Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–13.
ISSN-0037-8712
doi:10.5269/bspm.65218

On quasi-ternary hyperideals and bi-ternary hyperideals in ternary hypersemirings

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ABSTRACT: In this article we introduce the notion of quasi-ternary hyperideal in ternary hypersemirings which is the generalization of one-sided hyperideal and study some properties of it. Also, we obtain some characterizations of quasi-ternary hyperideals in a ternary hypersemiring. Again, we introduce the notion of bi-ternary hyperideals of ternary hypersemiring and relation between quasi-ternary hyperideals and bi-ternary hyperideals is established. We characterize regular ternary hypersemirings in terms of of quasi-ternary hyperideals and bi-ternary hyperideals. Lastly, we introduce the notions of prime quasi-ternary hyperideals and semiprime quasi-ternary hyperideals in a ternary hypersemiring.

Key Words: Ternary hypersemirings, quasi-ternary hyperideal, minimal left (right, lateral) hyperideal, bi-ternary hyperideal, regular ternary hypersemirings, prime quasi-ternary hyperideal.

Contents

1. Introduction

In the beginning of 19th century a new branch of mathematics was opened by the pathfinder F. Marty [17], which was known to be Hyperstructures Theory. After that many mathematicians and researchers like Corsini [3,4,5], De salvo [9], Dehkordi et al. [10], Krasner [14] introduced and studied different type of hyperstuctures. These days an astounding progression of algebraic hyperstructure theory has been observed. The theory of hyperstructure has many applications. In [6], P. Corsini and V. Leoreanu-Fotea and in [8] B. Davvaz and V. Leoreanu-Fotea points out their applications in different fields like theoretical physics, computer science etc. In 1982, R. Rota [20] introduced a special type of hyperstructure $(H, +, \otimes)$ in which $+ : H \times H \longrightarrow H$ is a binary mapping and $\otimes : H \times H \longrightarrow P(H)$ is a binary hyperoperation satisfying the following conditions: (1) $h_1 \otimes (h_2 \otimes h_3) = (h_1 \otimes h_2) \otimes h_3$, (2) $(h_1 + h_2) \otimes h_3 \subseteq h_1 \otimes h_2 + h_1 \otimes h_3$ (3) $h_1 \otimes (h_2 + h_3) \subseteq h_1 \otimes h_2 + h_1 \otimes h_3$, (4) $(-h_1) \otimes h_2 = h_1 \otimes (-h_2) = -(h_1 \otimes h_2)$ for all $h_1, h_2, h_3 \in H$, is called multiplicative hyperring. In [7], U. Dasgupta studied multiplicative hypersemirings in his Ph.D thesis, which is a generalization of multiplicative hyperrings. Ternary semiring were introduced by Dutta and Kar [12], which is a generalized concept of ternary ring introduced by W. G. Lister [15]. In 2015, Salim et al. [22] presented the class of multiplicative ternary hyperring, which is a generalization of multiplicative hyperring. Again, in [26], Tamang et al. generalized the concept of multiplicative ternary hyperring and bring up ternary hypersemiring.

In [25], Steinfeld introduced the notion of quasi-ideals. Quasi-ideals of semirings were studied by Dönges in [13], Sionson [24] and Dixit et al. [11] studied some properties of quasi-ideals and bi-deals of ternary semigroups. Naka and Hila [18], studied the structure of ternary semihypergroups in terms of quasi-hyperideals and bi-hyperideals. Many other reseracher like T. Changphas and B. Davvaz [2], S. Omidi and B. Davvaz [19], S. Abdullah et al. [1], Rao [21], Shao [23] also studied the quasi-hyperideals in ordered semihypergroups, ordered Γ-semihypergroups, Γ-semihypergroups and ordered semihyperrings. In this paper, we raise special type of hyperideals in ternary hypersemirings which is a generalized notion of

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Submitted September 29, 2022. Published October 23, 2023
2010 *Mathematics Subject Classification:* 16Y60, 20N20, 16Y99.

one-sided hyperideal, called quasi-ternary hyperideal and bi-ternary hyperideal in ternary hypersemirings and investigate their properties. We obtain some characterizations of quasi-ternary hyperideals and bi-ternary hyperideals in ternary hypersemirings. Also, we prove that in a regular ternary hypersemiring the notions of quasi-ternary hyperideal and bi-ternary hyperideal are equivalent. Finally we mention the prime quasi-ternary hyperideal and semiprime quasi-ternary hyperideal and characterize them.

2. Preliminaries

Definition 2.1 *Let A be a nonempty set and a mapping $\circledast : A \times A \times A \longrightarrow \wp^*(A)$ is called a ternary hyperoperation on A, where $\wp^*(A)$ is the set of all nonempty subsets of A. The image of $(a_1, a_2, a_3) \in A \times A \times A$ will be denoted by $a_1 \circledast a_2 \circledast a_3$ (which is known to be ternary hyperproduct of $a_1, a_2, a_3 \in A$).*

Definition 2.2 *[26] Let $(T, +, \circledast)$ be a ternary hypersemiring, that is, a commutative semigroup $(T, +)$ together with a ternary hyperproduct '$\circledast$' satisfying the following conditions:*

(i) $(t_1 \circledast t_2 \circledast t_3) \circledast t_4 \circledast t_5 = t_1 \circledast (t_2 \circledast t_3 \circledast t_4) \circledast t_5 = t_1 \circledast t_2 \circledast (t_3 \circledast t_4 \circledast t_5)$

(ii) $(t_1 + t_2) \circledast t_3 \circledast t_4 \subseteq t_1 \circledast t_3 \circledast t_4 + t_2 \circledast t_3 \circledast t_4$;

(iii) $t_1 \circledast (t_2 + t_3) \circledast t_4 \subseteq t_1 \circledast t_2 \circledast t_4 + t_1 \circledast t_3 \circledast t_4$;

(iv) $t_1 \circledast t_2 \circledast (t_3 + t_4) \subseteq t_1 \circledast t_2 \circledast t_3 + t_1 \circledast t_2 \circledast t_4$ *for all $t_1, t_2, t_3, t_4, t_5 \in T$,*

Instead of '$\subseteq$' if '$=$' holds, then $(T, +, \circledast)$ is called strongly distributive ternary hypersemiring.

We have the following remark.

Remark 2.1 *[22] For $t_1, t_2, t_3 \in T$ such that $|t_1 \circledast t_2 \circledast t_3| = 1$, then the concepts of ternary hypersemirings and ternary semirings are identical.*

Thus from the above remark we can say that the class of ternary hypersemirings is the generalized concept of the class of ternary semirings.

Example 2.1 Consider the ternary semiring $(\mathbb{Z}_0^-, +, \cdot)$ of the set of all negative integers with respect to the usual addition and multiplication of integers. Corresponding to any subset A of the set of negative integers, there exists a ternary hypersemiring induced by A is $(\mathbb{Z}_A^-, +, \cdot)$, where $Z_A^- \subseteq \mathbb{Z}_0^-$ and for any $x, y, z \in \mathbb{Z}_A^-$, $+$ is the usual addition of integers and $x \circledast y \circledast z = \{x \cdot a \cdot y \cdot b \cdot z : a, b \in A\}$.

