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On Essential and Closed Fuzzy Ideals of a Semiring

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ABSTRACT: In this paper, existence of essential fuzzy ideal of a semiring S is shown and some of the properties of these ideals are investigated. The study of essentiality helps to develop the concepts like closed fuzzy ideal and relative fuzzy complement of S. Fuzzy ideal μ of S (or δ) is said to be closed ideal of S (or δ) if μ has no non-constant (proper) essential extension. Various results on closed fuzzy ideal of S are studied. Also relation between complementary summand and relative complement is established.

Key Words: Fuzzy ideal, essential fuzzy ideal, closed fuzzy ideal, relative complement.

Contents

1. Introduction

In the direction of remarkable extension of fuzzy set theory, fuzzy algebra has become an important area of research. The concept of fuzziness was first introduced mathematically by L. A. Zadeh [15] in 1965. Using this concept Rosenfeld [1] introduced the notion of fuzzy subgroup, which further inspired many researchers to carry this concept to various algebraic structures in fuzzy setting. Around 1982 Liu [13] commenced the study on fuzzy subring and fuzzy ideal of a ring. Consequently many researchers like Mukherjee and Sen [27], Abou-Zaid [22], Kumar et al. [20,21] and Malik et al. [3] have carried out spacious work on fuzzy ideals of ring. After generalizing this concept to group and ring many researchers applied this idea to module, near ring, semiring and so on. The concept of fuzzy submodule was first introduced by Negotia and Ralescu [2] and later on Pan [4] carried out the study of fuzzy finitely generated modules and fuzzy quotient modules. Subsequently this study motivated numerous researchers viz Zahedi [29], Sidki [5], Mukherjee, Sen and Roy [26], etc to further generalise the concept of module in fuzzy setting. Later on Kalita and Saikia [17,18,19] has studied the concepts of fuzzy essential ideal, fuzzy annihilator, fuzzy singular ideal of ring. Recently, Medhi, Saikia and Davvaz [28] has studied the concept of fuzzy $\mathcal{F}-$closure of a module. Likewise, fuzzy ideal in near-ring were discussed in [23,12].

H. S. Vandiver [7] in 1934 first introduced the notion of semiring, which is an algebraic structure with two associative binary operations where one distributes over the other. Semiring has many applications in the field of optimization theory, graph theory, fuzzy computation, automata theory etc. Many researchers have studied the concept of semiring in fuzzy setting. J. Ahsan, J. N. Mordeson and M. Shabir [8] were first to investigate the concept of fuzzy sets in semiring and studied various properties of this structure. Later on many other researchers like Ghosh [24], J. Zhan and Z. Tan [11] had studied the concept of fuzzy k-ideal of semiring and many other concepts like fuzzy soft Γ- semiring, fuzzy soft k-ideal over Γ- semiring, h-ideal, L-fuzzy ideals. The concept of intuitionistics fuzzy h-ideals of semiring were studied by Rahman and Saikia [25]. In this paper an effort is made to study fuzzy semiring in which the existence of essential fuzzy ideal in semiring is shown, then by using the concept of essential

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fuzzy ideal the notions of relative complement and closed ideal in semiring is fuzzified and some of their properties are studied.
Throughout this paper S is a semiring with zero unless otherwise mentioned.

2. Basic definitions and notations

In this section, some basic definitions and notations of fuzzy set theory, fuzzy algebra and semiring theory are presented which will be used in the subsequent sections. General references for fuzzy set theory, fuzzy algebra and semiring theory are presented in [6,9,8] respectively.

Definition 2.1 *A semiring is a non-empty set S on which operations of addition and multiplication have been defined such that the following conditions are satisfied:*
(i) $(S,+)$ is a commutative monoid with identity element 0;
(ii) $(S,.)$ is a monoid with identity element 1;
(iii) Multiplication distributes over addition from both sides;
(iv) $0s = 0 = s0$ for all $s \in S$.

Definition 2.2 *An ideal I of a semiring S is said to be an essential ideal of S if $I \cap J \neq 0$ for every nonzero ideal J of S. It is denoted by $I \leq_e S$.*

Definition 2.3 *An ideal I of a semiring S is said to be closed ideal of S if I has no non-constant (proper) essential extension.*

Lemma 2.1 *Let S be a semiring and $A, A^{'}, B, B^{'}$ be ideals of S. Then the following hold:*
(i) If $A \subseteq B \subseteq S$ then $A \subseteq_e S$ if and only if $A \subseteq_e B \subseteq_e S$.
(ii) If $A \subseteq_e B \subseteq S$ and $A^{'} \subseteq_e B^{'} \subseteq S$ then $A \cap A^{'} \subseteq_e B \cap B^{'}$.
(iii) If $f : B \to S$ is a homomorphism and $A \subseteq_e S$ then $f^{-1}(A) \subseteq_e B$.
(iv) If $\{A_\alpha\}$ is an independent family of ideals of S and if $A_\alpha \subseteq_e B_\alpha \subseteq S$ for each α, then $\{B_\alpha\}$ is an independent family and $\oplus A_\alpha \subseteq_e \oplus B_\alpha$.

Definition 2.4 *Let X be any non-empty set. A mapping $\mu : X \to [0,1]$ is called a fuzzy subset of X. The class of all fuzzy subset of X is denoted by $[0,1]^X$.*

Definition 2.5 *Let X be any non-empty set and μ be any fuzzy subset of X. Let $t \in [0,1]$. The set $\mu_t = \{x \in X|\ \mu(x) \geq t\}$ is called a level subset of μ. Clearly $\mu_t \subseteq \mu_s$ whenever $t > s$.*

Definition 2.6 *Let μ be a fuzzy subset of a nonempty set X. Then $\mu(X)$ or $Im(\mu) = \{\mu(x)\ |x \in X\}$ is called the image of μ and $\mu^* = \{x \in X|\ \mu(x) > 0\}$ is called the support of μ.*

Definition 2.7 *Let $\mu \in [0,1]^X$. Then a fuzzy point $x_t,\ x \in X,\ t \in (0,1]$ is defined as the fuzzy subset x_t of X such that $x_t(x) = t$ and $x_t(y) = 0$ for all $y \in X - \{x\}$. We write $x_t \in \mu$ if and only if $x \in \mu_t$.*
If $Y \subseteq X$, then χ_Y is a characteristic function on Y. Also χ_0 denotes characteristic function on 0.

