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Non-linear new product $\frac{1}{2}(AB^*C+CB^*A)$ derivations on $*$-algebras

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ABSTRACT: Let $\mathcal{A}$ be a prime $*$-algebra with unit I and a nontrivial projection. Then the map $\Phi:\mathcal{A}\to\mathcal{A}$ satisfies in the following condition

$$\Phi(\{ABC\})=\{\Phi(A)BC\}+\{A\Phi(B)C\}+\{AB\Phi(C)\}$$

where $\{ABC\}=\frac{1}{2}(AB^*C+CB^*A)$ for all $A,B,C\in\mathcal{A}$, is additive. Moreover, if $\Phi(I)$ is self-adjoint, then Φ is a $*$-derivation.

Key Words: Prime $*$-algebra; $*$-derivation; Jordan derivation.

Contents

1. Introduction

Let $\mathcal{A}$ be a $*$-algebra. For $A,B\in\mathcal{A}$, denoted by $A\circ B=AB+BA$ and $A\bullet B=AB+BA^*$, which are Jordan product and $*$-Jordan product, respectively. These products are found playing a more and more important role in some research topics, and its study has recently attracted many author's attention (for example, see [2,5,6]). Recall that a map $\Phi:\mathcal{A}\to\mathcal{A}$ is said to be an additive derivation if $\Phi(A+B)=\Phi(A)+\Phi(B)$ and $\Phi(AB)=\Phi(A)B+A\Phi(B)$ for all $A,B\in\mathcal{A}$. A map Φ is additive $*$-derivation if it is an additive derivation and $\Phi(A^*)=\Phi(A)^*$. Derivations are very important maps both in theory and applications, and have been studied intensively [1,8,9,10,11,12]. in [4], Taghavi, Rohi and Darvish proved that every nonlinear $*$-Jordan derivation between factor von Neumann algebra is an additive derivation.

In recent years, many mathematicians devoted themselves to study the new products, ABA and AB^*A, which are called Jordan triple product and $*$-Jordan triple product, respectively. From the work done in this field in [3], Taghavi, Nouri, Razeghi and Darvish proved that if the map $\Phi:\mathcal{A}\to\mathcal{B}$ is bijective and preserves Jordan or $*$-Jordan triple product, then it is additive. Moreover, if Φ preserves Jordan triple product, they prove the multiplicativity or anti-multiplicativity of Φ. Finally, they prove that if $\mathcal{A}$ and $\mathcal{B}$ are two prime operator $*$-algebras, $\Psi:\mathcal{A}\to\mathcal{B}$ is bijective and preserves $*$-Joran triple product, then Ψ is a $\mathbb{C}$-linear or conjugate $\mathbb{C}$-linear $*$-isomorphism. Another definition that needs to be said here is Jordan derivation, which is a mapping like derivation mentioned earlier, which in this case is $D(A^2)=D(A)A+AD(A)$. It can be seen in [7]. In [13], Taghavi and Tavakoli defined a new product as $\frac{1}{2}(AB^*C+CB^*A)$ for all $A,B,C\in\mathcal{A}$. They proved that if the map $\Phi:\mathcal{A}\to\mathcal{B}$ is bijective and preserves the mentioned product, then it is additive. Also, if $\Phi(I)$ is a positive element, then Φ is a $*$-isomorphism. In this paper, we use $\{ABC\}=\frac{1}{2}(AB^*C+CB^*A)$. In the next section, we state the main results of the present paper.

2. MAIN RESULTS

Our main theorem is as follows:

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Theorem 2.1 *Let $\mathcal{A}$ be a prime $*$-algebra with I and a nontrivial projection. Then the map $\Phi : A \to A$ satisfies in the following condition*

$$\Phi(\{ABC\}) = \{\Phi(A)BC\} + \{A\Phi(B)C\} + \{AB\Phi(C)\} \tag{2.1}$$

where $\{ABC\} = \frac{1}{2}(AB^*C + CB^*A)$ for all $A, B, C \in \mathcal{A}$, is additive. Moreover, if $\Phi(I)$ is self-adjoint, then Φ is a $*$-derivation.