Definition 2.3 *[26] Let $(T, +, \circledast)$ be a ternary hypersemiring and '0' be an additive identity element of T. Then '0' is called a zero(strong zero) if $0 \in t_1 \circledast t_2 \circledast 0 = t_1 \circledast 0 \circledast t_2 = 0 \circledast t_1 \circledast t_2$ (resp.$\{0\} = t_1 \circledast 0 \circledast t_2 = 0 \circledast t_1 \circledast t_2 = t_1 \circledast t_2 \circledast 0$) for all $t_1, t_2 \in T$.*

Definition 2.4 *[26] An additive subsemigroup S of a ternary hypersemiring $(T, +, \circledast)$ is called a ternary subhypersemiring of T if $s_1 \circledast s_2 \circledast s_3 \subseteq S$ for all $s_1, s_2, s_3 \in S$.*

Definition 2.5 *[26] An additive subsemigroup A of a ternary hypersemiring $(T, +, \circledast)$ is said to be a right hyperideal (a lateral hyperideal or a left hyperideal) of T, if $a \circledast t_1 \circledast t_2 \subseteq A$ (resp. $t_1 \circledast a \circledast t_2 \subseteq A$ or $t_1 \circledast t_2 \circledast a \subseteq A$) for all $a \in A$ and for all $t_1, t_2 \in T$.*

An additive subsemigroup A which is a right hyperideal, a lateral hyperideal and a left hyperideal of T, then A is called a hyperideal of T.

Let T be a ternary hypersemiring. If I, J and K are nonempty subsets of T, then $I \circledast J \circledast K = \bigcup\{\sum_{finite} a_i \circledast b_i \circledast c_i : a_i \in I, b_i \in J, c_i \in K\}$.

Proposition 2.1 *[26] Let 't' be an element of a ternary hypersemring $(T,+,\circledast)$. Then the principal right hyperideal generated by 't' is denoted by $\langle t\rangle_r$, defined by*

$$\langle t\rangle_r = t\circledast T\circledast T + \{kt : k\in\mathbb{Z}_0^+\}$$

The principal lateral hyperideal generated by 't' is denoted by $\langle t\rangle_m$, defined by

$$\langle t\rangle_m = T\circledast t\circledast T + T\circledast T\circledast t\circledast T\circledast T + \{kt : k\in\mathbb{Z}_0^+\}$$

The principal left hyperideal generated by 't' is denoted by $\langle t\rangle_l$, defined by

$$\langle t\rangle_l = T\circledast T\circledast t + \{kt : k\in\mathbb{Z}_0^+\}$$

The principal hyperideal generated by 't' is denoted by $\langle t\rangle$, defined by

$$\langle t\rangle = T\circledast T\circledast t + t\circledast T\circledast T + T\circledast t\circledast T + T\circledast T\circledast t\circledast T\circledast T + \{kt : k\in\mathbb{Z}_0^+\}$$

Proposition 2.2 *Let A be a nonempty subsets of a ternary hypersemiring $(T,+,\circledast)$. Then,*

- *The principal right hyperideal generated by A is defined by*

$$\langle A\rangle_r = A\circledast T\circledast T + A.$$

- *The principal left hyperideal generated by A is defined by*

$$\langle A\rangle_l = T\circledast T\circledast A + A.$$

- *The principal lateral hyperideal generated by A is defined by*

$$\langle A\rangle_m = T\circledast A\circledast T + T\circledast T\circledast A\circledast T\circledast T + A.$$

- *The principal hyperideal generated by A is defined by*

$$\langle A\rangle = A\circledast T\circledast T + T\circledast T\circledast A + T\circledast A\circledast T + T\circledast T\circledast A\circledast T\circledast T + A.$$

Definition 2.6 *[26] Let $(T,+,\circledast)$ be a ternary hypersemiring. Then $t\in T$ is said to be idempotent if $t\circledast t\circledast t=\{t\}$.*

Throught this paper, T will be a ternary hypersemiring with zero.

3. Quasi-ternary hyperideal in ternary hypersemirings

Definition 3.1 *An additive subsemigroup Q of a ternary hypersemiring $(T,+,\circledast)$ is called a quasi-ternary hyperideal of T if $Q\circledast T\circledast T\cap(T\circledast Q\circledast T + T\circledast T\circledast Q\circledast T\circledast T)\cap T\circledast T\circledast Q\subseteq Q$.*

Remark 3.1

- *In a ternary hypersemiring $(T,+,\circledast)$, the improper hyperideal T and trivial hyperideal $\langle 0\rangle$ both are quasi-ternary hyperideals of T.*
- *In a ternary hypersemiring $(T,+,\circledast)$, every quasi-ternary hyperideal is a ternary subhypersemiring of T.*

Example 3.1 $(\mathbb{Z}_5^-,+,\cdot)$ is a ternary semiring. Let $A=\{2,3\}$. On $\mathbb{Z}_5^-$, '$\circledast$' defined as follows $z_1\circledast z_2\circledast z_3=\{z_1\cdot a_1\cdot z_2\cdot a_2\cdot z_3 : a_1,a_2\in A\}$ for all $z_1,z_2,z_3\in\mathbb{Z}_5^-$. Then $(\mathbb{Z}_5^-,+,\circledast)$ is a ternary hypersemiring. Let $Q=\{0,-2,-3\}$. Then Q is a quasi-ternary hyperideal of $\mathbb{Z}_5^-$.

Lemma 3.1 *Let T be a ternary hypersemiring, then*

(a) *Every left, right and lateral hyperideal of a ternary hypersemiring T is a quasi-ternary hyperideal of T.*

(b) *The intersection of ternary subhypersemiring S of T and the quasi-ternary hyperideal Q of T is either empty or a quasi-ternary hyperideal of S.*

(c) *The intersection of family of quasi-ternary hyperideals of a ternary hypersemiring S is either empty or a quasi-ternary hyperideal of S.*

Proof: (a) It is obvious.

(b) Let $Q\cap S$ is a non-empty subset of S. Now $((Q\cap S)\circledast S\circledast S)\cap(S\circledast(Q\cap S)\circledast S+S\circledast S\circledast(Q\cap S)\circledast S\circledast S)\cap S\circledast S\circledast(Q\cap S)\subseteq(S\circledast S\circledast S)\cap(S\circledast S\circledast S+S\circledast S\circledast S\circledast S\circledast S)\cap(S\circledast S\circledast S)\subseteq S\circledast S\circledast S\subseteq S$. Again $((Q\cap S)\circledast S\circledast S)\cap(S\circledast(Q\cap S)\circledast S+S\circledast S\circledast(Q\cap S)\circledast S\circledast S)\cap(S\circledast S\circledast(Q\cap S))\subseteq(Q\circledast S\circledast S)\cap(S\circledast Q\circledast S+S\circledast S\circledast Q\circledast S\circledast S)\cap(S\circledast S\circledast Q)\subseteq(Q\circledast T\circledast T)\cap(T\circledast Q\circledast T+T\circledast T\circledast Q\circledast T\circledast T)\cap(T\circledast T\circledast Q)\subseteq Q$. Consequently $Q\cap S$ is a quasi-ternary hyperideal of S.

(c) Let $\{Q_i : i\in I\}$ be a family of quasi-ternary hyperideal of T. Suppose that $\bigcap\limits_{i\in I}Q_i\neq\phi$. Then $(\bigcap\limits_{i\in I}Q_i)\circledast T\circledast T\cap(T\circledast(\bigcap\limits_{i\in I}Q_i)\circledast T+T\circledast T\circledast(\bigcap\limits_{i\in I}Q_i)\circledast T\circledast T)\cap T\circledast T\circledast(\bigcap\limits_{i\in I}Q_i)\subseteq Q_i\circledast T\circledast T\cap(T\circledast Q_i\circledast T+T\circledast T\circledast Q_i\circledast T\circledast T)\cap T\circledast T\circledast Q_i\subseteq Q_i$ for all $i\in I$. Thus $\bigcap\limits_{i\in I}Q_i$ is a quasi-ternary hyperideal of T. □

Remark 3.2 *Quasi-ternary hyperideals of a ternary hypersemirings $(T,+,\circledast)$ may not be a right hyperideal or a lateral hyperideal or a left hyperideal of T.*

To justify the above remark we give the following example.

Example 3.2 Consider the semigroup $(M_{2\times2}(\mathbb{Z}_0^-),+)$, where $\mathbb{Z}_0^-$ is set of all non-positive integers.

