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Lightlike sweeping surface and singularities in Minkowski 3–space $\mathbb{E}_1^3$

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ABSTRACT: In this work, we give the parametric equation of a lightlike sweeping surface in Minkowski 3–Space $\mathbb{E}_1^3$. We introduce a new geometric invariant to explain the geometric possessions and local singularities of this lightlike sweeping surface. We extract the sufficient and necessary conditions for this lightlike sweeping surface to be a spacelike/timelike developable ruled surface. Afterwards, we take advantage of singularity theory to give the categorization of singularities of this spacelike/timelike developable surface. Finally, we give several representative examples to offer the implementations of the theoretical results.

Key Words: Rotation minimizing frame, Local singularities, Function germs.

Contents

1. Introduction

Singularity theory of curves and surfaces is a practical area of examine in several parts of physics and mathematics. In the view of differential geometry, curves and surfaces are completed by maps with one variable and two variables, respectively. In recent years, singularity theory for curves and surfaces has become a essential instrument for various enjoyable fields such as computer vision and medical imaging (see e.g. [1-5]).

As we know, a sweeping surface is the surface trace by the movement of a planar curve (the generatrix or profile curve) while the plane is moved through space in such method that the movement of the plane is constantly in the orientation of the normal to the plane. Sweeping is a very powerful, significant, and diffuse method in geometric modelling. The primary idea is to pick some geometrical object (generator), which is then swept through a spine curve (trajectory) in the space. The outcome of such evolution, be composed of intrinsic shape deformation and movement through space, is a sweep object. The sweep object style is specified by the choice of the generator and the trajectory. Thus, sweeping a curve on the other curve traces a sweeping surface [6]. There are several denominations of sweeping surface that we are familiar with, such as pipe surface, tubular surface, string, and canal surface For example, a survey of the base geometric countenances of canal surfaces has been studied by Xu et al. in [7]. Furthermore, the authors offered sufficient conditions for canal surfaces without local self-intersection. Moreover, they obtained a straightforward term for the area and Gaussian curvature of canal surfaces. Izumiya et al. [8] examined several geometric possessions and singularities of circular surfaces by comparing with conforming aspects of ruled surfaces. JS. Ro and DW Yoon examined the study of a tube in Euclidean 3-space, fulfilling some equation in expressions of the Gaussian curvature, the mean curvature, and the second Gaussian curvature in [9]. L. Cui et al. in [10] initiated circular surfaces defined as one-parameter family of circles in the Euclidean 3-space, they gained the kinematic-geometry such surfaces and also

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discussed the geometric explanations from the viewpoint of mechanisms and robots operate. RA. Abdel-Baky, and Y. Ünlütürk in [11] defined a complete system of invariants to define spacelike circular surfaces with fixed radius in Minkowski 3-space $\mathbb{E}_1^3$. Furthermore, the authors simplified the study of spacelike circular surfaces into two curves: the Lorentzian spherical indicatrix of the unit normals of circle planes and the spacelike spine curve. Also, the authors derived spacelike roller coaster surfaces as a special class of spacelike circular surfaces. Abdel-Baky et al. in [12] examined some corresponding properties of timelike circular surfaces with classical ruled surfaces. In [13]. MF. Naghi and RA. Abdel-Baky considered a timelike sweeping surface with rotation minimizing frames in Minkowski 3-Space $\mathbb{E}_1^3$. They introduced a new geometric "invariant", which demonstrates the geometric properties and local singularities of the surface. Furthermore, the authors gave the sufficient and necessary conditions for this surface to be a developable ruled surface. Finally, the singularities of these ruled surfaces are investigated. Yanlin et al. [14], constructed equations of timelike circular surfaces and timelike roller coaster surfaces by using a complete system of invariants.

One of the most appropriate methods to analyzing curves and surfaces in differential geometry, Serret–Frenet frame, but not unique, there are also the other frame fields such as rotation minimizing frame (RMF) or Bishop frame [15]. Some applications of the Bishop frame can be found in [16-18]. Corresponding to the Bishop frame in Euclidean space, there exists a Minkowski version's frame that is called a Minkowski Bishop frame as applied to Minkowski geometry. When we investigate a space curve, it is more convenient for us to use the Minkowski Bishop frame along the curve as the basic tool than the Serret–Frenet frame type frame in Lorentzian space. There are several papers about Minkowski Bishop frame, for example [19-21].

However, to the best of the authors' knowledge, no literature exists regarding the singularities and lightlike sweeping surfaces according to the Minkowski Bishop frame. Thus, the present work hopes to serve such a need, and it is inspired by the works [13, 14]. In this paper, we consider the Bishop frame along a unit speed spacelike curve and improve the local differential geometry of lightlike sweeping surface. Applying the unfolding theory, we classify the generic properties, and present new invariant related to the singularities of this surface. It is demonstrate that the generic singularities are cuspidal edge and swallowtail, and the types of these singularities can be characterize by this invariant, respectively. In addition, we reveal some enjoyable relations among the contacts of original curve with osculating lightcone. Finally, examples are illustrated to explain the applications of the theoretical results.

2. Preliminaries

We introduce in this section some basic notions on Minkowski 3-space. For basic concepts and properties, see [22, 23].