Definition 2.8 *A fuzzy subset λ of a semiring S is called a fuzzy left (right) ideal of S if:*
(a) $\lambda(x+y) \geq \lambda(x) \wedge \lambda(y)$
(b) $\lambda(xy) \geq \lambda(x)(\lambda(xy) \geq \lambda(y)),\ \text{for all } x, y \in S$.
A fuzzy subset λ of a semiring S is called a fuzzy ideal if it is a fuzzy left ideal and a fuzzy right ideal of S.

The class of all fuzzy ideals of S is denoted by $FI(S)$.

Definition 2.9 *Let $\mu, \sigma \in [0,1]^S$. Then sum of μ and σ is defined as $(\mu+\sigma)(x) = \vee\{\mu(y) \wedge \sigma(z)|\ y, z \in S,\ x = y+z\}$.*

Definition 2.10 *Let $\mu, \sigma \in [0,1]^S$. Then product of μ and σ is defined as $(\sigma\mu)(x) = \vee\{\mu(y) \wedge \sigma(z) |\ y, z \in S,\ x = yz\}$.*

Lemma 2.2 *Let $\gamma_1, \gamma_2, \sigma \in FI(S)$ be such that $\gamma_1 \subseteq \sigma$ and $\gamma_2 \subseteq \sigma$. Then $\gamma_1 + \gamma_2 \subseteq \sigma$*

Note 1 *If $\mathscr{F}$ is the set of fuzzy ideal in S then $\mathscr{F}^*$ denotes the set of support of fuzzy ideal of S.*

Lemma 2.3 *If A is a fuzzy ideal of semiring S then χ_A is fuzzy ideal of S.*

3. Essential Fuzzy Ideal

In this section essential fuzzy ideal of a semiring S and some of its properties are studied. Essential fuzzy ideal plays an important role in various structure theorems. We begin this section with the definition of essential fuzzy ideal of a semiring S.

Definition 3.1 *Let μ and σ be two non-zero fuzzy ideals of a semiring S such that $\mu \subseteq \sigma$. Then μ is called essential in σ, denoted by $\mu \subseteq_e \sigma$ if for every non-zero fuzzy ideal $\gamma(\subseteq \sigma)$ of S, $\mu \cap \gamma \neq \chi_0$.*

Definition 3.2 *A fuzzy ideal μ of a semiring S is called an essential fuzzy ideal of S, denoted by $\mu \subseteq_e S$ if for every non-zero fuzzy ideal σ of S, $\mu \cap \sigma \neq \chi_0$, i.e. a fuzzy ideal of S is essential in S when S is considered as an ideal of itself.*

Theorem 3.1 *Let μ be an essential fuzzy ideal of S. Then μ_t is an essential ideal of S for some $t \neq 0$.*

Proof: Let μ be an essential fuzzy ideal of S. Let $A(\neq 0)$ be an ideal of S. This implies $\chi_A(\neq \chi_0)$ is a fuzzy ideal of S. We have, $\mu \subseteq_e S$, which implies that there exist $x(\neq 0) \in S$ such that $x_t \in \mu$ and $x_t \in \chi_A$ where $t \neq 0$. From this we have $\mu(x) \geq t$ and $\chi_A(x) \geq t > 0$ and this implies $x \in \mu_t$ and $x \in A$. Thus $\mu_t \subseteq_e S$.

Theorem 3.2 *Let μ be a fuzzy ideal of S. If $\mu_t \subseteq_e S$ for all $t \neq 0$ then $\mu \subseteq_e S$.*

Proof: Let $\sigma(\neq \chi_0)$ be a fuzzy ideal of S. This implies that $\exists\ x(\neq 0) \in S$ such that $\sigma(x) = s(\neq 0)$. Therefore $\sigma_s \neq \{0\}$, is an ideal of S. So $\mu_t \cap \sigma_s \neq \{0\}$. This implies $\exists\ y(\neq 0)$ such that $\mu(y) \geq t$ and $\sigma(y) \geq s$. Let $t_0 = \wedge\{s, t\}$. Then $\mu(y) \geq t_0$, $\sigma(y) \geq t_0$. So, $(\mu \cap \sigma)(y) \geq t_0$. $\mu \cap \sigma \neq \chi_0$. Thus $\mu \subseteq_e S$.

Theorem 3.3 *An ideal A of S is essential in S iff χ_A is essential fuzzy ideal of S.*

Proof: Let $A \subseteq_e S$. Let $\mu(\neq \chi_0)$ be fuzzy ideal of S. So $\exists\ x(\neq 0) \in S$ such that $\mu(x) = t(\neq 0)$. This implies $\mu_t \neq \{0\}$ is an ideal of S. Therefore $\mu_t \cap A \neq \{0\}$ which implies $\exists\ y(\neq \{0\}) \in \mu_t \cap A$ and so $\mu(y) \geq t,\ \chi_A(y) = 1$. From this we get $(\mu \cap \chi_A)(y) \geq t > 0$ and this directly implies that $\chi_A \subseteq_e S$.
Conversely, let B be non-zero ideal of S. This implies $\chi_B(\neq \chi_0)$ is a fuzzy ideal of S. So $(\chi_A \cap \chi_B)y \neq 0$, for some $y(\neq 0) \in S$ which implies $\chi_A(y) = 1,\ \chi_B(y) = 1$. From this it is clear that $A \subseteq_e S$.