Let $A=\left\{\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$. Let $\circledast$ be a ternary hyperoperation on $(M_{2\times2}(\mathbb{Z}_0^-),+)$ defined by $a\circledast b\circledast c=\{a\cdot i\cdot b\cdot j\cdot c : i,j\in A\}$ where $a,b,c\in(M_{2\times2}(\mathbb{Z}_0^-),+)$. Then $(M_{2\times2}(\mathbb{Z}_0^-),+,\circledast)$ is a ternary hypersemiring. Let $Q=\left\{\begin{pmatrix}a&0\\0&0\end{pmatrix}: a\in\mathbb{Z}_0^-\right\}$. Then we can easily verify that Q is a quasi-ternary hyperideal, but Q is not a right hyperideal, a lateral hyperideal or a left hyperideal of $(M_{2\times2}(\mathbb{Z}_0^-),+,\circledast)$.

Theorem 3.1 *Let Q be a ternary subhypersemiring of $(T,+,\circledast)$. If Q is the intersection of a right hyperideal, a lateral hyperideal and a left hyperideal, then Q is a quasi-ternary hyperideal of T.*

Proof: Let Q be an additive subsemigroup of a ternary hypersemiring T and $Q=I\cap J\cap K$, where I is a right hyperideal, J is a lateral hyperideal and K is a left hyperideal of T. Then by Lemma 3.1, we can say that Q is a quasi-ternary hyperideal of T. □

Remark 3.3 *From Theorem 3.1, we can say that Q has an intersection property.*

Lemma 3.2 *Let $(T,+,\circledast)$ be a ternary hypersemiring and t be an element of T. Then the following statements hold:*

(i) *$T\circledast T\circledast t$ is a left hyperideal.*

(ii) *$T\circledast t\circledast T+T\circledast T\circledast t\circledast T\circledast T$ is a lateral hyperideal.*

(iii) *$t\circledast T\circledast T$ is a right hyperideal,*

(iv) *$(T\circledast T\circledast t)\cap(T\circledast t\circledast T+T\circledast T\circledast t\circledast T\circledast T)\cap(t\circledast T\circledast T)$ is a quasi-ternary hyperideal.*

Proof: (i) Since $T \circledast T \circledast (T \circledast T \circledast t) \subseteq (T \circledast T \circledast T) \circledast T \circledast t \subseteq T \circledast T \circledast t$, then $T \circledast T \circledast t$ is a left hyperideal of T.

(ii) and (iii) are similar to (i).

(iv) It follows from Theorem 3.1. □

Theorem 3.2 *Let $(T, +, \circledast)$ be a ternary hypersemiring. An additive subsemigroup Q of ternary hypersemiring T is a minimal quasi-ternary hyperideal of S if and only if $Q = I \cap J \cap K$, where I is a minimal right hyperideal, J a minimal lateral hyperideal and K is a minimal left hyperideal of T.*

Proof: Let $Q = I \cap J \cap K$, where I is a minimal right hyperideal, J is a minimal lateral hyperideal and a L is a minimal left hyperideal of T. Then by Theorem 3.1, we have that Q is a quasi-ternary hyperideal of T. Now we show that Q is minimal. Suppose $Q' \subseteq Q$ be any other quasi-ternary hyperideal of T. Then $Q' \circledast T \circledast T$ is a right hyperideal of T and $Q' \circledast T \circledast T \subseteq Q \circledast T \circledast T \subseteq I \circledast T \circledast T \subseteq I$. Then we have $Q' \circledast T \circledast T = I$, since I is a minimal right hyperideal of T. Again for a minimal lateral hyperideal J of T, we get $T \circledast Q' \circledast T + T \circledast T \circledast Q' \circledast T \circledast T = J$ and for a minimal left hyperideal K of T we get $T \circledast T \circledast Q' = K$. Therefore $Q = I \cap J \cap K = Q' \circledast T \circledast T \cap (T \circledast Q' \circledast T + T \circledast T \circledast Q' \circledast T \circledast T) \cap T \circledast T \circledast Q' \subseteq Q'$. Thus $Q = Q'$ and hence the quasi-ternary hyperideal Q is minimal in T.

For the converse, we have $(Q \circledast T \circledast T) \cap (T \circledast Q \circledast T + T \circledast T \circledast Q \circledast T \circledast T) \cap (T \circledast T \circledast Q) \subseteq Q$. Let $q \in Q$. Then by Lemma 3.2, $I = q \circledast T \circledast T$ is a right hyperideal, $J = (T \circledast q \circledast T + T \circledast T \circledast q \circledast T \circledast T)$ is a lateral hyperideal and $K = T \circledast T \circledast q$ is a left hyperideal of T. Therefore by Theorem 3.1, $I \cap J \cap K$ is a quasi-ternary hyperideal of T. Now $q \circledast T \circledast T \cap (T \circledast q \circledast T + T \circledast T \circledast q \circledast T \circledast T) \cap T \circledast T \circledast q \subseteq Q \circledast T \circledast T \cap (T \circledast Q \circledast T + T \circledast T \circledast Q \circledast T \circledast T) \cap T \circledast T \circledast Q \subseteq Q$. By minimality of Q, we obtain $q \circledast T \circledast T \cap (T \circledast q \circledast T + T \circledast T \circledast q \circledast T \circledast T) \cap T \circledast T \circledast q = Q$. Now we show that $q \circledast T \circledast T$ is a right minimal hyperideal of T , $(T \circledast q \circledast T + T \circledast T \circledast q \circledast T \circledast T)$ is a lateral minimal hyperideal of T and $T \circledast T \circledast q$ is a left minimal hyperideal of T. Suppose that I be any right hyperideal of T such that $I \subseteq q \circledast T \circledast T$. Then $I \circledast T \circledast T \subseteq I \subseteq q \circledast T \circledast T$. Now $I \circledast I \circledast T \cap (T \circledast q \circledast T + T \circledast T \circledast q \circledast T \circledast T) \cap T \circledast T \circledast q \subseteq q \circledast T \circledast T \cap (T \circledast q \circledast T + T \circledast T \circledast q \circledast T \circledast T) \cap T \circledast T \circledast q = Q$. By minimality of Q of a ternary hypersemiring S, we obtain $I \circledast T \circledast T \cap (T \circledast q \circledast T + T \circledast T \circledast q \circledast T \circledast T) \cap T \circledast T \circledast q = Q$. This implies that $Q \subseteq I \circledast T \circledast T$. Again $q \circledast T \circledast T \subseteq Q \circledast T \circledast T \subseteq (I \circledast T \circledast T) \circledast T \circledast T \subseteq I \circledast T \circledast T$. Thus $q \circledast T \circledast T \subseteq I \circledast T \circledast T \subseteq I$ and hence $I = q \circledast T \circledast T$. Hence $q \circledast T \circledast T$ is a minimal right hyperideal of T. Similarly we can prove that $(T \circledast q \circledast T + T \circledast T \circledast q \circledast T \circledast T)$ and $T \circledast T \circledast q$ are minimal lateral hyperideal and minimal left hyperideal of T respectively. □

Proposition 3.1 *Let $(T, +, \circledast)$ be a ternary hypersemiring and I be a minimal lateral hyperideal of T. Then I is a minimal hyperideal of T.*