Let $\mathbb{R}^3 = \{(a_1, a_2, a_3) \mid, a_i \in \mathbb{R}$ (i=1, 2, 3)$\}$ be a 3-dimensional Cartesian space. For any $\mathbf{a}=(a_1, a_2, a_3)$, and $\mathbf{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$, the pseudo scalar product of $\mathbf{a}$, and $\mathbf{b}$ is defined by

$$< \mathbf{a}, \mathbf{b} > = a_1 b_1 + a_2 b_2 - a_3 b_3. \tag{2.1}$$

We call $(\mathbb{R}^3, <,>)$ Minkowski 3-space. We write $\mathbb{E}_1^3$ instead of $(\mathbb{R}^3, <,>)$. We say that a non-zero vector $\mathbf{a} \in \mathbb{E}_1^3$ is spacelike, lightlike or timelike if $< \mathbf{a}, \mathbf{a} > > 0$, $< \mathbf{a}, \mathbf{a} > = 0$ or $< \mathbf{a}, \mathbf{a} > < 0$ respectively. The norm of the vector $\mathbf{a} \in \mathbb{E}_1^3$ is defined to be $\|\mathbf{a}\| = \sqrt{|< \mathbf{a}, \mathbf{a} >|}$. For a fixed point $\mathbf{p} \in \mathbb{E}_1^3$, we define

$$\mathbb{LC}_p^* = \{\mathbf{a} \in \mathbb{E}_1^3 \mid < \mathbf{a} - \mathbf{p}, \mathbf{a} - \mathbf{p} > = 0\} - \{\mathbf{p}\}, \tag{2.2}$$

and we call it the (open) lightcone at the vertex $\mathbf{p}$. When $\mathbf{p} = \mathbf{0}$, and $r = 1$, we simply denote $\mathbb{LC}_0^*$. For any two vectors $\mathbf{a}$, $\mathbf{c} \in \mathbb{E}_1^3$, we define a vector $\mathbf{a} \times \mathbf{c}$ by

$$\mathbf{a} \times \mathbf{c} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & -\mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \left((a_2 c_3 - a_3 c_2), (a_3 c_1 - a_1 c_3), -(a_1 c_2 - a_2 c_1)\right), \tag{2.3}$$

where $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ is the canonical basis of $\mathbb{E}_1^3$. We can easily check that

$$\det(\mathbf{a}, \mathbf{c}, \mathbf{b}) = < \mathbf{a} \times \mathbf{c}, \mathbf{b} >, \tag{2.4}$$

so that $\mathbf{a}\times\mathbf{c}$ is pseudo orthogonal to any $\mathbf{b}=(b_1,b_2,b_3)\in\mathbb{E}_1^3$.

Definition 2.1. i) *Spacelike angle*: Let $\mathbf{a}$ and $\mathbf{c}$ be spacelike vectors in $\mathbb{E}_1^3$ that define a spacelike vector subspace; then we have $|<\mathbf{a},\mathbf{c}>|\leq\|\mathbf{a}\|\,\|\mathbf{c}\|$, and hence, there is a unique real number $\theta\geq 0$ such that $<\mathbf{a},\mathbf{a}>=\|\mathbf{a}\|\,\|\mathbf{c}\|\cos\theta$.
Central angle: Let $\mathbf{a}$ and $\mathbf{c}$ be spacelike vectors in $\mathbb{E}_1^3$ that define a timelike vector subspace; then we have $|<\mathbf{a},\mathbf{c}>|>\|\mathbf{a}\|\,\|\mathbf{c}\|$, and hence, there is a unique real number $\theta\geq 0$ such that $<\mathbf{a},\mathbf{a}>=\|\mathbf{a}\|\,\|\mathbf{c}\|\cosh\theta$. This number is named the central angle between the vectors $\mathbf{a}$, and $\mathbf{c}$.
iii) *Lorentzian timelike angle:* Let $\mathbf{a}$ be spacelike vector and $\mathbf{c}$ be timelike vector in $\mathbb{E}_1^3$. Then there is a unique real number $\theta\geq 0$ such that $<\mathbf{a},\mathbf{a}>=\|\mathbf{a}\|\,\|\mathbf{c}\|\sinh\theta$. This number is named the Lorentzian timelike angle between the vectors $\mathbf{a}$, and $\mathbf{c}$.

Let $\beta=\beta(s)$ be a unit speed spacelike curve with timelike principal normal in $\mathbb{E}_1^3$; by $\kappa(s)$ and $\tau(s)$ we indicate the natural curvature and torsion of $\beta(s)$, respectively. Let $\{\zeta_1(s),\,\zeta_2(s),\,\zeta_3(s)\}$ be the Serret–Frenet frame linked with the curve $\beta(s)$, then the Serret–Frenet formulae read:

$$\begin{pmatrix}\zeta_1'\\ \zeta_2'\\ \zeta_3'\end{pmatrix}=\begin{pmatrix}0&\kappa&0\\ \kappa&0&\tau\\ 0&\tau&0\end{pmatrix}\begin{pmatrix}\zeta_1\\ \zeta_2\\ \zeta_3\end{pmatrix}=\psi\times\begin{pmatrix}\zeta_1\\ \zeta_2\\ \zeta_3\end{pmatrix}, \tag{2.5}$$