Theorem 3.4 *If A and B are two ideals of S, then A is essential in B iff χ_A is essential in χ_B.*

Proof: We first suppose that $A \subseteq_e B$. Then we have, $A \subseteq B$. So $\chi_A \subseteq \chi_A$ and χ_A is fuzzy ideal of χ_B. Let γ be non-zero subset of χ_B. Then $\gamma(x) = t(\neq 0)$, for some $x(\neq 0) \in B$. So $\gamma_t(\neq \{0\})$ is an ideal of B. Now, since $A \subseteq_e B$, this implies for the ideal γ_t of B, $A \cap \gamma_t \neq \chi_0$. This implies $y(\neq 0) \in A \cap \gamma_t$. This implies $y \in A$ and $\gamma(y) \geq t > 0$. This implies $\chi_A \cap \gamma \neq \chi_0$. Thus $\chi_A \subseteq_e \chi_B$.
Conversely, let $\chi_A \subseteq_e \chi_B$. This implies $A \subseteq B$. Let $C \neq \{0\}$ be an ideal of S such that C is contained in B. Then for nonzero fuzzy ideal χ_C of χ_B, $\chi_A \cap \chi_C \neq \chi_0$. So, $(\chi_A \cap \chi_C)(y) > 0$, for some $y(\neq 0) \in S$. This implies $\chi_A(y) = 1$ and $\chi_C(y) = 1$. This implies $y \in A$ and $y \in C$ and so $y \in A \cap C$. From this $A \cap C \neq \{0\}$. Hence $A \subseteq_e B$.

Theorem 3.5 *Every non-zero fuzzy ideal of S is an essential ideal of itself.*

Theorem 3.6 *Let $\mu_1,\ \mu_2,\ \sigma_1,\ \sigma_2$ be fuzzy ideals of a semiring S. If $\mu_1 \subseteq_e \sigma_1$, $\mu_2 \subseteq_e \sigma_2$ then $\mu_1 \cap \mu_2 \subseteq_e \sigma_1 \cap \sigma_2$.*

Proof: Let $\gamma(\neq \chi_0)$ be any fuzzy ideal of S such that $\gamma \subseteq \sigma_1 \cap \sigma_2$. It is easy to see that $\mu_2 \cap \gamma$ is a non-zero fuzzy ideal of σ_1. Since, $\mu_1 \subseteq_e \sigma_1$, so $\mu_1 \cap (\mu_2 \cap \gamma) \neq \chi_0$. Which implies $(\mu_1 \cap \mu_2) \cap \gamma \neq \chi_0$. Hence $\mu_1 \cap \mu_2 \subseteq_e \sigma_1 \cap \sigma_2$.

Theorem 3.7 *Let μ_1 and μ_2 be two fuzzy ideals of a semiring S. If $\mu_1 \subseteq_e S$, $\mu_1 \subseteq_e S$ then $\mu_1 \cap \mu_2 \subseteq_e S$.*

Theorem 3.8 *Let μ and σ be two non-zero fuzzy ideal of S such that $\mu \subseteq_e \sigma$. Then for any non-zero fuzzy ideal γ of S, $\mu \cap \gamma \subseteq_e \sigma \cap \gamma$.*

Theorem 3.9 *Let μ, γ, σ be non-zero fuzzy ideals of S such that $\mu \subseteq \gamma \subseteq \sigma$ then $\mu \subseteq_e \sigma$ if and only if $\mu \subseteq_e \gamma \subseteq_e \sigma$.*

Proof: Let μ be essential in σ. Let $\lambda(\neq \chi_0)$ be any fuzzy ideal of S such that $\lambda \subseteq \gamma$. Then $\lambda \subseteq \sigma$. $\mu \subseteq_e \sigma$ implies $\mu \cap \lambda \neq \chi_0$. Therefore $\mu \subseteq_e \gamma$. Given that $\gamma \subseteq \sigma$. Let $\eta(\neq \chi_0)$ be a fuzzy ideal of S such that $\eta \subseteq \sigma$. Now, we have to show $\gamma \cap \eta \neq \chi_0$. Since $\mu \subseteq_e \sigma$, therefore $\mu \cap \eta \neq \chi_0$. And this implies $\gamma \cap \eta \neq \chi_0$, as $\mu \subseteq \gamma$. Hence, $\gamma \subseteq_e \sigma$.
Conversely, let ψ be any non-zero fuzzy ideal of S such that $\psi \subseteq \sigma$. Now, $\gamma \subseteq_e \sigma$ implies $\gamma \cap \psi \neq \chi_0$. So $\gamma \cap \psi$ is a fuzzy ideal of S with $\gamma \cap \psi \subseteq \gamma$. Now, since $\mu \subseteq_e \gamma$ so for some $y(\neq 0) \in S$, $s \neq 0$ we have $y_s \in \mu$ and so $y_s \in \gamma \cap \psi$. This implies $\mu(y) \geq s$, $\psi(y) \geq s$. This implies $\mu \cap \psi \neq \chi_0$. Thus $\mu \subseteq_e \sigma$.

Theorem 3.10 *If A is an essential ideal of S then the fuzzy subset μ of S defined by:*

$$\mu(x) = \begin{cases} 1, & \text{if } x = 0 \\ t, & \text{if } x \in A - \{0\}, t \neq 0 \\ 0, & \text{if } x \notin A \end{cases}$$

is an essential fuzzy ideal of S.

Proof: Let $A \subseteq_e S$ and $\sigma(\neq \chi_0)$ be any fuzzy ideal of S. Then $\exists\ x(\neq 0) \in S$ such that $\sigma(x) = s(\neq 0)$. So, $\sigma_s \neq \{0\}$, is an ideal of S. Since, $A \subseteq_e S$, therefore $A \cap \sigma_s \neq \{0\}$. This implies $\exists y(\neq 0) \in S$ such that $y \in A \cap \sigma_s$, which implies $y \in A, \sigma(y) \geq s$. Which again implies $\mu(y) = t, \sigma(y) \geq s$.
Let $t_0 = \wedge(t, s) \neq 0$. Then $\mu(y) \geq t_0$, $\sigma(y) \geq t_0$. Which gives $\mu \cap \sigma \neq \chi_0$.
Thus μ is an essential fuzzy ideal of S.