Proof: Let P_1 be a nontrivial projection in $\mathcal{A}$ and $P_2 = I_{\mathcal{A}} - P_1$. Denote $\mathcal{A}_{ij} = P_i \mathcal{A} P_j$, $i, j = 1, 2$, then $\mathcal{A} = \sum_{i,j=1}^{2} \mathcal{A}_{ij}$. For every $A \in \mathcal{A}$ we may write $A = A_{11} + A_{12} + A_{21} + A_{22}$. In all that follow, when we write $\mathcal{A}_{ij}$, it indicates that $A_{ij} \in \mathcal{A}_{ij}$. For showing additivity of Φ on $\mathcal{A}$, we use above partition of $\mathcal{A}$ and give some lemmas that prove Φ is additive on each $\mathcal{A}_{ij}$, $i, j = 1, 2$.
The proof of the theorem is organized as a series lemmas. We begin with the following lemma with a simple proof. □

Lemma 2.1 $\Phi(0) = 0$

Proof: $\Phi(0) = \Phi(\{000\}) = \{\Phi(0)00\} + \{0\Phi(0)0\} + \{00\Phi(0)\} = 0.$ □

Lemma 2.2 *For every $A_{ij} \in \mathcal{A}_{ij}$, $A_{ji} \in \mathcal{A}_{ji}$, we have $\Phi(A_{ij} + A_{ji}) = \Phi(A_{ij}) + \Phi(A_{ji})$ for $1 \le i \ne j \le 2$.*

Proof: Let $T = \Phi(A_{ij} + A_{ji}) - \Phi(A_{ij}) - \Phi(A_{ji})$, we should prove that $T = 0$. Using Lemma 2.1 we have $\Phi(\{I(P_i - P_j)(A_{ij} + A_{ji})\}) = \Phi(\{I(P_i - P_j)A_{ij}\}) + \Phi(\{I(P_i - P_j)A_{ji}\})$. From this, we get

$$\begin{aligned}
&\{\Phi(I)(P_i - P_j)(A_{ij} + A_{ji})\} + \{I\Phi(P_i - P_j)(A_{ij} + A_{ji})\} + \{I(P_i - P_j)\Phi(A_{ij} + A_{ji})\} = \\
&\{\Phi(I)(P_i - P_j)A_{ij}\} + \{I\Phi(P_i - P_j)A_{ij}\} + \{I(P_i - P_j)\Phi(A_{ij})\} \\
&+ \{\Phi(I)(P_i - P_j)A_{ij}\} + \{I\Phi(P_i - P_j)A_{ji}\} + \{I(P_i - P_j)\Phi(A_{ji})\}.
\end{aligned}$$

So, we obtain $\{I(P_i - P_j)(\Phi(A_{ij} + A_{ji}) - \Phi(A_{ij}) - \Phi(A_{ji}))\} = 0$. That is $\{I(P_i - P_j)T\} = \frac{1}{2}((P_i - P_j)T + T(P_i - P_j)) = 0$. So, we have $T_{ii} = T_{jj} = 0$. For every $C_{ij} \in \mathcal{A}_{ij}$, since $\{C_{ij}I(A_{ij} + A_{ji})\} = \{C_{ij}IA_{ij}\} + \{C_{ij}IA_{ji}\}$, we have $\Phi(\{C_{ij}I(A_{ij} + A_{ij})\}) = \Phi(\{C_{ij}IA_{ij}\}) + \Phi(\{C_{ij}IA_{ji}\})$. From this, we get

$$\begin{aligned}
&\{\Phi(C_{ij})I(A_{ij} + A_{ji})\} + \{C_{ij}\Phi(I)(A_{ij} + A_{ji})\} + \{C_{ij}I\Phi(A_{ij} + A_{ji})\} = \\
&\{\Phi(C_{ij})IA_{ij}\} + \{C_{ij}\Phi(I)A_{ij}\} + \{C_{ij}I\Phi(A_{ij})\} \\
&+ \{\Phi(C_{ij})IA_{ji}\} + \{C_{ij}\Phi(I)A_{ji}\} + \{C_{ij}I\Phi(A_{ji})\}.
\end{aligned}$$

So, we obtain $\{C_{ij}I((\Phi(A_{ij} + Aji) - \Phi(A_{ij}) - \Phi(A_{ji}))\} = 0$. That is $\{C_{ij}IT\} = \frac{1}{2}(C_{ij}T + TC_{ij}) = 0$. Hence $P_jTC_{ij} = 0$. Then $T_{ji}C_{ij} = 0$ for all $C_{ij} \in \mathcal{A}_{ij}$, that is $T_{ji}CP_j = 0$ for all $C \in \mathcal{A}$. By primness, it follow that $T_{ji} = 0$. Similarly, by putting C_{ji} in place of C_{ij}, we get $T_{ij} = 0$, so $T = 0$. □