where

$$\begin{gathered}<\zeta_1,\zeta_1>=<\zeta_3,\zeta_3>=1,\ <\zeta_2,\zeta_2>=-1,\\ \zeta_1\times\zeta_2=\zeta_3,\ \zeta_1\times\zeta_3=\zeta_2,\ \zeta_2\times\zeta_3=\zeta_1.\end{gathered} \tag{2.6}$$

and $\psi(s)=\tau\zeta_1-\kappa\zeta_3$ is the Darboux vector of the Serret–Frenet frame. In this paper, $\beta'(s)$ indicate the derivatives of β with respect to arc-length parameter.

Definition 2.1. A moving pseudo orthogonal frame $\{\xi_1,\,\xi_3,\,\xi_3\}$, along a non null space curve $\alpha(s)$, is rotation minimizing frame (RMF) with respect to ξ_1 if its angular velocity ω satisfies $<\omega,\xi_1>=0$ or, equivalently, the derivatives of ξ_2 and ξ_3 are both parallel to ξ_1. A comparable characterization holds when ξ_2 or ξ_3 is taken as the reference orientation.

According to the Definition 2.1, we notice that the Serret–Frenet frame is RMF with respect to the principal normal ξ_2, but not with respect to the tangent ζ_1 and the binormal ζ_3. Although the Serret–Frenet frame is not RMF with respect to ζ_1, one can readily obtain such a RMF from it. New normal plane vectors ($\mathbf{N}_1$,$\mathbf{N}_2$) are assigned by a rotation of (ζ_2,ζ_3) according to

$$\begin{pmatrix}\mathbf{T}_1\\ \mathbf{N}_1\\ \mathbf{N}_2\end{pmatrix}=\begin{pmatrix}1&0&0\\ 0&\cosh\theta&\sinh\theta\\ 0&\sinh\theta&\cosh\theta\end{pmatrix}\begin{pmatrix}\zeta_1\\ \zeta_2\\ \zeta_3\end{pmatrix}, \tag{2.7}$$

where

$$\begin{gathered}<\mathbf{T}_1,\mathbf{T}_1>=<\mathbf{N}_2,\mathbf{N}_2>=1,\ <\mathbf{N}_1,\mathbf{N}_1>=-1,\\ \mathbf{T}_1\times\mathbf{N}_1=\mathbf{N}_2,\ \mathbf{T}_1\times\mathbf{N}_2=\mathbf{N}_1,\ \mathbf{N}_1\times\mathbf{N}_2=\mathbf{T}_1,\end{gathered}$$

and $\theta(s)\geq 0$ is a Lorentzian timelike angle. Here, we will call the set $\{\mathbf{T}_1,\,\mathbf{N}_1,\,\mathbf{N}_2\}$ as RMF or Bishop frame. Therefore, the Bishop formulae read:

$$\begin{pmatrix}\mathbf{T}_1'\\ \mathbf{N}_1'\\ \mathbf{N}_2'\end{pmatrix}=\begin{pmatrix}0&\kappa_1&\kappa_2\\ \kappa_1&0&0\\ -\kappa_2&0&0\end{pmatrix}\begin{pmatrix}\mathbf{T}_1\\ \mathbf{N}_1\\ \mathbf{N}_2\end{pmatrix}=\widetilde{\omega}\times\begin{pmatrix}\mathbf{T}_1\\ \mathbf{N}_1\\ \mathbf{N}_2\end{pmatrix}, \tag{2.8}$$

where $\widetilde{\omega}(s)=-\kappa_2\mathbf{N}_1-\kappa_1\mathbf{N}_2$ is Bishop Darboux vector. Here, the Bishop curvatures are given by

$\kappa_1(s) = \kappa \cosh\vartheta$, and $\kappa_2(s) = -\kappa \sinh\vartheta$. One can show that

$$\left.\begin{array}{l} \kappa_1^2 - \kappa_2^2 = \kappa^2, \text{ and } \vartheta = -\tanh^{-1}\left(\frac{\kappa_2}{\kappa_1}\right);\ \kappa_1 \neq 0, \\ \theta(s) = -\int\limits_{s_0}^{s} \tau ds + \theta_0,\ \theta_0 = \theta(0). \end{array}\right\} \tag{2.9}$$

Consequently, the Serret–Frenet frame and the RMF identical iff $\beta(s)$ is a planar, i.e. $\tau = 0$.

A sweeping surface along β is a surface defined by

$$M : \mathbf{R}(s,u) = \beta(s) + F(s)\mathbf{r}(u) = \alpha(s) + r_1(u)\mathbf{N}_1(s) + r_2(u)\mathbf{N}_2(s), \tag{2.10}$$

where $\beta(s)$ is named the (at least C^1-continuos) spine curve, $0 \leq s \leq T$, s is the arc length parameter. $\mathbf{r}(u)$ is the planar profile (cross-section) curve given by the parametrization $\mathbf{r}(u) = (0,\ r_1(u),\ r_2(u))^t$, the symbol 't' indicates to transposition, with other parameter $u \in I \subseteq \mathbb{R}$. The special orthogonal matrix $F(s) = \{\mathbf{T}_1(s),\ \mathbf{N}_1(s),\ \mathbf{N}_2(s)\}$ specifies the RMF along $\beta(s)$.