Proof: *First of all, we show that I is both right and left hyperideal of a ternary hypersemiring T. Let $a \in I$. Then, $T \circledast a \circledast T + T \circledast T \circledast a \circledast T \circledast T$ is a letaral hyperideal of T and $T \circledast a \circledast T + T \circledast T \circledast a \circledast T \circledast T \subseteq T \circledast I \circledast T + T \circledast T \circledast I \circledast T \circledast T \subseteq I$. By minimality of I in a ternary hypersemiring of T, we get $I = T \circledast a \circledast T + T \circledast T \circledast a \circledast T \circledast T$. Now $I \circledast T \circledast T = (T \circledast a \circledast T + T \circledast T \circledast a \circledast T \circledast T) \circledast T \circledast T \subseteq T \circledast a \circledast T + T \circledast T \circledast a \circledast T \circledast T \subseteq I$ and $T \circledast T \circledast I = T \circledast T \circledast (T \circledast a \circledast T + T \circledast T \circledast a \circledast T \circledast T) \subseteq T \circledast a \circledast T + T \circledast T \circledast a \circledast T \circledast T \subseteq I$. Thus M is both right and left hyperideal of a ternary hypersemiring T. Now it remains to show that I is a minimal hyperideal of T. If possible, let I' be any other hyperideal of T such that $I' \subseteq I$. Since I' is a hyperideal of T, then it is a lateral hyperideal also. By hypothesis, we have $I' = I$. Consequently I is a minimal hyperideal of T.* □

Corollary 3.1 *Let $(T, +, \circledast)$ be a ternary hypersemiring and Q be a minimal quasi-ternary hyperideal of T. Then Q contained in a minimal hyperideal of T.*

Proof: For a right hyperideal R, a lateral hyperideal M and a left hyperideal L of a ternary hypersemiring T, we have $Q = R \cap M \cap L$, by Theorem 3.2. Then $Q \subseteq M$. From Proposition 3.1, it follows that M is a minimal hyperideal of S. □

4. Quasi-ternary hyperideals and regularity of ternary hypersemiring

Definition 4.1 *[26] An element 't' of a ternary hypersemiring $(T,+,\circledast)$ is called regular if $t \in t \circledast T \circledast t$. Then $t \in \sum_{i=1}^{n} t \circledast t_i \circledast t$ for $t_i \in T (i = 1,2,3,\cdots)$.*

A ternary hypersemiring $(T,+,\circledast)$ is called regular if each of its elements is regular. If T is strongly distributive ternary hypersemiring, an element $t \in T$ is regular if only if there exists $a \in T$ such that $t \in t \circledast a \circledast t$.

Definition 4.2 *A ternary subhypersemiring B of a ternary hypersemiring $(T,+,\circledast)$ is called a bi-ternary hyperideal of T if $B \circledast T \circledast B \circledast T \circledast B \subseteq B$.*

Lemma 4.1 *Let $(T,+,\circledast)$ be a ternary hypersemiring and Q be a quasi-ternary hyperideal of T. Then Q is a bi-ternary hyperideal of T.*

Proof: Since Q is a quasi-ternary hyperideal of a ternary hypersemiring T, then we have $Q \circledast T \circledast Q \circledast T \circledast Q \subseteq Q \circledast (T \circledast T \circledast T) \circledast T \subseteq Q \circledast T \circledast T$, $Q \circledast T \circledast Q \circledast T \circledast Q \subseteq T \circledast (T \circledast T \circledast T) \circledast Q \subseteq T \circledast T \circledast Q$ and $Q \circledast T \circledast Q \circledast T \circledast Q \subseteq T \circledast T \circledast Q \circledast T \circledast T$. Again $\{0\} \in T \circledast Q \circledast T$, so $Q \circledast T \circledast Q \circledast T \circledast Q \subseteq T \circledast Q \circledast T + T \circledast T \circledast Q \circledast T \circledast T$. Consequently, it follows that $Q \circledast T \circledast Q \circledast T \circledast Q \subseteq Q \circledast T \circledast T \cap (T \circledast Q \circledast T + T \circledast T \circledast Q \circledast T \circledast T) \cap T \circledast T \circledast Q \subseteq Q$ (since Q is a quasi-ternary hyperideal) and hence Q is a bi-ternary hyperideal of T. □

Theorem 4.1 *Let $(T,+,\circledast)$ be a ternary hypersemiring. Then the following statements are equivalent:*

(i) *T is a regular;*

(ii) *For any right hyperideal U, lateral hyperideal V and left hyperideal W of T, $U \circledast V \circledast W = U \cap V \cap W$;*

(iii) *Each right hyperideal U, each lateral hyperideal V and each left hyperideal W of T satisfies*

- *(a) $U \circledast U \circledast U = U$;*
- *(b) $V \circledast V \circledast V = V$;*
- *(c) $W \circledast W \circledast W = W$;*
- *(d) $U \circledast V \circledast W$ is a quasi-ternary hyperideal of T;*

(iv) *The set $\mathcal{Q}$ of all quasi-ternary hyperideals of T is a regular ternary hypersemigroup with respect to ternary hyperoperation '$\circledast$';*

(v) *$Q = Q \circledast T \circledast Q \circledast T \circledast Q$, for any quasi-ternary hyperideal Q of a ternary hypersemiring T.*

Proof: $(i) \Rightarrow (ii)$ Let T be a regular ternary hypersemiring. Let U be a right hyperideal of T, V be a lateral hyperideal of T and W be a left hyperideal of T. Clearly $U \circledast V \circledast W \subseteq U \cap V \cap W$ (1). Suppose $t \in U \cap V \cap W \subseteq T$. Now regularity of T implies that $t \in t \circledast T \circledast t \Rightarrow t \in (t \circledast T \circledast t) \circledast (T \circledast t \circledast T) \circledast (t \circledast T \circledast t) \subseteq U \circledast V \circledast W$. Thus we have $U \cap V \cap W \subseteq U \circledast V \circledast W$ (2). From (1) and (2), it follows that $U \cap V \cap W = U \circledast V \circledast W$.

$(ii) \Rightarrow (iii)$ For (a), let U be a right hyperideal of T and $\langle U \rangle_l = U + T \circledast T \circledast U$ be a left hyperideal of T. By (2), $U = U \cap \langle U \rangle_m \cap \langle U \rangle_l = U \circledast \langle U \rangle_m \circledast \langle U \rangle_l \subseteq U \circledast U \circledast U \subseteq U$. Hence $U \circledast U \circledast U = U$.

Similarly we can prove (b) and (c). For (d), by Theorem 3.1, $U \circledast V \circledast W = U \cap V \cap W$ is a quasi-ternary hyperideal of T.

$(iii) \Rightarrow (iv)$. Let $\mathcal{Q}$ be the set of all quasi-ternary hyperideals of T. Let $\langle Q \rangle_r$ be a right hyperideal of S. Then by $(iii)(a)$, $Q \subseteq \langle Q \rangle_r = \langle Q \rangle_r \circledast \langle Q \rangle_r \circledast \langle Q \rangle_r \subseteq Q \circledast T \circledast T$. Again, if we consider $\langle Q \rangle_m$ be a lateral hyperideal and $\langle Q \rangle_l$ is a left hyperideal of T, then $\langle Q \rangle_m = T \circledast Q \circledast T + T \circledast T \circledast Q \circledast T \circledast T + Q$ and $\langle Q \rangle_l = T \circledast T \circledast Q + Q$ respectively. Then by (iii)(a), (b) we have $Q \subseteq T \circledast Q \circledast T + T \circledast T \circledast Q \circledast T \circledast T$ and $Q \subseteq T \circledast T \circledast Q$ respectively. Since Q is quasi-ternary hyperideal of T, then $Q \subseteq Q \circledast T \circledast T \cap (T \circledast Q \circledast T + T \circledast T \circledast Q \circledast T \circledast T) \cap T \circledast T \circledast Q \subseteq Q$. Hence $Q = Q \circledast T \circledast T \cap (T \circledast Q \circledast T + T \circledast T \circledast Q \circledast T \circledast T) \cap T \circledast T \circledast Q \cdots (A)$. Let

Q_1, Q_2, Q_3 be three quasi-ternary hyperideals of a ternary hypersemiring T and $U = Q_1 \circledast Q_2 \circledast Q_3 \circledast T \circledast T$ be right hyperideal of T, $V = T \circledast Q_1 \circledast Q_2 \circledast Q_3 \circledast T + T \circledast T \circledast Q_1 \circledast Q_2 \circledast Q_3 \circledast T \circledast T$ be a lateral hyperideal of T and $W = T \circledast T \circledast Q_1 \circledast Q_2 \circledast Q_3$ be a left hyperideal of T. Then by (iii)(d), $U \circledast V \circledast W$ is a quasi-ternary hyperideal of T. Now by (A), we get $U \circledast V \circledast W = (U \circledast V \circledast W) \circledast T \circledast T \cap (T \circledast (U \circledast V \circledast W) \circledast T + T \circledast T \circledast (U \circledast V \circledast W) \circledast T \circledast T) \cap T \circledast T \circledast (U \circledast V \circledast W) \subseteq U \circledast V \circledast W \subseteq Q_1 \circledast Q_2 \circledast Q_3$ (1) by using Lemma 4.1. Therefore $Q_1 \circledast Q_2 \circledast Q_3$ is a quasi-ternary hyperideal of S. Obviously $\circledast$ is associative on $\mathcal{Q}$. Hence $\mathcal{Q}$ is ternary hypersemigroup.