Example 3.1 *Let us consider the semiring $B(n, i) = \{0, 1, 2, \ldots, n-1\}$, where addition and multiplication are defined as follows: if $a, b \in B(n, i)$ then $a \oplus b = a + b$, if $a + b \leq n - 1$ and otherwise $a \oplus b$ is the unique element $c = a + b (\text{mod } n - i)$, the operation multiplication is defined similarly as addition. Considering $n = 12,\ i = 6$ gives $S = B(12, 6)$ is a semiring. In semiring $B(12, 6)$ the ideal $A = \{0, 2, 4, 6, 8, 10\}$ is an essential ideal of S and so the fuzzy set μ of S defined as above is an essential fuzzy ideal of S.*

Theorem 3.11 *A non-zero fuzzy ideal μ of S is essential in S iff μ is essential fuzzy ideal of χ_s.*

Theorem 3.12 *Let μ be a non-zero fuzzy ideal of S. Then $\mu \subseteq_e S$ iff $\mu^* \subseteq_e S$.*

Proof: Let $\mu \subseteq_e S$. Let A be any non-zero ideal of S. Then χ_A is non-zero fuzzy ideal of S. Since, $\mu \subseteq_e S$, so $\exists x(\neq 0)$ such that $\mu \cap \chi_A(x) > 0$ and this implies that $\mu(x) > 0,\ \chi_A(x) > 0$. Which implies $x \in \mu^*,\ x \in A$ and so $x \in \mu^* \cap A$. Thus $\mu^* \subseteq_e S$.
Conversely, let γ be any non-zero fuzzy ideal of S. Then γ^* is a non-zero ideal of S and this implies $\mu^* \cap \gamma^* \neq \{0\}$. So $\exists x \neq 0$ such that $\mu(x) > 0,\ \gamma(x) > 0$. This implies $\mu \cap \gamma \neq \chi_0$. Therefore $\mu \subseteq_e S$.

Theorem 3.13 *Let μ and σ be two non-zero fuzzy ideal of a semiring S. Then $\mu \subseteq_e \sigma$ iff $\mu^* \subseteq_e \sigma^*$.*

Proof: We consider $\mu \subseteq_e \sigma$. Since $\mu \neq \chi_0$, so $\exists(x \neq 0)$ such that $\mu(x) > 0$. This implies $\mu^* \neq \{0\}$. Now, $\mu \subseteq \sigma$ this implies $\mu^* \subseteq \sigma^*$. Let $(\{0\}) \neq A$ be any ideal of σ^*. We define $\gamma(x) = \begin{cases} \sigma(x), & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$. Then,$\gamma^* = A$. Also $A \neq \{0\}$, $\exists x(\neq 0) \in A$ such that, $\gamma(x) = \sigma(x) > 0$. So, $\gamma \neq \chi_0$. Also, $\gamma \subseteq \sigma$. Hence, $\mu \subseteq_e \sigma$. This implies $\mu \cap \gamma \neq \chi_0$. Therefore $\exists y(\neq 0)$ such that $\mu(y) > 0$ and $\gamma(y) > 0$. This implies $y \in \mu^*$ and $y \in \gamma^* = A$ and so $y \in \mu^* \cap A$. Therefore $\mu^* \subseteq_e \sigma^*$.
Conversely, let $\mu^* \subseteq_e \sigma^*$. We consider a fuzzy ideal $\gamma(\neq \chi_0)$ such that $\gamma \subseteq \sigma$. So $\gamma(x) > 0$. This implies $\gamma^* \neq \{0\}$. Since $\gamma \subseteq \sigma$, we have $\gamma^* \subseteq \sigma^*$. Given, $\mu^* \subseteq_e \sigma^*$. So for the ideal $\gamma^*(\neq 0)$ of σ^* we have $\gamma^* \cap \sigma^* \neq \{0\}$. This implies $\mu(z) > 0$ and $\gamma(z) > 0$ for some $z \neq 0$ and so $\mu \cap \gamma \neq \chi_0$. Thus $\mu \subseteq_e \sigma$.

Note 2 *The above theorem give an easy way to check the condition of fuzzy essentiality: to check $\mu \subseteq_e \sigma$ it is sufficient to check whether $\mu^* \subseteq_e \sigma^*$ or not.*

Example 3.2 *We consider the semiring $B(n, i)$ as defined in Example 3.1. We take $n = 12$, $i = 6$. Then $S = B(12, 6) = \{0, 1, 2, 3, \ldots, 11\}$ is a semiring. We consider the following two fuzzy ideals of $B(12, 6)$:*

$$\mu(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0.5, & \text{if } x \in \{4, 8\} \\ 0, & \text{if } x \notin \{0, 4, 8\} \end{cases}$$

$$\sigma(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0.5, & \text{if } x \in \{2, 4, 6, 8, 10\} \\ 0, & \text{if } x \notin \{0, 2, 4, 6, 8, 10\} \end{cases}$$

Here $\mu \subseteq \sigma$ but $\mu \nsubseteq_e \sigma$. As for the non-zero fuzzy ideal $\gamma \subseteq \sigma$, $\mu \cap \gamma = \chi_0$ where, $\gamma(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0.5, & \text{if } x = 6 \\ 0, & \text{if } x \notin \{0, 6\} \end{cases}$

Also $\gamma \subseteq_e S$ but $\mu \nsubseteq_e S$.

Example 3.3 *We define fuzzy subsets μ_n, $(n = 2, 3, 4, \ldots..)$ of N, the semiring of natural numbers by:*

$$\mu_n(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0.6, & \text{if } x \in nN - \{0\} \\ 0, & \text{if } x \notin nN \end{cases}$$

μ_n is fuzzy ideal of N. $\mu_n^* = nN \subseteq_e N$. So $\mu \subseteq_e N$. Also $\cap\mu_n$ is fuzzy ideal of N and $(\cap\mu_n)^* \subseteq \cap\mu_n^* = \cap nN = \{0\}$. This implies $(\cap\mu_n)^* = \{0\}$, which is not essential. So, $\cap\mu_n$ is not essential. From the above example we can conclude that theorem 3.6 is not true for infinite intersection.

Theorem 3.14 *If $f : S \to K$ is a semiring homomorphism and μ is an essential fuzzy ideal of K then $f^{-1}(\mu)$ is an essential fuzzy ideal of S.*

Proof: Let, $y \in f^{-1}(\mu^*)$ iff $f(y) \in \mu^*$ iff $\mu(f(y)) > 0$ iff $[f^{-1}(\mu)](y) > 0$ iff $y \in [f^{-1}(\mu)]^*$. Therefore $[f^{-1}(\mu)]^* = f^{-1}(\mu^*)$. We have $\mu \subseteq_e K$ iff $\mu^* \subseteq_e K$ [by theorem 3.12] iff $f^{-1}(\mu^*) \subseteq_e S$ [by lemma 2.1(iii)] iff $[f^{-1}(\mu)]^* \subseteq_e S$ iff $f^{-1}(\mu) \subseteq_e S$.