3. Lightlike sweeping surface and singularities

In this section, we introduce lightlike sweeping surface in Minkowski 3-space $\mathbb{E}_1^3$. Consider the planar profile (cross-section) $\mathbf{r}(u) = (0, u, u)$. By applying Eq. (2.13), it follows that

$$M^{\pm} : \mathbf{R}(s,u) = \beta(s) + u\mathbf{N}_1(s) \pm u\mathbf{N}_2(s). \tag{3.1}$$

By the formulae in Eq. (2.11), we can calculate

$$\left.\begin{array}{l} \mathbf{R}_u(s,u) = \mathbf{N}_1(s) \pm \mathbf{N}_2(s), \\ \mathbf{R}_s(s,u) = \left(1 + u\left(-\kappa_2 \pm \kappa_1\right)\right)\mathbf{T}_1. \end{array}\right\} \tag{3.2}$$

The normal vector of M is

$$\mathbf{R}_u \times \mathbf{R}_s = \left(1 + u\left(-\kappa_2 \pm \kappa_1\right)\right)\left(\mathbf{N}_2(s) \pm \mathbf{N}_1(s)\right). \tag{3.3}$$

Note that $\left\|\mathbf{R}_u \times \mathbf{R}_s\right\|^2 = 0$ means that $M^{\pm}$ is a lightlike surface. Furthermore, $M^{\pm}$ has singular points iff the first derivatives are linearly dependent, that is, $1 + u\left(-\kappa_2 \pm \kappa_1\right) = 0$. Therefore, $(s,u) \in I \times \mathbb{R}$ is singular point of M iff

$$u = \frac{1}{\kappa_2 \mp \kappa_1}.$$

Hence, there are two singular points on every cross-section. Connecting these two sets of singular points gives two curves that contain all the singular points of a lightlike sweeping surface. From Eq. (3.1) it follows that the expression of the two curves is

$$\gamma^{\pm}(s) = \beta(s) + \frac{1}{\kappa_2 \mp \kappa_1}\left(\left(\mathbf{N}_1(s) \mp \mathbf{N}_2(s)\right)\right), \tag{3.4}$$

and we call it the striction curve of the lightlike sweeping surface M.

Corollary 3.1. Let $M^{\pm}$ be the lightlike sweeping surface expressed by Eq. (3.1). If $\kappa_1^2 - \kappa_2^2 \neq 0$, then $M^{\pm}$ has no singular points iff

$$u \neq \frac{1}{\kappa_2 \mp \kappa_1}.$$

From Eqs. (2.8) , and (3.4), the following can be obtained:

$$\gamma^{\pm\prime}(s) = \delta_{\pm}(s)\left[\frac{-1}{-\kappa_2 \pm \kappa_1}\left(\mathbf{N}_1(s) \pm \mathbf{N}_2(s)\right)\right],$$

where

$$\delta_{\pm}(s) = \frac{-\kappa_2^{'} \pm \kappa_1^{'}}{-\kappa_2 \pm \kappa_1}.$$

So we gain $\delta_{\pm}(s)$ as a new geometric invariant of the surface $M^{\pm}$. As a result the following lemma can be given

Lemma 3.1. For the lightlike sweeping surface in Eq. (3.1), we have $\delta_{\pm}(s) = 0$ iff

$$\gamma^{\pm}(s) = \beta(s) + \frac{-1}{-\kappa_2 \pm \kappa_1} \left(\mathbf{N}_1(s) \pm \mathbf{N}_2(s)\right)$$

is a constant vector.
Proof. By simple calculations, we have that

$$\gamma^{\pm'}(s) = \delta_{\pm}(s) \left[\frac{-1}{-\kappa_2 \pm \kappa_1} \left(\mathbf{N}_1(s) \pm \mathbf{N}_2(s)\right)\right].$$

Thus $\gamma^{\pm'}(s) = \mathbf{0}$ iff $\delta_{\pm}(s) = 0$ ■.

From now on, we shall often not write the parameter s explicitly in our formulae. The main aim of this paper are in the following theorem:

Theorem 3.1. For the lightlike sweeping surface in Eq. (3.1), with $\kappa_1^2 - \kappa_2^2 \neq 0$, and $\mathbf{r} \in \mathbb{LC}_0^*$, one has the following:
A- (1) $\beta(\mathbf{s})$ is locally diffeomorphic to a line $\{\mathbf{0}\}\times\mathbb{R}$ at s_0 iff $\delta_{\pm}(s_0) \neq 0$.
(2) $\beta(\mathbf{s})$is locally diffeomorphic to the cusp $C \times \mathbb{R}$ at s_0 iff $\delta_{\pm}(s_0) = 0$, and $\delta_{\pm}{}'(s_0) \neq 0$.
B- (1) M is locally diffeomorphic to Cuspidal edge CE at (s_0, u_0) iff $\mathbf{r} = \pm\gamma^{\pm}(s_0)$, and $\delta_{\pm}(s_0) \neq 0$.
(2) M is locally diffeomorphic to Swallowtail SW at (s_0, u_0) iff $\mathbf{r} = \pm\gamma^{\pm}(s_0)$, $\delta_{\pm}(s_0) = 0$, and $\delta_{\pm}{}'(s_0) \neq 0$.
Her, $C \times \mathbb{R} = \left\{(r_1, r_2)|r_1^2 = r_2^3\right\} \times \mathbb{R}$, $CE = \left\{(r_1, r_2, r_3)|\mathrm{r}_1{=}\mathrm{u},\ \mathrm{r}_2{=}\mathrm{v}^2,\ \mathrm{r}_3{=}\mathrm{v}^3\right\}$, and $SW = \left\{(r_1, r_2, r_3)|r_1 = u,\ r_2 = 3v^2 + uv^2,\ r_3 = 4v^3 + 2uv\right\}$. The pictures of $C \times \mathbb{R}$, CE, and SW will be seen in Figs 1, 2, 3.
1. latex

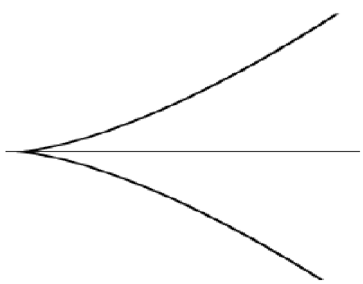

Figure 1: $C \times \mathbb{R}$

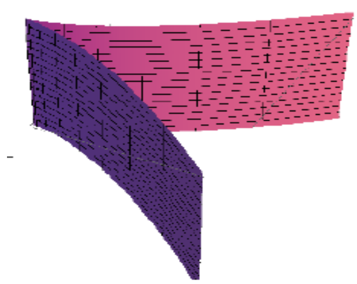

Figure 2: CE

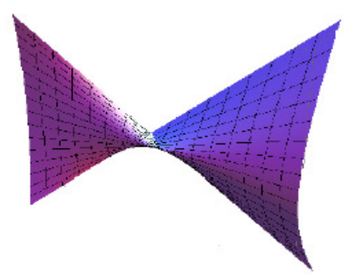

Figure 3: SW

3.1. Lightlike distance Bishop functions

Now we will consider lightlike distance Bishop function that will be helpful to research the singularties of M as follows: $\mathfrak{D} : I \times \mathbb{E}_1^2 \to \mathbb{R}$, by $\mathfrak{D}(s, \mathbf{r}) =< \mathbf{r} - \beta(\mathbf{s}), \mathbf{r} - \beta(\mathbf{s})$. We call it lightlike distance Bishop height function. We use the notation $\mathfrak{d}_{\mathbf{r}}(s) = \mathfrak{D}(s, \mathbf{r})$ for any fixed $\mathbf{r} \in \mathbb{LC}_0^*$. Then, we have the following proposition:

Proposition 3.1. For the lightlike sweeping surface in Eq. (3.1), with $\kappa_1^2-\kappa_2^2\neq 0$, the following holds:

1- $\mathfrak{d}_r(s)=0$ iff there exist a, a_1, and $a_2\in\mathbb{R}$, with $a^2-a_1^2+a_2^2=0$ such that $\mathbf{r}=\beta+a\mathbf{T}_1+a_1\mathbf{N}_1+a_2\mathbf{N}_2$.

2- $\mathfrak{d}_r(s)=\mathfrak{d}_r^{'}(s)=0$ iff $\mathbf{r}=\beta+u\left(\mathbf{N}_1\pm\mathbf{N}_2\right)$

3- $\mathfrak{d}_r(s)=\mathfrak{d}_r^{'}(s)=\mathfrak{d}^{''}(s)=0$ iff $\mathbf{r}=\beta+\frac{-1}{\kappa_1\mp\kappa_2}\left(\mathbf{N}_1\pm\mathbf{N}_2\right).$

4- $\mathfrak{d}_r(s)=\mathfrak{d}_r^{'}(s)=\mathfrak{d}^{''}(s)=\mathfrak{d}_r^{'''}(s)=0$ iff $\delta_\pm(s)=0$, and $\mathbf{r}=\beta+\frac{-1}{\kappa_1\mp\kappa_2}\left(\mathbf{N}_1\pm\mathbf{N}_2\right).$

5- $\mathfrak{d}_r(s)=\mathfrak{d}_r^{'}(s)=\mathfrak{d}^{''}(s)=\mathfrak{d}_r^{'''}(s)=\mathfrak{d}_r^{(4)}(s)=0$ iff $\delta_\pm=\delta_\pm^{'}=0$, and $\mathbf{r}$=$\beta(s)+\frac{-1}{-\kappa_2\pm\kappa_1}\left(\mathbf{N}_1(s)\pm\mathbf{N}_2(s)\right)$.