Lemma 2.3 *For every $A_{ii} \in \mathcal{A}_{ii}$, $A_{ij} \in \mathcal{A}_{ij}$, $A_{ji} \in \mathcal{A}_{ji}$, we have $\Phi(A_{ii} + A_{ij} + A_{ji}) = \Phi(A_{ii}) + \Phi(A_{ij}) + \Phi(A_{ji})$ for $1 \le i \ne j \le 2$.*

Proof: Let $T = \Phi(A_{ii} + A_{ij} + A_{ji}) - \Phi(A_{ii}) - \Phi(A_{ij}) - \Phi(A_{ji})$. Using Lemmas 2.1 and 2.2, we have $\Phi(\{IP_j(A_{ii} + A_{ij} + A_{ji})\}) = \Phi(\{IP_jA_{ii}\}) + \Phi(\{IP_jA_{ij}\}) + \Phi(\{IP_jA_{ji}\})$. We can write that

$$\begin{aligned}
&\{\Phi(I)P_j(A_{ii} + A_{ij} + A_{ji})\} + \{I\Phi(P_j)(A_{ii} + A_{ij} + A_{ji})\} + \{IP_j\Phi(A_{ii} + A_{ij} + A_{ji})\} = \\
&\{\Phi(I)P_jA_{ii}\} + \{I\Phi(P_j)A_{ii}\} + \{IP_j\Phi(A_{ii})\} + \{\Phi(I)P_jA_{ij}\} + \{I\Phi(P_j)A_{ij}\} \\
&+ \{IP_j\Phi(A_{ij})\} + \{\Phi(I)P_jA_{ji}\} + \{I\Phi(P_j)A_{ji}\} + \{IP_j\Phi(A_{ji})\}.
\end{aligned}$$

So, we obtain $\{IP_j((\Phi(A_{ii}+A_{ij}+A_{ji})-\Phi(A_{ii})-\Phi(A_{ij})-\Phi(A_{ji})))\}=0$. That is $\{IP_jT\}=\frac{1}{2}(P_jT+TP_j)=0$, from which we get that $T_{jj}=T_{ij}=T_{ji}=0$. Using Lemma 2.1, we have $\Phi(\{I(P_i-P_j)(A_{ii}+A_{ij}+A_{ji})\})=\Phi(\{I(P_i-P_j)A_{ii}\})+\Phi(\{I(P_i-P_j)A_{ij}\})+\Phi(\{I(P_i-P_j)A_{ji}\})$. We can write that

$$\begin{aligned}&\{\Phi(I)(P_i-P_j)(A_{ii}+A_{ij}+A_{ji})\}+\{I\Phi(P_i-P_j)(A_{ii}+A_{ij}+A_{ji})\}\\&+\{I(P_i-P_j)\Phi(A_{ii}+A_{ij}+A_{ji})\}\\&=\{\Phi(I)(P_i-P_j)A_{ii}\}+\{I\Phi(P_i-P_j)A_{ii}\}+\{I(P_i-P_j)\Phi(A_{ii})\}\\&+\{\Phi(I)(P_i-P_j)A_{ij}\}+\{I\Phi(P_i-P_j)A_{ij}\}+\{I(P_i-P_j)\Phi(A_{ij})\}\\&+\{\Phi(I)(P_i-P_j)A_{ji}\}+\{I\Phi(P_i-P_j)A_{ji}\}+\{I(P_i-P_j)\Phi(A_{ji})\}.\end{aligned}$$

So, we obtain $\{I(P_i-P_j)(\Phi(A_{ii}+A_{ij}+A_{ji})\Phi(A_{ii})-\Phi(A_{ij})-\Phi(A_{ji}))\}=0$. That is $\{I(P_i-P_j)T\}=\frac{1}{2}((P_i-P_j)T+T(P_i-P_j))=0$, so $T_{ii}=0$. Then $T=0$. □

Lemma 2.4 *For every $A_{ii}\in\mathcal{A}_{ii},A_{ij}\in\mathcal{A}_{ij},A_{ji}\in\mathcal{A}_{ji},A_{jj}\in\mathcal{A}_{jj}$ we have $\Phi(A_{ii}+A_{ij}+A_{ji}+A_{jj})=\Phi(A_{ii})+\Phi(A_{ij})+\Phi(A_{ji})+\Phi(A_{jj})$ for $1\le i\ne j\le 2$.*