We now provide an example to show that the converse of above theorem is not true.

Example 3.4 *We consider the semiring $S = B(n, i) = \{0, 1, 2, \ldots, n-1\}$ where addition and multiplication are defined as in Example 1. Here we take $n = 6$, $i = 2$. So $S = B(6, 2) = \{0, 1, 2, 3, 4, 5\}$ is a semiring. Also we consider the mapping $f : S \to S$ defined by:*

$$f(x) = \begin{cases} 0, & \text{if } x = 2n,\ n = 0, 1, 2 \\ 2, & \text{if } x \text{ is odd} \end{cases}$$

It can be easily verified that f is semiring homomorphism. We consider fuzzy ideal μ defined by: $\mu(0) = 1$, $\mu(4) = 1/3$, $\mu(2) = 1/2$, $\mu(x) = 0$, otherwise. $f(\mu)(0) = \vee\{\mu(x) | x \in S, f(x) = 0\} = 1$. Similarly, $f(\mu)(1) = 0$, $f(\mu)(2) = 0$, $f(\mu)(3) = 0$, $f(\mu)(4) = 0$, $f(\mu)(5) = 0$. $\mu^* = \{0, 2, 4\} \subseteq_e S$ so, $\mu \subseteq_e S$. $f(\mu) = \chi_0 \nsubseteq_e S$.

Theorem 3.15 *Let P be a non-empty subset of S. Then $\langle \alpha_P \rangle = \alpha_{\langle P \rangle}$ where $\langle P \rangle$ is the ideal of S generated by P and $\alpha \in [0, 1]$.*

Proof: As in [9] the result can be analogously obtained in semiring.

Corollary 3.1 *Let $x \in S$ and $\alpha \in [0, 1]$. Then $\langle x_\alpha \rangle = \alpha_{\langle x \rangle}$.*

Definition 3.3 *Let $s \in S$, $s \neq 0$ and γ be essential fuzzy ideal of S. We define a fuzzy set $\sigma \in [0, 1]^S$ by: $\sigma = \cup\{\delta \mid \delta \in [0, 1]^S, \delta a_p \subseteq \gamma\}$.*

The above definition 3.3 can be equivalently constructed as follows:

Lemma 3.1 $\sigma = \cup\{s_\alpha \mid s \in S, \alpha \in [0, 1], s_\alpha a_p \subseteq \gamma\}$.

Proof: $\{s_\alpha \mid s \in S, \alpha \in [0, 1]\} \subseteq [0, 1]^S$.
Therefore, $\{s_\alpha \mid s \in S, \alpha \in [0, 1], s_\alpha a_p \subseteq \gamma\} \subseteq \{\delta | \delta \in [0, 1]^S, \delta a_p \subseteq \gamma\}$
This implies,
$\cup\{s_\alpha \mid s \in S, \alpha \in [0, 1], s_\alpha a_p \subseteq \gamma\} \subseteq \cup\{\delta | \delta \in [0, 1]^S, \delta a_p \subseteq \gamma\} = \sigma$
Let $\delta \in [0, 1]^S$ such that $\delta a_p \subseteq \gamma$.
Let, $s \in S$ and $\delta(s) = \alpha$.
Now,

$$\begin{aligned}
(s_\alpha a_p)(x) &= \vee\{s_\alpha(z) \wedge a_p(y) | z \in S, y \in S, zy = x\} \\
&= \vee\{\delta(s) \wedge a_p(y) | y \in S, sy = x\} \\
&\leq \vee\{\delta(z) \wedge a_p(y) | z \in S, y \in S, zy = x\} \\
&= (\delta a_p)(x) \\
&\leq \gamma(x) \text{ for all } x \in S \\
\text{Thus, } & s_\alpha a_p \subseteq \gamma \\
\text{So, } \quad & \sigma \subseteq \cup\{s_\alpha \mid s \in S, \alpha \in [0, 1], s_\alpha a_p \subseteq \gamma\} \\
& \sigma = \cup\{s_\alpha \mid s \in S, \alpha \in [0, 1], s_\alpha a_p \subseteq \gamma\}.
\end{aligned}$$

Lemma 3.2 $\sigma = \cup\{\delta | \ \delta \in FI(S), \delta a_p \subseteq \gamma\}$.

Proof: Clearly, $\{\delta | \ \delta \in FI(S), \delta a_p \subseteq \gamma\} \subseteq \{\delta | \delta \in [0, 1]^S, \delta a_p \subseteq \gamma\} = \sigma$
Let, $s \in S, \alpha \in [0, 1]$ and $s_\alpha a_p \subseteq \gamma$
Let $\delta = \langle s_\alpha \rangle$

Now, $\langle s_\alpha\rangle a_p = (\chi_0 \cup \alpha_{\langle s\rangle})a_p = \chi_0 a_p \cup \alpha_{\langle s\rangle} a_p = \chi_0 a_p \cup \alpha_{\langle s\rangle} a_p \subseteq \chi_0 \cup \alpha_{\langle s\rangle} a_p$ Again,

$$\begin{aligned}
(\alpha_{\langle s\rangle} a_p)(x) &= \vee\{\alpha_{\langle s\rangle}(z) \wedge a_p(y)|\ z \in S, y \in S, zy = x\} \\
&= \vee\{\alpha \wedge a_p(y)|\ z \in \langle s\rangle, y \in S, zy = x\} \\
&\leq \vee\{(s_\alpha a_p)(sy)|\ t \in S, y \in S, t(sy) = x\} \\
&\leq \vee\{(\gamma(sy)|\ t \in S, y \in S, t(sy) = x\} \\
&\leq \vee\{(\gamma(t(sy))|\ t \in S, y \in S, t(sy) = x\} \\
&= \gamma(x) \text{ for all } x \in S
\end{aligned}$$

$$\text{Therefore,} \quad \alpha_{\langle s\rangle} a_p \subseteq \gamma$$

$$\text{So,} \quad \langle s_\alpha\rangle a_p \subseteq \chi_0 \cup \gamma = \gamma$$

$$\text{Hence, } \{\delta|\ \delta \in FI(S),\ \delta a_p \subseteq \gamma\} \supseteq \{s_\alpha | s \in S, \alpha \in [0,1], s_\alpha a_p \subseteq \gamma\} = \sigma.$$

$$\text{Therefore,} \quad \sigma = \cup\{\delta|\ \delta \in FI(S),\ \delta a_p \subseteq \gamma\}.$$