Proof. According to Eq. (2.8) we have that $\left\|\mathbf{T}_1^{'}\right\|^2\neq 0$ iff $\kappa_1^2-\kappa_2^2\neq 0$. Direct computation gives

$$\left.\begin{array}{l}\mathfrak{d}_{\mathbf{r}}(s)=<\mathbf{r}-\beta,\mathbf{r}-\beta>,\\ \mathfrak{d}_{\mathbf{r}}^{'}(s)=-2<\mathbf{T}_1,\mathbf{r}-\beta>,\\ \mathfrak{d}_{\mathbf{r}}^{''}(s)=-2\left(<\mathbf{T}_1^{'},\mathbf{r}-\beta>-1\right),\\ \mathfrak{d}_{\mathbf{r}}^{'''}(s)=-2<\mathbf{T}_1^{''},\mathbf{r}-\beta>,\\ \mathfrak{d}_{\mathbf{r}}^{(4)}(s)=-2\left(<\mathbf{T}_1^{'''},\mathbf{r}-\beta>-<\mathbf{T}_1^{''},\mathbf{T}_1>\right).\end{array}\right\}$$

Since $\{\mathbf{T}_1(s),\ \mathbf{N}_1(s),\ \mathbf{N}_2(s)\}$ is RMF along $\beta(s)$, then there exist a, a_1, and $a_2\in\mathbb{R}$ such that $\mathbf{r}=\beta+a\mathbf{T}_1+a_1\mathbf{N}_1+a_2\mathbf{N}_2$.

1- From $\mathfrak{d}_{\mathbf{r}}(s)=<\mathbf{r}-\beta,\mathbf{r}-\beta>=0$, we get $a^2-a_1^2+a_2^2=0$, thus assertion (1) holds.

2- Since $\mathfrak{d}_{\mathbf{r}}(s)=0$, we have $\mathfrak{d}_{\mathbf{r}}^{'}=-2<\mathbf{T}_1,\mathbf{T}_1+a_1\mathbf{N}_1+a_2\mathbf{N}>=0\Leftrightarrow a=0$. Hence, $-a_1^2+a_2^2=0$, letting $a_1=u\in\mathbb{R}$, we obtain that $\mathbf{r}=\beta+u\left(\mathbf{N}_1\pm\mathbf{N}_2\right)$, this completes the proof of assertion (2).

3- Under the condition $\mathfrak{d}_{\mathbf{r}}(s)=\mathfrak{d}_{\mathbf{r}}^{'}(s)=0$, we calculate that

$$\mathfrak{d}_{\mathbf{r}}^{''}(s)=0\Leftrightarrow -2\left(<\kappa_1\mathbf{N}_1+\kappa_2\mathbf{N}_2,u\mathbf{N}_1\pm u\mathbf{N}_2-1\right)=0.$$

It follows that $u=\frac{-1}{\kappa_1\mp\kappa_2}$, that is, $\mathbf{r}=\beta(s)+\frac{-1}{\kappa_1\mp\kappa_2}\left(\mathbf{N}_1\pm\mathbf{N}_2\right)$, assertion (3) holds.

4- Under the condition $\mathfrak{d}_{\mathbf{r}}(s)=\mathfrak{d}_{\mathbf{r}}^{'}(s)=\mathfrak{d}_{\mathbf{r}}^{''}(s)=0$, we calculate that

$$\begin{array}{rcl}\mathfrak{d}_{\mathbf{r}}^{'''}(s)&=&-2<\left(\kappa_1^2-\kappa_2^2\right)\mathbf{T}_1+\kappa_1^{'}\mathbf{N}_1+\kappa_2^{'}\mathbf{N}_2,\dfrac{-1}{\kappa_1\mp\kappa_2}\left(\mathbf{N}_1\pm\mathbf{N}_2\right)>\\ &=&2\dfrac{-\kappa_2^{'}\pm\kappa_1^{'}}{-\kappa_2\pm\kappa_1}\\ &=&\delta_\pm(s).\end{array}$$

Thus $\mathfrak{d}_{\mathbf{r}}^{'''}(s)=0$ iff $\delta_\pm(s)=0$, therefor $\mathfrak{d}_{\mathbf{r}}(s)=\mathfrak{d}_{\mathbf{r}}^{'}(s)=\mathfrak{d}_{\mathbf{r}}^{''}(s)=\mathfrak{d}_{\mathbf{r}}^{'''}(s)=0$ iff $\delta_\pm(s)=0$, that is, assertion (4) holds.

5- Suppose that $\mathfrak{d}_{\mathbf{r}}(s)=\mathfrak{d}_{\mathbf{r}}^{'}(s)=\mathfrak{d}_{\mathbf{r}}^{''}(s)=\mathfrak{d}_{\mathbf{r}}^{'''}(s)=0$, we calculate that

$$\begin{array}{rcl}\mathfrak{d}_{\mathbf{r}}^{(4)}(s)&=&-2\left(<\mathbf{T}_1^{'''},\dfrac{-1}{\kappa_1\mp\kappa_2}\left(\mathbf{N}_1\pm\mathbf{N}_2\right)>+\kappa_2^2-\kappa_1^2\right)\\ &=&-2\left(\begin{array}{c}<3(-\kappa_2\kappa_2^{'}+\kappa_1\kappa_1^{'})\mathbf{T}_1+(\kappa_1^3-\kappa_1\kappa_2^2+\kappa_1^{''})\mathbf{N}_1+(-\kappa_2^3+\kappa_2\kappa_1^2+\kappa_2^{''})\mathbf{N}_2,\\ \frac{-1}{\kappa_1\mp\kappa_2}\left(\mathbf{N}_1\pm\mathbf{N}_2\right)>+\kappa_2^2-\kappa_1^2\end{array}\right)\\ &=&2\dfrac{\delta_\pm^{'}(s)}{\kappa_1\mp\kappa_2}.\end{array}$$