Again $Q = Q \circledast T \circledast T \cap (T \circledast Q \circledast T + T \circledast T \circledast Q \circledast T \circledast T) \cap T \circledast T \circledast Q = Q \circledast T \circledast T \circledast (T \circledast Q \circledast T + T \circledast T \circledast Q \circledast T \circledast T) \circledast T \circledast T \circledast Q$ by (1) $\subseteq Q \circledast T \circledast Q$. i.e $Q \subseteq Q \circledast T \circledast Q$. Thus Q is regular in $\mathcal{Q}$. Consequently $\mathcal{Q}$ is a regular ternary hypersemigroup.

$(iv) \Rightarrow (v)$ For each quasi-ternary hyperideal Q of T such that $Q \subseteq Q \circledast T \circledast Q$, since Q is a regular in $\mathcal{Q}$, then it implies that $Q \subseteq Q \circledast T \circledast Q \subseteq Q \circledast T \circledast Q \circledast T \circledast Q \subseteq Q$ by Lemma 4.1.

$(v) \Rightarrow (i)$ For each element $t \in T$, by Theorem 3.1, $\langle t\rangle_r \cap \langle t\rangle_m \cap \langle t\rangle_l$ is a quasi-ternary hyperideal of T. Now by (v) $t \in \langle t\rangle_r \cap \langle s\rangle_m \cap \langle t\rangle_l \subseteq (\langle t\rangle_r \cap \langle t\rangle_m \cap \langle t\rangle_l) \circledast T \circledast (\langle t\rangle_r \cap \langle t\rangle_m \cap \langle t\rangle_l) \circledast T \circledast (\langle t\rangle_r \cap \langle t\rangle_m \cap \langle t\rangle_l) \subseteq \langle t\rangle_r \circledast T \circledast \langle t\rangle_m \circledast T \circledast \langle t\rangle_l \subseteq t \circledast T \circledast t \circledast T \circledast t \subseteq t \circledast T \circledast t$. i.e $t \in t \circledast T \circledast t$. Hence $t \in T$ is regular in T. Consequently T is a regular ternary hypersemiring. □

Definition 4.3 *A hyperideal I of a ternary hypersemiring $(R, +, \circledast)$ is called idempotent if $I \circledast I \circledast I = I$.*

Theorem 4.2 *If every quasi-ternary hyperideal Q of a ternary hypersemiring $(T, +, \circledast)$ is idempotent, then T is regular.*

Proof: Let U be a right hyperideal, V be a lateral hyperideal and W be a left hyperideal of T. Then by Lemma 3.1, $U \cap V \cap W$ is a quasi-ternary hyperideal of T. Obviously $U \circledast V \circledast W \subseteq U \cap V \cap W$. Now by hypothesis $U \cap V \cap W = (U \cap V \cap W) \circledast (U \cap V \cap W) \circledast (U \cap V \cap W) \subseteq U \circledast V \circledast W$. Thus $U \cap V \cap W = U \circledast V \circledast W$. Then by Theorem 4.1, T is a regular ternary hypersemiring. □

Lemma 4.2 *Every lateral hyperideal M of a regular ternary hypersemiring $(T, +, \circledast)$ is a regular ternary subhypersemiring.*

Proof: Each element $m \in M \subseteq T$ is regular in T, so $m \in m \circledast T \circledast m \subseteq m \circledast (T \circledast m \circledast T) \circledast m \subseteq m \circledast M \circledast m$ (since M is lateral hyperideal of T and $T \circledast m \circledast T \subseteq M$). This follows that M is a regular ternary subhypersemiring. □

Remark 4.1 *Since every right (lateral, left) hyperideal of T is a quasi-ternary hyperideal of T, then it follows that every right(resp. lateral, left) hyperideal of T is a bi-ternary hyperideal of S, but the converse is not true.*

Proposition 4.1 *The intersection of a bi-ternary hyperideal and a ternary subhypersemiring of a ternary hypersemiring $(S, +, \circledast)$ is again a bi-ternary hyperideal of S.*

Proof: Let B be a bi-ternary hyperideal and T be a ternary subhypersemiring of ternary hypersemiring S. Let $\mathfrak{B} = B \cap T$. Now $(B \cap T) \circledast T \circledast (B \cap T) \circledast T \circledast (B \cap T) \subseteq (B \cap T) \circledast S \circledast (B \cap T) \circledast S \circledast (B \cap T) \subseteq B \circledast S \circledast B \circledast S \circledast B \subseteq B$ and $(B \cap T) \circledast T \circledast (B \cap T) \circledast T \circledast (B \cap T) \subseteq T \circledast (T \circledast T \circledast T) \circledast T \subseteq T \circledast T \circledast T \subseteq T$. Hence $\mathfrak{B} \circledast T \circledast \mathfrak{B} \circledast T \circledast \mathfrak{B} \subseteq \mathfrak{B}$. □

Lemma 4.3 *Let $(T, +, \circledast)$ be a ternary hypersemiring and B be a bi-ternary hyperideal of T. If T_1, T_2 are two ternary subhypersemiring of S, then $B \circledast T_1 \circledast T_2$, $T_1 \circledast T_2 \circledast B \circledast T_1 \circledast T_2$ and $T_1 \circledast T_2 \circledast B$ are bi-ternary hyperideals of T.*

Corollary 4.1 *If B_1, B_2 and B_3 are bi-ternary hyperideals of a ternary hypersemiring $(T, +, \circledast)$, then $B_1 \circledast B_2 \circledast B_3$ is a bi-ternary hyperideal of ternary hypersemiring T.*

Corollary 4.2 *If Q_1, Q_2 and Q_3 are quasi-ternary hyperideals of a ternary hypersemiring $(T, +, \circledast)$, then $Q_1 \circledast Q_2 \circledast Q_3$ is a bi-ternary hyperideal of T.*

Theorem 4.3 *Let $(T, +, \circledast)$ be a ternary hypersemiring, $\mathfrak{B}$ be a bi-ternary hyperideal of T and B be a bi-ternary hyperideal of $\mathfrak{B}$ such that $B \circledast B \circledast B = B$. Then B is a bi-ternary hyperideal of T.*

Proof: We have $\mathfrak{B} \circledast T \circledast \mathfrak{B} \circledast T \circledast \mathfrak{B} \subseteq \mathfrak{B}$, since $\mathfrak{B}$ is a bi-ternary hyperideal of T. Again since B is a bi-ternary hyperideal in $\mathfrak{B}$, then $B \circledast \mathfrak{B} \circledast B \circledast \mathfrak{B} \circledast B \subseteq B$.