Proof: Let $T=\Phi(A_{ii}+A_{ij}+A_{ji}+A_{jj})-\Phi(A_{ii})-\Phi(A_{ij})-\Phi(A_{ji})-\Phi(A_{jj})$. Using Lemmas 2.1 and 2.3, we obtain $\Phi(\{IP_j(A_{ii}+A_{ij}+A_{ji}+A_{jj})\})=\Phi(\{IP_jA_{ii}\})+\Phi(\{IP_jA_{ij}\})+\Phi(\{IP_jA_{ji}\})+\Phi(\{IP_jA_{jj}\})$. From this, we can write that

$$\begin{aligned}&\{\Phi(I)P_j(A_{ii}+A_{ij}+A_{ji}+A_{jj})\}\\&+\{I\Phi(P_j)(A_{ii}+A_{ij}+A_{ji}+A_{jj})\}+\{IP_j\Phi(A_{ii}+A_{ij}+A_{ji}+A_{jj})\}\\&=\{\Phi(I)P_jA_{ii}\}+\{I\Phi(P_j)A_{ii}\}+\{IP_j\Phi(A_{ii})\}\\&+\{\Phi(I)P_jA_{ij}\}+\{I\Phi(P_j)A_{ij}\}+\{IP_j\Phi(A_{ij})\}\\&+\{\Phi(I)P_jA_{ji}\}+\{I\Phi(P_j)A_{ji}\}+\{IP_j\Phi(A_{ji})\}\\&+\{\Phi(I)P_jA_{jj}\}+\{I\Phi(P_j)A_{jj}\}+\{IP_j\Phi(A_{jj})\}.\end{aligned}$$

So, we get $\{IP_j(\Phi(A_{ii}+A_{ij}+A_{ji}+A_{jj})-\Phi(A_{ii})-\Phi(A_{ij})-\Phi(A_{ji})-\Phi(A_{jj}))\}=0$. That is $\{IP_jT\}=\frac{1}{2}(P_jT+TP_j)=0$, we have $T_{jj}=T_{ij}=T_{ji}=0$. Similarly, by putting P_i in place of P_j, we get $T_{ii}=0$. So, $T=0$. □

Lemma 2.5 *For every $A_{ii}\in\mathcal{A}_{ii},B_{ji}\in\mathcal{A}_{ji}$, we have $\Phi(A_{ii}+B_{ji})=\Phi(A_{ii})+\Phi(B_{ji})$ for $1\le i\ne j\le 2$.*

Proof: Let $T=\Phi(A_{ii}+B_{ji})-\Phi(A_{ii})-\Phi(B_{ji})$. We obtain $\Phi(\{P_jI(A_{ii}+B_{ji})\})=\Phi(\{P_jIA_{ii}\})+\Phi(\{P_jIB_{ji}\})$. From this, we can write that

$$\begin{aligned}\{\Phi(P_j)I(A_{ii}+B_{ji})\}+\{P_j\Phi(I)(A_{ii}+B_{ji})\}+\{P_jI\Phi(A_{ii}+B_{ji})\}&=\{\Phi(P_j)IA_{ii}\}+\{P_j\Phi(I)A_{ii}\}\\&+\{P_jI\Phi(A_{ii})\}+\{\Phi(P_j)IB_{ji}\}\\&+\{P_j\Phi(I)B_{ji}\}+\{P_jI\Phi(B_{ji})\}.\end{aligned}$$

So, we get $\{P_jI(\Phi(A_{ii}+B_{ji})-\Phi(A_{ii})-\Phi(B_{ji}))\}=0$ That is $\{P_jIT\}=0$, we have $T_{jj}=T_{ij}=T_{ji}=0$. Similarly, by putting (P_i-P_j) in place of P_j, we get $T_{ii}=0$. So, $T=0$. □

Lemma 2.6 *For every $A_{ij},B_{ij}\in\mathcal{A}_{ij}$, we have $\Phi(A_{ij}+B_{ij})=\Phi(A_{ij})+\Phi(B_{ij})$ for $1\le i\ne j\le 2$.*

Proof: Using Lemma 2.5, we obtain

$$\begin{aligned}\Phi(A_{ij}+B_{ij}) = \Phi(\{(P_i+2A_{ij})I(P_j+2B_{ij})\}) &= \{\Phi(P_i+2A_{ij})I(P_j+2B_{ij})\} \\ &+ \{(P_i+2A_{ij})\Phi(I)(P_j+2B_{ij})\} \\ &+ \{(P_i+2A_{ij})I\Phi(P_j+2B_{ij})\} \\ &= \{\Phi(P_i+2A_{ij})IP_j\} + \{\Phi(P_i+2A_{ij})I(2B_{ij})\} \\ &+ \{(P_i+2A_{ij})\Phi(I)P_j\} + \{(P_i+2A_{ij})\Phi(I)(2B_{ij})\} \\ &+ \{(P_i+2A_{ij})I\Phi(P_j)\} + \{(P_i+2A_{ij})I\Phi(2B_{ij})\} \\ &= \Phi(\{(P_i+2A_{ij})IP_j\}) + \Phi(\{(P_i+2A_{ij})I(2B_{ij})\}) \\ &= \Phi(A_{ij}) + \Phi(B_{ij}).\end{aligned}$$