Then we have $\mathfrak{d}_r(s)=\mathfrak{d}_{\mathbf{r}}^{'}(s)=\mathfrak{d}_{\mathbf{r}}^{''}(s)=\mathfrak{d}_{\mathbf{r}}^{'''}(s)=\mathfrak{d}_{\mathbf{r}}^{(4)}(s)=0$ iff $\delta_\pm=\delta_\pm^{'}=0$, and $\mathbf{r}=\beta+\frac{-1}{-\kappa_2\pm\kappa_1}\left(\mathbf{N}_1(s)\pm\mathbf{N}_2(s)\right)$ ■.

3.2. Unfolding of functions by one-variable

In this subsection, we employ some general results on the singularity theory for families of function germs [20, 21]. Let F: $(\mathbb{R}\times\mathbb{R}^r,(s_0,\mathbf{x}_0))\to\mathbb{R}$ be a smooth function, and $f(s)=F_{x_0}(s,\mathbf{x}_0)$. Then F is called an r-parameter unfolding of $f(s)$. We say that $f(s)$ has A_k-singulaity at s_0 if $f^{(p)}(s_0)=0$ for all $1\le p\le k$, and $f^{(k+1)}(s_0)\neq 0$. We also say that f has $A_{\geqslant k}$-singulaity $(k\geqslant 1)$ at s_0. Let the $(k-1)$-jet of the partial derivative $\frac{\partial F}{\partial x_i}$ at s_0 be $j^{(k-1)}\left(\frac{\partial F}{\partial x_i}(s,\mathbf{x}_0)\right)(s_0)=\Sigma_{j=0}^{k-1}L_{ji}\,(s-s_0)^j$ (without the constant term), for $i=1,...,r$. Then $F(s)$ is named an p-versal unfolding if the $k\times r$ matrix of coefficients (L_{ji}) has rank k $(k\le r)$.

We now introduce the following important sets concerning the unfolding:

$$\mathfrak{T}_F^j=\left\{\mathbf{x}\in\mathbb{R}^r|\text{ there exists } s \text{ with } F\left(s,\mathbf{x}\right)=...=\frac{\partial^j F}{\partial s^j}\left(s,\mathbf{x}\right)=0 \text{ at } \left(s,\mathbf{x}\right)\right\}. \tag{3.5}$$

which is named a discriminant set of order j. Of course, $\mathfrak{T}_F^1=\mathfrak{L}$, and $\mathfrak{T}_F^2$ is the set of singular points of $\mathfrak{L}$. According to Proposition 3.1 and the definition of A_k-singularity, we have the following:

Corollary.3.2. For the lightlike sweeping surface in Eq. (3.1), with $\kappa_1^2-\kappa_2^2\neq 0$, the following holds:

1- $\mathfrak{d}_r(s)$ has A_1-singularity at s iff $\mathbf{r}=\beta+\frac{-1}{-\kappa_2\pm\kappa_1}\left(\mathbf{N}_1(s)\pm\mathbf{N}_2(s)\right)$

2- $\mathfrak{d}_r(s)$ has A_2-singularity at s iff $\mathbf{r}=\beta+\frac{-1}{-\kappa_2\pm\kappa_1}\left(\mathbf{N}_1(s)\pm\mathbf{N}_2(s)\right)$, and $\delta_\pm(s)\neq 0$.

3- $\mathfrak{d}_r(s)$ has A_3-singularity at s iff $\mathbf{r}=\mathbf{r}=\beta+\frac{-1}{-\kappa_2\pm\kappa_1}\left(\mathbf{N}_1(s)\pm\mathbf{N}_2(s)\right)$, $\delta_\pm(s)=0$, and $\delta_\pm^{'}(s)\neq 0$.

Then similar to [1-3], we state the following theorem:

We now introduce the following useful sets regarding the unfolding:

$$\mathfrak{L}_F^j=\left\{\mathbf{x}\in\mathbb{R}^r|\text{ there exists } s \text{ with } F\left(s,\mathbf{x}\right)=...=\frac{\partial^j F}{\partial s^j}\left(s,\mathbf{x}\right)=0 \text{ at } \left(s,\mathbf{x}\right)\right\}. \tag{3.6}$$

which is named a discriminant set of order j. Of course, $\mathfrak{L}_F^1=\mathfrak{L}_F$, and $\mathfrak{L}_F^2$ is the set of singular points of $\mathfrak{L}_F$. Then, we state the following theorem [20, 21]:

Theorem 3.2. Let F: $(\mathbb{R}\times\mathbb{R}^r,(s_0,\mathbf{x}_0))\to\mathbb{R}$ be an r-parameter unfolding of $f(s)$, which has the A_k singularity at s_0. Suppose that F is a p-versal unfolding.