Lemma 3.3 *$\sigma \in FI(S)$ and $\sigma(0) = 1$.*

Proof: Clearly, $\chi_0 a_p = \chi_0 \subseteq \gamma$. So, $\chi_0 \subseteq \sigma$.
Now,

$$\begin{aligned}
\sigma(s_1) \wedge \sigma(s_2) &= (\vee\{\delta_1(s_1)|\ \delta_1 \in FI(S), \delta_1 a_p \subseteq \gamma\}) \wedge (\vee\{\delta_2(s_2)|\ \delta_2 \in FI(S), \delta_2 a_p \subseteq \gamma\}) \\
&= (\vee\{\delta_1(s_1) \wedge \delta_2(s_2)|\ \delta_1, \delta_2 \in FI(S),\ \delta_1 a_p \subseteq \gamma,\ \delta_2 a_p \subseteq \gamma) \\
&\leq (\vee\{(\delta_1 + \delta_2)(s_1) \wedge (\delta_1 + \delta_2)(s_2)|\ \delta_1, \delta_2 \in FI(S), \delta_1 a_p \subseteq \gamma, \delta_2 a_p \subseteq \gamma) \\
&\leq \vee\{(\delta_1 + \delta_2)(s_1 + s_2)|\ \delta_1, \delta_2 \in FI(S),\ (\delta_1 + \delta_2)a_p \subseteq \delta_1 a_p + \delta_2 a_p \subseteq \gamma + \gamma = \gamma\} \\
&\leq \vee\{\delta(s_1 + s_2)|\ \delta \in FI(S), \delta a_p \subseteq \gamma\} \\
&= \sigma(s_1 + s_2) \text{ for all } s_1, s_2 \in S
\end{aligned}$$

Now,

$$\begin{aligned}
\sigma(zs) &= \vee\{\delta(zs)|\ \delta \in FI(S),\ \delta a_p \subseteq \gamma\} \\
&\geq \vee\{\delta(s)|\ \delta \in FI(S),\ \delta a_p \subseteq \gamma\} \\
&= \delta(s) \text{ for all } z, s \in S.
\end{aligned}$$

$$\text{Therefore,} \quad \sigma \in FI(S).$$

Theorem 3.16 *If γ is an essential fuzzy ideal of semiring S and $a \neq \{0\}$ then there exists an essential fuzzy ideal σ such that $a_p\sigma \neq \chi_0$, where $p \in (0,1]$.*

Proof: Let, $\sigma = \cup\{s_\alpha\ | s \in S, \alpha \in [0,1], s_\alpha a_p \subseteq \gamma\}$
By lemma [3.3] $\sigma \in FI(S)$.
Since γ is essential fuzzy ideal, therefore γ^* is also an essential ideal of S, so for $a \neq \{0\}$ and γ^*, there exist an essential ideal $L = \{s \in S | sa \in \gamma^*\}$ such that $aL \neq 0$ and $aL \subseteq \gamma^*$.
Now,

$$\begin{aligned}
(s_\alpha a_p)(m) &= \vee\{s_\alpha(x) \wedge a_p(y) | x \in S, y \in S, xy = m\} \\
&= \begin{cases} 0, & \text{if } m \neq sa \\ \alpha \wedge p, & \text{if } m = sa \end{cases}
\end{aligned}$$

$$\text{Therefore,} \quad (s_\alpha a_p) = (sa)_{\alpha\wedge p}$$

$$\text{So,} \quad \sigma = \cup\{s_\alpha\ | s \in S, \alpha \in [0,1], (sa)_\alpha \in \gamma\}.$$

$x \in \sigma^*$ implies that there exist $\alpha \in (0,1]$ such that $(xa)_{\alpha\wedge p} \in \gamma$ which gives $\gamma(xa) \geq \alpha \wedge p > 0$ i.e $xa \in \gamma^*$ i.e $x \in L$. Therefore, $\sigma^* \subseteq L$.

Conversly, $x \in L$ implies $xa \in \gamma^*$.

$$\begin{aligned}
&\text{i.e } \gamma(xa) = \alpha, \text{ say, where } \alpha \neq 0 \\
&\text{i.e } \gamma(xa) \geq \alpha \wedge p \\
&\text{i.e } (xa)_{\alpha \wedge p} \in \gamma \\
&\text{i.e } x_\alpha \in \{x_\alpha | x \in S, \alpha \in [0,1], (xa)_{\alpha \wedge p} \in \gamma\} \\
&\text{i.e } x_\alpha \in \sigma \\
&\text{i.e } \sigma(x) \geq \alpha > 0 \\
&\text{i.e } x \in \sigma^* \\
&\text{Therefore, } L \subseteq \sigma^* \\
&\text{So, } \sigma^* = L
\end{aligned}$$

Therefore, from above $a\sigma^* \neq \{0\}$, $a\sigma^* \subseteq \gamma^*$ and σ^* is essential and hence σ is essential.
From definition of σ, $\sigma a_p \subseteq \gamma$
Since, $a\sigma^* \neq \{0\}$, there exist $s \in \sigma^*$ such that $sa \neq \{0\}$.
Now, $(\sigma a_p)(sa) \geq \sigma(s) \wedge a_p(a) > 0$.
So, $\sigma a_p \neq \chi_0$.

In the next part of this section we define relative complement for a fuzzy ideal of a semiring and discuss its properties.