Now $B \circledast T \circledast B \circledast T \circledast B = (B \circledast B \circledast B) \circledast T \circledast B \circledast T \circledast (B \circledast B \circledast B) = B \circledast B \circledast (B \circledast T \circledast B \circledast T \circledast B) \circledast B \circledast B \subseteq B \circledast B \circledast (\mathfrak{B} \circledast T \circledast \mathfrak{B} \circledast T \circledast \mathfrak{B}) \circledast B \circledast B \subseteq B \circledast B \circledast \mathfrak{B} \circledast B \circledast B = B \circledast B \circledast \mathfrak{B} \circledast B \circledast (B \circledast B \circledast B) \subseteq B \circledast (B \circledast \mathfrak{B} \circledast B \circledast \mathfrak{B} \circledast B) \circledast B \subseteq B \circledast B \circledast B = B$. Hence the proof is completed. □

Proposition 4.2 *Let $(T, +, \circledast)$ be a ternary hypersemiring and A, B, and C be ternary subhypersemirings of a ternary hypersemiring T. Then $\mathfrak{B} = A \circledast B \circledast C$ is a bi-ternary hyperideal of T if at least one of A, B and C a right, a lateral and a left hyperideal of T.*

Proof: Let $\mathfrak{B} = A \circledast B \circledast C$. Suppose that A is a left hyperideal of T. Then we get

$$\begin{aligned}
&(A \circledast B \circledast C) \circledast T \circledast (A \circledast B \circledast C) \circledast T \circledast (A \circledast B \circledast C)\\
&\subseteq (T \circledast T \circledast T) \circledast T \circledast (T \circledast T \circledast T) \circledast T \circledast (A \circledast B \circledast C)\\
&\subseteq (T \circledast T \circledast T) \circledast T \circledast (A \circledast B \circledast C)\\
&\subseteq (T \circledast T \circledast A) \circledast B \circledast C\\
&\subseteq A \circledast B \circledast C.
\end{aligned}$$

Hence $\mathfrak{B} = A \circledast B \circledast C$ is a bi-ternary hyperideal of T.

Suppose that B is a left hyperideal of T. Then we get

$$\begin{aligned}
&(A \circledast B \circledast C) \circledast T \circledast (A \circledast B \circledast C) \circledast T \circledast (A \circledast B \circledast C)\\
&\subseteq A \circledast (T \circledast T \circledast T) \circledast (T \circledast T \circledast T) \circledast T \circledast (A \circledast B \circledast C)\\
&\subseteq A \circledast (T \circledast T \circledast T) \circledast (A \circledast B \circledast C)\\
&\subseteq A \circledast (T \circledast A \circledast B) \circledast C\\
&\subseteq A \circledast B \circledast C.
\end{aligned}$$

This shows that $\mathfrak{B} = A \circledast B \circledast C$ is a bi-ternary hyperideal of T.

Again, if C is a left hyperideal of T, then

$$\begin{aligned}
&(A \circledast B \circledast C) \circledast T \circledast (A \circledast B \circledast C) \circledast T \circledast (A \circledast B \circledast C)\\
&\subseteq (A \circledast B \circledast C) \circledast (T \circledast T \circledast T) \circledast (T \circledast T \circledast T) \circledast T \circledast T\\
&\subseteq (A \circledast B \circledast C) \circledast (T \circledast T \circledast T) \circledast T\\
&\subseteq A \circledast B \circledast (C \circledast T \circledast T)\\
&\subseteq A \circledast B \circledast C.
\end{aligned}$$

Therefore $\mathfrak{B} = A \circledast B \circledast C$ is a bi-ternary hyperideal of T.

Other cases are also similar to above. □

Corollary 4.3 *The ternary hyperproduct of a right hyperideal, a lateral hyperideal and a left hyperideal of $(T, +, \circledast)$ is a bi-ternary hyperideal of T.*

Corollary 4.4 *Let $(T, +, \circledast)$ be a regular ternary hypersemiring. Then $R \cap M \cap L$ is bi-ternary hyperideal of T, for any right hyperideal R, lateral hyperideal M and left hyperideal L of T.*

Proof: It follows from Theorem 4.1 and Proposition 4.2. □

Proposition 4.3 *Let $(T, +, \circledast)$ be a ternary hypersemiring and B be a ternary subhypersemiring. If B satisfies the condition that $U \circledast V \circledast W \subseteq B \subseteq U \cap V \cap W$, where U is a right hyperideal, V is a lateral hyperideal and W is a left hyperideal of T, then B is a bi-ternary hyperideal of T.*

Proof: It follows from $B \circledast T \circledast B \circledast T \circledast B \subseteq (U \cap V \cap W) \circledast T \circledast (U \cap V \cap W) \circledast T \circledast (U \cap V \cap W) \subseteq U \circledast (T \circledast V \circledast T) \circledast W \subseteq U \circledast V \circledast W \subseteq B$. □

Theorem 4.4 *Let $(T, +, \circledast)$ be a ternary hypersemiring. Then the following conditions are equivalent:*

(i) *T is regular;*

(ii) *every hyperideal I of T is an idempotent;*

(iii) *$B \circledast T \circledast B \circledast T \circledast B = B$, for every bi-ternary hyperideal B of T;*

(iv) *$Q \circledast T \circledast Q \circledast T \circledast Q = Q$, for every quasi-ternary hyperideal Q of T;*

Proof: $(i) \Leftrightarrow (ii)$ Clearly $I \circledast I \circledast I \subseteq I$. Since T is regular ternary hypersemiring, then for $x \in I \subseteq T$ implies that $x \in x \circledast T \circledast x \subseteq x \circledast (T \circledast x \circledast T) \circledast x \subseteq I \circledast I \circledast I$. Thus $I \circledast I \circledast I = I$.

For the converse, let U, V, and W be a right hyperideal, a lateral hyperideal and a left hyperideal of T respectively. Obviously $U \circledast V \circledast W \subseteq U \cap V \cap W$. Again $U \cap V \cap W = (U \cap V \cap W) \circledast (U \cap V \cap W) \circledast (U \cap V \cap W) \subseteq U \circledast V \circledast W$. Thus $U \circledast V \circledast W = U \cap V \cap W$. Therefore, by Theorem 4.1, T is a regular ternary hypersemiring.

$(i) \Rightarrow (iii)$ Suppose S is regular ternary hypersemiring. Let B be a bi-ternary hyperideal of T. Then $B \circledast T \circledast B \circledast T \circledast B \subseteq B \quad (1)$. Let $b \in B \subseteq T$. Then $b \in b \circledast T \circledast b \subseteq b \circledast T \circledast b \circledast T \circledast b \subseteq B \circledast T \circledast B \circledast T \circledast B$, that is, $B \subseteq B \circledast T \circledast B \circledast T \circledast B \quad (2)$. From (1) and (2), it follows that $B = B \circledast T \circledast B \circledast T \circledast B$.

$(iii) \Rightarrow (iv)$ It follows from the Lemma 4.1.

$(iv) \Rightarrow (i)$ Suppose the condition (iv) holds. Let U be a right hyperideal, V be a lateral hyperideal and W be a left hyperideal of T. Then by Theorem 3.1, $Q = U \cap V \cap W$ is a quasi-ternary hyperideal of ternary hypersemiring T. By hypothesis, $Q \circledast T \circledast Q \circledast T \circledast Q = Q$. Now $U \cap V \cap W = Q = Q \circledast T \circledast Q \circledast T \circledast Q \subseteq U \circledast T \circledast V \circledast T \circledast W \subseteq U \circledast V \circledast W$. Again, clearly $U \circledast V \circledast W \subseteq U \cap V \cap W$. Therefore $U \cap V \cap W = U \circledast V \circledast W$. Hence by Theorem 4.1, T is a regular ternary hypersemiring. □

Theorem 4.5 *Let $(T, +, \circledast)$ be a ternary hypersemiring and B be a ternary subhypersemiring of T. Then B is bi-ternary hyperideal of T if and only if $B = B \circledast S \circledast B$.*

Proof: Let B be a bi-ternary hyperideal of T and $b \in B \subseteq T$, then $b \in b \circledast T \circledast b \subseteq B \circledast T \circledast B$. Thus $B \subseteq B \circledast T \circledast B$. Again, $B \circledast T \circledast B \subseteq B \circledast T \circledast B \circledast T \circledast B \subseteq B$. Hence $B = B \circledast T \circledast B$.