The proof is complete. □

Lemma 2.7 *For every $A_{ii}, B_{ii} \in \mathcal{A}_{ii}$, we have $\Phi(A_{ii}+B_{ii}) = \Phi(A_{ii}) + \Phi(B_{ii})$ for $1 \le i \le 2$.*

Proof: Let $T = \Phi(A_{ii}+B_{ii}) - \Phi(A_{ii}) - \Phi(B_{ii})$. Using Lemma 2.1, we obtain $\Phi(\{IP_j(A_{ii}+B_{ii})\}) = \Phi(\{IP_jA_{ii}\}) + \Phi(\{IP_jB_{ii}\})$. From this, we can write that

$$\begin{aligned}&\{\Phi(I)P_j(A_{ii}+B_{ii})\} + \{I\Phi(P_j)(A_{ii}+B_{ii})\} + \{IP_j\Phi(A_{ii}+B_{ii})\} \\ &= \{\Phi(I)P_jA_{ii}\} + \{I\Phi(P_j)A_{ii}\} + \{IP_j\Phi(A_{ii})\} \\ &+ \{\Phi(I)P_jB_{ii}\} + \{I\Phi(P_j)B_{ii}\} + \{IP_j\Phi(B_{ii})\}.\end{aligned}$$

So, we get $\{I(P_j)(\Phi(A_{ii}+B_{ii}) - \Phi(A_{ii}) - \Phi(B_{ii}))\} = 0$. That is $\{IP_jT\} = \frac{1}{2}(P_jT + TP_j) = 0$, we have $T_{jj} = T_{ij} = T_{ji} = 0$. For every $C_{ij} \in \mathcal{A}_{ij}$, by Lemma 2.6 we obtain $\Phi(\{C_{ij}I(A_{ii}+B_{ii})\}) = \Phi(\{C_{ij}IA_{ii}\}) + \Phi(\{C_{ij}IB_{ii}\})$. We get the same as the previous lemmas $\{C_{ij}IT\} = 0$. Since $T_{jj} = 0$ and $\mathcal{A}$ is the prime, we have $T_{ii} = 0$. So, $T = 0$. □

Lemma 2.8 *If $\Phi(I)$ is self-adjoint then $\Phi(I) = 0$.*

Proof: We can write $\Phi(\{III\}) = \{\Phi(I)II\} + \{I\Phi(I)I\} + \{II\Phi(I)\}$. We get it easily $\Phi(I) + \Phi(I)^* = 0$. Since $\Phi(I)$ is self-adjoint, therefore $\Phi(I) = 0$. □

Lemma 2.9 *Φ is a $*$-derivation.*

Proof: For every $A \in \mathcal{A}$ we can write $\Phi(\{IAI\}) = \{\Phi(I)AI\} + \{I\Phi(A)I\} + \{IA\Phi(I)\}$. Since, $\Phi(I) = 0$, we have $\Phi(\frac{1}{2}(IA^*I + IA^*I)) = \frac{1}{2}(I\Phi(A)^*I + I\Phi(A)^*I)$. So, $\Phi(A^*) = \Phi(A)^*$. Now, considering $A = C$ and $B = I$, we have $\Phi(\{AIA\}) = \{\Phi(A)IA\} + \{A\Phi(I)A\} + \{AI\Phi(A)\}$. Since $\Phi(I) = 0$, we get $\Phi(A^2) = \Phi(A)A + A\Phi(A)$. We get that Φ is a Jordan derivation and since according to [7], every Jordan derivation of a unital prime algebra $\mathcal{A}$ with a nontrivial idempotent P is a derivation. Therefore, the proof is complete. The following example shows that the self-adjoint condition of $\Phi(I)$ in theorem is necessary. □

Example 2.1 *Let $\mathcal{A}$ be a prime $*$-algebra with unit I and nontrivial projection. Define a map $\Phi : \mathcal{A} \to \mathcal{A}$ where $\Phi(A) = iA$ for all $A \in \mathcal{A}$. In this mapping $\Phi(I)$ is not self-adjoint. It can be easily shown that the mapping Φ in (2.1) applies, but is not a derivation.*

Conflict of Interest The authors declare no conflict of interest.

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