Definition 3.4 *Let μ and δ be fuzzy ideals of a semiring S. A relative complement for μ in S is any ideal σ of δ which is maximal with respect to the property $\mu \cap \sigma = \chi_0$.*

Remark 3.1 *We can show characteristic function χ_B is maximal with respect to the property $\mu \cap \chi_B = \chi_0$. If $\mu \cap \gamma = \chi_0$ then γ can be enlarged to relative complement of μ: $\mu \cap \gamma = \chi_0 \Rightarrow \{(\mu \cap \gamma)\}^* = \{0\}$ $\Rightarrow \mu^* \cap \gamma^* = \{0\}$. So,there exists an ideal B maximal w.r.t the property $\mu^* \cap B = \{0\}$. Thus χ_B is a fuzzy ideal. Also $\gamma^* \subseteq B \Rightarrow \gamma \subseteq \chi_B$. Let $x \in \{(\mu \cap \chi_B)\}^* = \mu^* \cap B = \{0\}$. This implies that $x = 0$. So, $\mu \cap \chi_B = \chi_0$. Let σ be a fuzzy ideal such that $\chi_B \subseteq \sigma$ and $\mu \cap \sigma = \chi_0$. Then $B \subseteq \sigma^*$ and $\mu^* \cap \sigma^* = \{0\}$. By maximality of B, $\sigma^* = B$. Now, $\sigma(x) = 0$ this implies $\sigma(x) \leq \chi_B(x)$ and for $\sigma(x) \neq 0$ this implies $x \in \sigma^* = B$ this implies $\chi_B(x) = 1$. So, $\sigma(x) \leq \chi_B(x)$. Hence, $\sigma \subseteq \chi_B$. Consequently $\sigma = \chi_B$. So χ_B is maximal with respect to the property $\mu \cap \chi_B = \chi_0$.*

4. Closed fuzzy ideal

The fuzzification of essential ideal of semiring motivates one to fuzzify the concept of closed ideal, relative complement in semiring. In this section closed ideal of S is fuzzyfied. We start with the definition of closed fuzzy ideal of S.

Definition 4.1 *A fuzzy ideal μ of S(or δ) is said to be closed ideal of S(or δ) if μ has no non-constant (proper) essential extension i.e if the only solution of the relation $\mu \subseteq_e \sigma(\subseteq \delta)$ is $\mu = \sigma$.*

Example 4.1 *Let us consider a semiring $B(n,i) = \{0,1,2,\ldots,n-1\}$, where addition and multiplication are defined as in Example 3.1. Considering $n = 12$, $i = 6$ gives $S = B(12,6)$ is a semiring. Here $A = \{0,2,4,6,8,10\}$ is an essential ideal of S. It can be easily seen from Example 3.1 that the fuzzy ideal*

$$\sigma = \begin{cases} 1, & \text{if } x = 0 \\ 0.5, & \text{if } x \in \{2,4,6,8,10\} \\ 0, & \text{if } x \notin \{0,2,4,6,8,10\} \end{cases}$$

is essential in S and since there is only one such essential fuzzy ideal in S so it is a closed fuzzy ideal of S.

The following results are obvious:

Lemma 4.1 *χ_0 and χ_S are always closed ideal of S.*

Lemma 4.2 *χ_0 and δ are always closed ideal of δ.*

Lemma 4.3 *Let μ and σ be two non-zero fuzzy ideals of a semiring S. Then μ is closed in σ iff μ_t is closed in σ_t.*

Lemma 4.4 *Let μ and σ be two non-zero fuzzy ideals of a semiring S. Then μ is closed in σ iff μ^* is closed in σ^*.*

Theorem 4.1 *Let δ be a fuzzy ideal of semiring S. Every direct summand of δ is a closed ideal of δ.*

Proof: Let $\delta = \mu \oplus \sigma$
To show μ is closed ideal of δ.
Let, if possible, $\mu \subseteq_e \mu' \subseteq \delta$.
Then, $(\mu' \cap \sigma) \cap \mu = \mu' \cap (\mu \cap \sigma) = \mu' \cap \chi_0 = \chi_0$.
$\Rightarrow \mu' \cap \sigma = \chi_0$ $\quad [\because \mu \subseteq_e \mu', \mu' \cap \sigma \subseteq \mu']$
Now, $\delta = \mu \oplus \sigma \subseteq \mu' \oplus \sigma \subseteq \delta$
$\Rightarrow \delta = \mu' \oplus \sigma$
So, $\mu \oplus \sigma = \mu' \oplus \sigma$. Again,

$$\begin{aligned}
&\mu \cap \sigma = \chi_0 \\
\Rightarrow\ &(\mu \cap \sigma)_t = (\chi_0)_t, t \neq 0. \\
\Rightarrow\ &\mu_t \cap \sigma_t = \{0\} \\
&\text{Let} \quad a \in (\mu \oplus \sigma)_t \\
\Rightarrow\ &(\mu \oplus \sigma)(a) \geq t \\
\Rightarrow\ &\mu(b) \wedge \sigma(c) \geq t, \text{where } b \in \mu^*,\ c \in \sigma^*, a = b + c \\
\Rightarrow\ &\mu(b) \geq t, \sigma(c) \geq t \\
\Rightarrow\ &b \in \mu_t, c \in \sigma_t \\
\Rightarrow\ &a = b + c \in \mu_t \oplus \sigma_t
\end{aligned}$$

Also, $a \in \mu_t \oplus \sigma_t \Rightarrow b' \in \mu_t,\ c' \in \sigma_t$, where $a = b' + c'$
$\Rightarrow \mu(b') \geq t,\ \sigma(c') \geq t \Rightarrow \mu(b') \wedge \sigma(c') \geq t$, where $a = b' + c'$
$\Rightarrow (\mu \oplus \sigma)(a) \geq t$
$\Rightarrow a \in (\mu \oplus \sigma)_t$
Therefore, $(\mu \oplus \sigma)_t = \mu_t \oplus \sigma_t$, where $t \neq 0$.
Hence, $\mu \oplus \sigma = \mu' \oplus \sigma$
$\Rightarrow \mu_t \oplus \sigma_t = \mu'_t \oplus \sigma_t$, where $t \neq 0$.
Let, $x_t \in \mu', t \neq 0$. Then $x \in \mu'_t$.
Now, $x + y \in \mu'_t \oplus \sigma_t = \mu_t \oplus \sigma_t$, where $y \in \sigma_t$
$\Rightarrow x \in \mu_t$
$\Rightarrow x_t \in \mu$
Now, $\mu'(x) = p(\neq 0)$
$\Rightarrow x_p \in \mu$
$\Rightarrow \mu(x) \geq p = \mu'(x) \geq \mu(x)$
$\Rightarrow \mu(x) = p$
Again, $\mu \subseteq \mu'$
$\Rightarrow (\mu'^*)'' \subseteq (\mu^*)''$, where $(\mu^*)'' = \{x \in S \mid \mu(x) = 0\}$
If, $\mu'(y) = 0$ then $y \in (\mu'^*)''$. So, $y \in (\mu^*)''$. Therefore, $\mu(y) = 0$. So, $\mu' = \mu$. Hence μ is a closed ideal of $\mu \oplus \sigma$.