Converse is straightforward. □

Theorem 4.6 *Let $(T, +, \circledast)$ be a regular ternary hypersemiring. Then the notions of bi-ternary hyperideals of ternary hypersemirings and quasi-ternary hyperideals of a ternary hypersemirings are equivalent.*

Proof: Let B be a bi-ternary hyperideal of regular ternary hypersemiring T. Then by Theorem 4.1, $U \cap V \cap W = U \circledast V \circledast W$, where U, V, W are a right hyperideal, a lateral hyperideal, and a left hyperideal of a ternary hypersemiring T respectively. Now

$$\begin{aligned} B \circledast T \circledast T \cap & (T \circledast B \circledast T + T \circledast T \circledast B \circledast T \circledast T) \cap T \circledast T \circledast B \\ & = B \circledast T \circledast T \circledast (T \circledast B \circledast T + T \circledast T \circledast B \circledast T \circledast T) \circledast T \circledast T \circledast B \\ & \subseteq B \circledast (T \circledast T \circledast T) \circledast B \circledast (T \circledast T \circledast T) \circledast B + \\ & B \circledast (T \circledast T \circledast T) \circledast T \circledast B \circledast (T \circledast T \circledast T) \circledast T \circledast B \\ & \subseteq B \circledast T \circledast B \circledast T \circledast B + B \circledast T \circledast T \circledast B \circledast T \circledast T \circledast B \\ & \subseteq B + B \circledast T \circledast B \\ & \subseteq B + B \\ & \subseteq B \end{aligned}$$

since B is bi-ternary hyperideal and using the Theorem 4.5. Consequently B is a quasi-ternary hyperideal of T.

Converse part follows from Lemma 4.1. □

Theorem 4.7 *Let $(T, +, \circledast)$ be a regular ternary hypersemiring. If A and B are two ternary subhypersemirings of T and C be a bi-ternary hyperideal if T, then $A \circledast B \circledast C$, $A \circledast C \circledast B$ and $C \circledast A \circledast B$ are quasi-ternary hyperideals of a ternary hypersemiring T.*

Proof: The theorem follows from Lemma 4.3 and Theorem 4.6. □

Corollary 4.5 *For any three quasi-ternary hyperideals A, B and C of a regular teranry hypersemiring $(T, +, \circledast)$, $A \circledast B \circledast C$ is a quasi-ternary hyperideal of T.*

Proof: Corollary follows from Corollary 4.2 and Theorem 4.7. □

In [16], Mandal et al. studied prime hyperideals in ternary hypersemiring. Now we introduce the notions of prime and semiprime quasi-ternary hyperideals in ternary hypersemirings.

Definition 4.4 *Let $(T, +, \circledast)$ be a ternary hypersemiring and Q be ternary subhypersemiring. Then Q is called a prime quasi-ternary hyperideal of T if $U \circledast V \circledast W \subseteq Q$ implies $U \subseteq Q$ or $V \subseteq Q$ or $W \subseteq Q$ for any three quasi-ternary hyperideals U, V and W of T.*

A ternary subhypersemiring Q of T is called a semiprime quasi-ternary hyperideal if $A \circledast A \circledast A \subseteq Q$ implies $A \subseteq Q$ for any quasi-ternary hyperideal A of S.

Remark 4.2 *Every prime quasi-ternary hyperideal of a ternary hypersemiring T is a semiprime quasi-ternary hyperideal of T. But the converse is not true.*

Example 4.1 Let $T = M_{2\times 2}(\mathbb{Z}_0^-)$ be a set all 2×2 matrices over $\mathbb{Z}_0^-$. Let $A = \left\{ \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \right\}$. On T, '$+$' is a matrix addition and '$\circledast$' defined as follows: $\alpha \circledast \beta \circledast \gamma = \{\alpha \cdot a_1 \cdot \beta \cdot a_2 \cdot \gamma : a_1, a_2 \in A\}$ for all $\alpha, \beta, \gamma \in T$. Then $(T_A, +, \circledast)$ forms a ternary hypersemiring. Let $Q = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in 15\mathbb{Z}_0^- \right\}$ be a semiprime quasi-ternary hyperideal but not prime quasi-ternary hyperideal of T_A, since $A = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in \mathbb{Z}_0^- \right\}$, $B = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in 3\mathbb{Z}_0^- \right\}$ and $C = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in 5\mathbb{Z}_0^- \right\}$ are quasi-ternary hyperideals of S such that $A \circledast B \circledast C \subseteq Q$ but $A \nsubseteq Q$, $B \nsubseteq Q$ and $C \nsubseteq Q$.

Proposition 4.4 *Let $(T, +, \circledast)$ be a ternary hypersemiring and $t \in T$. The principal quasi-ternary hyperideal generated by t is denoted by $\langle t \rangle_q = (t \circledast T \circledast T) \cap (T \circledast a \circledast T + T \circledast T \circledast a \circledast T \circledast T) \cap (T \circledast T \circledast a) + \{nt : t \in T, n \in \mathbb{Z}_0^+\}$.*

Proposition 4.5 *If a ternary subhypersemiring Q of a ternary hypersemiring $(T, +, \circledast)$ is a prime quasi-ternary hyperideal, then Q is a right hyperideal or a lateral hyperideal or a left hyperideal of T.*

Proof: Let Q be a prime quasi-ternary hyperideal of a ternary hypersemiring T. We have $(Q \circledast T \circledast T) \circledast (T \circledast Q \circledast T + T \circledast T \circledast Q \circledast T \circledast T) \circledast (T \circledast T \circledast Q) \subseteq (Q \circledast T \circledast T) \cap (T \circledast Q \circledast T + T \circledast T \circledast Q \circledast T \circledast T) \cap (T \circledast T \circledast Q) \subseteq Q$. Then by the hypothesis, $Q \circledast T \circledast T \subseteq Q$ or $T \circledast Q \circledast T + T \circledast T \circledast Q \circledast T \circledast T \subseteq Q$ or $T \circledast T \circledast Q \subseteq Q$. Hence Q is a right or a lateral or a left hyperideal of T. □

Proposition 4.6 *A ternary subhypersemiring Q of a commutative ternary hypersemiring $(T, +, \circledast)$ is a prime quasi-ternary hyperideal of T if and only if for any $x, y, z \in T$, $x \circledast y \circledast z \in Q$ implies $x \in Q$ or $y \in Q$ or $z \in Q$.*

Theorem 4.8 *Let Q be a quasi-ternary hyperideal of a ternary hypersemiring $(T, +, \circledast)$. Then the following statements are equivalent:*

(i) *Q is prime;*

(ii) *for any $x, y, z \in T$ such that $\langle x \rangle_q \circledast \langle y \rangle_q \circledast \langle z \rangle_q \subseteq Q$ implies $x \in Q$ or $y \in Q$ or $z \in Q$.*

Proof: $(i) \Rightarrow (ii)$ Suppose that Q is a prime quasi-ternary hyperideal and let $\langle x \rangle_q \circledast \langle y \rangle_q \circledast \langle z \rangle_q \subseteq Q$. By Proposition 4.4, $\langle x \rangle_q$, $\langle y \rangle_q$ and $\langle z \rangle_q$ are quasi-ternary hyperideal of T. Therefore $\langle x \rangle_q \subseteq Q$ or $\langle y \rangle_q \subseteq Q$ or $\langle z \rangle_q \subseteq Q \Rightarrow x \in Q$ or $y \in Q$ or $z \in Q$.

