sec)</td>
<td>0.0251</td>
<td>0.0321</td>
<td>0.0291</td>
</tr>
<tr>
<td>Case III</td>
<td>27</td>
<td>30</td>
<td>27</td>
</tr>
<tr>
<td>CPU time (sec)</td>
<td>0.0244</td>
<td>0.0358</td>
<td>0.0271</td>
</tr>
</tbody>
</table>

**Example 5.2.** Let \( H_1 = H_2 = l_2(\mathbb{R}) := \{x = (x_1, x_2, x_3, \cdots) : x_i \in \mathbb{R} : \sum_{i=1}^\infty |x_i|^2 < \infty \} \) and \( \|x\| = (\sum_{i=1}^\infty |x_i|^2)^{\frac{1}{2}} \) for all \( x \in l_2(\mathbb{R}) \). Suppose the operators \( F, G : l_2(\mathbb{R}) \to l_2(\mathbb{R}) \) are defined by
\[
G(x) - X - X_0,
\]
\[
F(x) = (5 - \|x\|)x \quad \forall x \in l_2(\mathbb{R}).
\]

It is easy to see that \( F \) is quasimonotone, Lipschitzian continuous and weakly sequentially continuous on \( l_2(\mathbb{R}) \), and \( G \) is \( \tau \)-strongly monotone, see [25]. Let \( C = Q = \{x \in l_2(\mathbb{R}) : \|x\| \leq 3\} \). Clearly, \( C \) is nonempty, closed and convex subsets of \( l_2(\mathbb{R}) \). Hence, we have
\[
P_C(x) = \begin{cases} 
\frac{x}{\|x\|}, & \text{if } \|x\| \leq 3, \\
\frac{3x}{\|x\|}, & \text{if otherwise.}
\end{cases}
\]

(5.1)
Figure 1: Example 5.1, Top Left: Case I; Top Right: Case II; Bottom Centered: case III.
In addition, we define $S_n, S : l_2(\mathbb{R}) \rightarrow l_2(\mathbb{R})$ are defined by $S_n x = (0, x_1, x_2, x_3, \cdots)$ and $S x = (0, \frac{x_1}{2}, \frac{x_2}{2}, \cdots)$. We choose $\beta_n = \frac{2}{200 n+5}, \theta_n = \theta, \alpha = 0.2, \nu = 0.3, \lambda_0 = \frac{1}{3}, \lambda_{n+1} = \frac{100}{(n+1)+3}, \epsilon_n = \frac{\alpha}{n+\theta}, \text{for all } n \in \mathbb{N}$. Also if we consider $\epsilon = ||x_n - x_{n+1}|| \leq 10^{-5}$ as the stopping criterion and choose the following as starting points:

Case I: $x_0 = (2, 2, 2, \cdots), x_1 = (0.5, 0.5, 0.5, \cdots)$;

Case II: $x_0 = (1, 2, 3, 4, \cdots), x_1 = (1, 1, 1, \cdots)$;

Case III: $x_0 = (0.1, 0.2, 0.3, \cdots), x_1 = (2, 4, 6, \cdots)$.

Table 2: Computation result for Example 5.2.

<table>
<thead>
<tr>
<th>Case</th>
<th>No of Iter.</th>
<th>CPU time (sec)</th>
<th>Algorithm 1</th>
<th>Algorithm 1 without ${S_n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>30</td>
<td>0.0390</td>
<td>39</td>
<td>0.0709</td>
</tr>
<tr>
<td>II</td>
<td>20</td>
<td>0.0212</td>
<td>44</td>
<td>0.0592</td>
</tr>
<tr>
<td>III</td>
<td>24</td>
<td>0.0326</td>
<td>30</td>
<td>0.0581</td>
</tr>
</tbody>
</table>

6. Conclusion

A modified Tseng with an inertial extrapolation step is introduced and studied for solving the BVIP (1.4) in infinite dimensional real Hilbert space when the cost operators are quasimonotone, and $\tau$-strongly monotone and Lipschitz continuous. In addition, we established that the proposed iterative method converges strongly to the solution set of BVIP (1.4). Our method uses stepsizes that are generated at each iteration by some simple computations, which allows it to be easily implemented without the prior knowledge of the operator norm or the coefficient of an underlying operator. In addition, we present some examples and numerical experiment to show the efficiency and implementation of our method in the framework of infinite and finite dimensional Hilbert spaces. We emphasize that one of the novelty of this work is in the use of a weaker operator (quasimonotone), modified inertial term introduced and the method of proof of the strong convergence of our iterative algorithm to the solution of problem (1.4).

References

Figure 2: Example 5.2, Top Left: Case I; Top Right: Case II; Bottom Centered: case III.


41. Y. Yao, Muglia, and G. L. Marino, A modified Korpelevichs method convergent to the minimum-norm solution of a variational inequality, Optimization, 63 (2014), 559–569.


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