Theorem 4.2 *If μ is a direct summand of δ, say $\delta = \mu \oplus \sigma$ then complementary summand of σ is a relative complement for μ in δ.*

Proof: Let, $\sigma \subseteq \sigma^{'} \subseteq \delta$ be such that $\mu \cap \sigma^{'} = \chi_0$
Now, $\delta = \mu \oplus \sigma \subseteq \mu \oplus \sigma^{'} \subseteq \delta$.
$\Rightarrow \delta = \mu \oplus \sigma = \mu \oplus \sigma^{'}$
$\Rightarrow \delta^* = \mu^* \oplus \sigma^* = \mu^* \oplus (\sigma^{'})^*$
Therefore, σ^* is a relative complement of μ^* in δ^* and $\sigma^* \subseteq (\sigma^{'})^* \subseteq \delta^*$
So, $(\sigma^{'})^* = \sigma^*$ or $(\sigma^{'})^* = \delta^*$
If $(\sigma^{'})^* = \sigma^*$ then $\sigma^* \subseteq_e (\sigma^{'})^*$ and hence $\sigma \subseteq_e \sigma^{'} \subseteq \delta$.
Since σ is closed ideal of δ. So, $\sigma^{'} = \sigma$ or $\sigma^{'} = \delta$.
Now, $\sigma^{'} = \delta$ and $\mu \cap \sigma^{'} = \chi_0 \Rightarrow \mu = \chi_0$.
Therefore,

$$\begin{aligned} &\chi_0 + \sigma = \delta \\ \Rightarrow\ &\delta(x) = \chi_0(y) \wedge \sigma(z),\ y \in \chi_0^*,\ z \in \sigma^*, x = y + z \\ &\quad\ \ = \chi_0(0) \wedge \sigma(x) \\ &\quad\ \ = \sigma(x) \end{aligned}$$

Therefore, $\sigma = \delta = \sigma^{'}$
If $(\sigma^{'})^* = \delta^*$, then $\mu \cap \sigma^{'} = \chi_0$, which implies $\mu^* \cap (\sigma^{'})^* = \{0\}$, which implies $\mu^* = \{0\}$, this will imply $\mu = \chi_0$ and so $\delta = \sigma$, as above and hence $\delta = \sigma = \sigma^{'}$.
Therefore σ is a relative complement for μ in δ.

Theorem 4.3 *If $\mu \subseteq \delta$ and μ, σ, δ are fuzzy ideals of S and*
(a) μ is fuzzy closed ideal of δ
(b) μ is a complement for some $\sigma \subseteq \delta$
(c) If σ is any complement of μ in δ then μ is a complement of σ in δ.
Then (c) $\Rightarrow$ (b) $\Rightarrow$ (a).

Proof: $(c) \Rightarrow (b)$ Let μ has complement σ i.e σ is relative complement of μ in δ. Then by (c) μ is relative complement of σ in δ.
$(b) \Rightarrow (a)$ Let, $\mu \subseteq_e \mu^{'} \subseteq \delta$
Now, $(\mu^{'} \cap \sigma) \cap \mu = \sigma \cap (\mu^{'} \cap \mu) = \sigma \cap \mu = \chi_0$.
As, $\mu \subseteq_e \mu^{'}$ and $\mu^{'} \cap \sigma \subseteq \mu^{'}$, this implies $\mu^{'} \cap \sigma = \chi_0$
By maximality of μ we have $\mu = \mu^{'}$. Therefore μ is closed.

Theorem 4.4 *Let μ, δ be fuzzy ideal of S and $\mu \subseteq \delta$ then δ has a closed ideal σ such that $\mu \subseteq_e \sigma$.*

Proof Let $\mu \subseteq \delta$ and let k is an ideal of a semiring S such that $\mathscr{F} = \{\chi_K |\ \mu \subseteq_e \chi_K\}$.
Since, $\mu \subseteq_e \chi_K$ this implies $\mu^* \subseteq_e K$. Therefore, $\mathscr{F}^* = \{K | \mu^* \subseteq_e K\}$
Clearly, $\mathscr{F}^* \neq \{0\}$.
Let, $K_i \subseteq K_{i+1} \subseteq K_{i+2} \ldots$ be arbitrary chain of $\mathscr{F}^*$.
Claim $\cup K_i \in \mathscr{F}^*$. Let $\delta^* \subseteq \cup K_i$ where $\delta^* = \cup \delta_i^*$ and $\delta_i^* \subseteq K_i$.
If not then $\mu^* \cap (\cup \delta_i^*) = \{0\}$, which implies $\mu^* = \{0\}$, a contradiction.
Thus $\cup K_i \in \mathscr{F}^*$. Hence $\mathscr{F}^*$ has maximal element by Zorn's lemma and let it be σ^*. Thus $\mu^* \subseteq_e \sigma^*$, which implies $\mu \subseteq_e \sigma$ $\quad [\because \mu \subseteq_e \sigma \iff \mu^* \subseteq_e \sigma^*]$
Next we claim σ^* is closed. If not there exist σ_1^* such that $\sigma^* \subseteq_e \sigma_1^* \subseteq \delta^*$ this implies $\sigma_1^* \in \mathscr{F}^*$, a contradiction to maximality of σ^*. So σ^* is closed and hence σ is closed.

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