$(ii) \Rightarrow (i)$ It is obvious. □

Theorem 4.9 *Let $(T, +, \circledast)$ be a ternary hypersemiring. Then the following statements are equivalent:*

(i) *quasi-ternary hyperideals of a ternary hypersemiring T are idempotent;*

(ii) *For any three quasi-ternary hyperideals U, V, W of T, $U \cap V \cap W \subseteq U \circledast V \circledast W$ with $U \cap V \cap W \neq \phi$;*

(iii) *For any $t \in T$, $\langle t \rangle_q = \langle t \rangle_q \circledast \langle t \rangle_q \circledast \langle t \rangle_q$.*

Proof: $(i) \Rightarrow (ii)$ It is obvious that $U \cap V \cap W$ is a quasi-ternary hyperideal of a ternary hypersemiring T. By the conjecture, $U \cap V \cap W = (U \cap V \cap W) \circledast (U \cap V \cap W) \circledast (U \cap V \cap W) \subseteq U \circledast V \circledast W$.

$(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (i)$ are obvious. □

Definition 4.5 *Let $(T, +, \circledast)$ be a ternary hypersemiring. A non-empty subset A of T is said to be m_q-system if for any $a, b, c \in A$, $(\langle a \rangle_q \circledast \langle b \rangle_q \circledast \langle c \rangle_q) \cap A \neq \phi$*

Definition 4.6 *Let $(T, +, \circledast)$ be a ternary hypersemiring. A nonempty subset A of T is said to be n_q-system if for any $a \in A$, $(\langle a \rangle_q \circledast \langle a \rangle_q \circledast \langle a \rangle_q) \cap A \neq \phi$*

Theorem 4.10 *Let Q be a quasi-ternary hyperideal of a ternary hypersemiring $(T, +, \circledast)$. Then Q is a prime quasi-ternary hyperideal of T if only if Q^c(complement of Q) is an m_q-system.*

Proof: Suppose that Q is a prime quasi-ternary hyperideal of T. Assume that Q^c is an m_q-system, if not, then there exist $a, b, c \in Q^c$ such that $(\langle a \rangle_q \circledast \langle b \rangle_q \circledast \langle c \rangle_q) \cap Q^c = \phi \Rightarrow \langle a \rangle_q \circledast \langle b \rangle_q \circledast \langle c \rangle_q \subseteq Q$. Then by Theorem 4.8, $a \in Q$ or $b \in Q$ or $c \in Q$. So we arrive a contradiction. Hence Q^c is an m_q-system.

Conversely, let Q^c is an m_q-system. Suppose $\langle a \rangle_q \circledast \langle b \rangle_q \circledast \langle c \rangle_q \subseteq Q \nRightarrow a \in Q$ or $b \in Q$ or $c \in Q$. Then $a, b, c \in Q^c \Rightarrow \langle a \rangle_q \circledast \langle b \rangle_q \circledast \langle c \rangle_q \cap Q^c \neq \phi$, a contradiction. Hence $a, b, c \in Q$. Then by Theorem 4.8, Q is a prime quasi-ternary hyperideal of T. □

Theorem 4.11 *Let Q be a quasi-ternary hyperideal of a ternary hypersemiring $(T, +, \circledast)$. Then Q is a semiprime quasi-ternary hyperideal of T if only if Q^c is an n_q-system.*

Proof: Parallel proof of the Theorem 4.10. □

References

1. Abdullah, S., Hila, K., Aslam, M., *On bi-Γ-hyperideals of Γ-semihypergroups*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., 74(4), 79-90, (2012).
2. Changphas, T., Davvaz, B., *Bi-hyperideals and quasi-hyperideals in ordered semihypergroups*, Italian Journal of Pure and applied Mathematics, 35, 439-508, (2015).
3. Corsini, P., *Hypergraphs and hypergroups*, Algebra Universalis 35, 548-555, (1996).
4. Corsini, P., *Hypergroups réguliers et hypermodules*, Ann. Univ. Ferrara-Sez. VII-Sc. Mat.20, 121-235, (1975).
5. Corsini, P., (1980). Sur les semihypergroupes. *Atti Soc. Pelor. Sc. Mat. Fis. Nat.* XXVI.
6. Corsini, P., Leoreanu, V., (2003). Applications of Hyperstructure Theory, Advances in Mathematics (Dordrecht), 5. Kluwer Academic Publishers, Dordrecht. xii+322 pp. ISBN: 1-4020-1222-5.
7. Dasgupta, U., *On certain classes of hypersemirings*, PhD. Thesis, University of Calcutta (2011).
8. Davvaz B. Leoreanu-Fotea V., (2007). Hyperring Theory and Applications. Palm Harbor, FL: International Academic Press (ISBN 1-57485-064-4). iv, 328 p.
9. De Salvo M., *Hyperrings and Hyperfields*, Annales Scientifiques de Universite de Clermont-Ferrand II. No. 22, (1984).
10. Ostadhadi-Dehkordi, S., Davvaz, B., *A Strong Regular Relation on Γ-semihyperrings*, Journal of Sciences,Islamic Republic of Iran, 22(3), 257-266, (2011).
11. Dixit, V.N., Dewan, S., *A note on quasi and bi-ideals in ternary semigroups.* Int. J. Math. Math. Sci. 18(3), 501-508, (1995).
12. Dutta, T.K., Kar, S., *A note on regular ternary semirings*, Kyungpook Mathematical Journal, 46(3), 357-365, (2006).
13. Dönges, C., *On Quasi-ideals of semirings*, Int. J. Math. Math. Sci, 17(1), 47-58, (1994).
14. Krasner, M., *A class of hyperrings and hyperfields*, Int. J. Math. Math. Sci, 6(2), 307-312, (1983).
15. Lister, W. G., *Ternary rings*, Trans. Amer. Math. Soc, 154, 37-55, (1971).
16. Mandal, M., Tamang, N., Das, S., *On some properties of hyperideals in ternary hypersemirings*, Asian-European Journal of Mathematics, Vol. 16, No. 01, 2250230, (2023).
17. Marty, F., *Sur une generalization de la notion de group*, 8^{th} Congress des Math. Scandenaves Stockholm, 45-49, (1934).
18. Naka, K., Hila, K., *On the structure of quasi-hyperideals and bi-hyperideals in ternary semihypergroups*, Afr. Mat. 26, No. 7-8, 1573-1591, (2015).
19. Omidi, S., Davvaz, B., *Bi-Γ-hyperideals and Greens relations in ordered Γ-Semihypergroups*, Eurasian Journal of Mathematics, 8(4), 63-73, (2017).
20. Rota, R., *Strongly distributive multiplicative hyperrings*, Journal of Geometry, 39, No. 1-2, 130-138, (1990).
21. Rao, Y., Xu P., Shao, Z., Kosari, S., *Left k-bi-quasi hyperideals in ordered semihyperrings*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. 83(1), 125–134, (2021).
22. Salim, Md., Dutta, T. K., *Regular equivalence and strongly regular equivalence on multiplicative ternary hyperring*, Journal of Hyperstructures, 4(1), 20-36, (2015).
23. Shao, Z., Chen, X., Kosari, S., Omidi, S., *On some properties of right pure (bi-quasi-)hyperideals in ordered semihyperrings*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., 83(4), 95-104, (2021).
24. Sioson, F. M., *Ideal theory in ternary semigroups*, Math. Japonica, 10, 63-84, (1965).
25. Steinfeld, O., *Uber die Quasiideale Halbgruppen*, Publ. Math. Debrecen, 4, 262-275, (1956).
26. Tamang, N., Mandal, M., *Hyperideals of a ternary hypersemiring*, Bull.Cal. Math. Soc, 110(5), 385-398, (2018).

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