(a) If $k=1$, then $\mathfrak{L}_F$ is locally diffeomorphic to $\{\mathbf{0}\}\times\mathbb{R}^{r-1}$, and $\mathfrak{L}_F^2=\emptyset$,

(b) If $k=2$, then $\mathfrak{L}_F$ is locally diffeomorphic to $C\times\mathbb{R}^{r-2}$, and $\mathfrak{L}_F^2$ is locally diffeomorphic to $\{\mathbf{0}\}\times\mathbb{R}^{r-1}$,

(c) If $k=3$, then $\mathfrak{L}_F$ is locally diffeomorphic to $SW\times\mathbb{R}^{r-3}$, and $\mathfrak{L}_F^2$ is locally diffeomorphic to $C\times\mathbb{R}^{r-2}$.

For the proof of Theorem 3.1, we have the following fundamental proposition:

Proposition 3.2. Let $\beta(s):I\subseteq\mathbb{R}\to\mathbb{E}_1^3$ be a unit speed spacelike curve with $\kappa_1^2-\kappa_2^2\neq 0$ If the $\mathfrak{d}_\mathbf{r}(s)=\mathfrak{D}(s,\mathbf{r})$ has an A_k-singularity $(k=1,\,2,\,3)$ at $s_0\in\mathbb{R}$, then $\mathfrak{D}$ is the $p-$versal unfolding of $d_\mathbf{r}(s_0)$.
Proof. Since $\mathbf{r}=(r_0,r_1,r_2)\in\mathbb{E}_1^3$, and $\beta(s)=(\beta_0(s),\beta_1(s),\beta_2(s))\in\mathbb{E}_1^3$. From now on, we shall often not write the parameter s explicitly in our formulae. Then, we have

$$\mathfrak{D}(s,\mathbf{r})=-(r_0-\beta_0)^2+(r_2-\beta_2)^2+(r_3-\beta_3)^2.$$

So

$$\left.\begin{array}{l}\frac{\partial\mathfrak{D}}{\partial r_0}=-2\left(r_0-\beta_0\right),\ \frac{\partial\mathfrak{D}}{\partial r_1}=2\left(r_1-\beta_1\right),\ \frac{\partial\mathfrak{D}}{\partial r_2}=2\left(r_2-\beta_2\right),\\ \frac{\partial^2\mathfrak{D}}{\partial s\partial r_0}=2\beta_0^{'},\ \frac{\partial^2\mathfrak{D}}{\partial s\partial r_1}=-2\beta_1^{'},\ \frac{\partial^2\mathfrak{D}}{\partial s\partial r_2}=-2\beta_2^{'},\\ \frac{\partial^3\mathfrak{D}}{\partial s^2\partial r_0}=2\beta_0^{''},\ \frac{\partial^3\mathfrak{D}}{\partial s^2\partial r_1}=-2\beta_1^{''},\ \frac{\partial^3\mathfrak{D}}{\partial s^2\partial r_2}=-2\beta_2^{''}.\end{array}\right\}$$

Therefore, the 2-jets of $\frac{\partial \mathfrak{D}}{\partial r_i}$ at s$_0$ (i=0, 1, 2) are:

$$\left.\begin{array}{l} j^2\left(\frac{\partial \mathfrak{D}}{\partial r_0}\right) = -2\left(r_0-\beta_0\right)+2\beta_0^{'}\left(s-s_0\right)+\frac{1}{2}\left(s-s_0\right)^2,\ \frac{\partial \mathfrak{D}}{\partial r_1} = 2\beta_0^{''}\left(\frac{1}{2}\left(s-s_0\right)^2\right), \\ j^2\left(\frac{\partial^2 \mathfrak{D}}{\partial s \partial r_0}\right) = 2\left(r_1-\beta_1\right)-2\beta_1^{'}\left(s-s_0\right)-2\beta_1^{''}\left(\frac{1}{2}\left(s-s_0\right)^2\right), \\ j^2\left(\frac{\partial^3 \mathfrak{D}}{\partial s^2 \partial r_0}\right) = 2\left(r_2-\beta_2\right)-2\beta_2^{''}\left(s-s_0\right)-2\beta_2^{''}\left(\frac{1}{2}\left(s-s_0\right)^2\right). \end{array}\right\}$$

(i) If $\mathfrak{d}_{\mathbf{r}}(s_0)$ has the A_1-singularity at $s_0 \in \mathbb{R}$, then (L_{ji}) is:

$$A = \left(\begin{array}{ccc} -2\left(r_0-\beta_0\right) & 2\left(r_1-\beta_1\right) & 2\left(r_2-\beta_2\right) \end{array} \right).$$

It is obvious that rank$(A)=1$, since $\mathbf{r}-\beta \neq \mathbf{0}$.
(ii) If $\mathfrak{d}_{\mathbf{r}}(s_0)$ has the A_2-singularity at $s_0 \in \mathbb{R}$, then $\mathfrak{d}_{\mathbf{r}}^{'}(s_0)=\mathfrak{d}_{\mathbf{r}}^{''}(s_0)=0$, and by Corollary 3.2:

$$\mathbf{r}-\beta = \frac{-1}{\kappa_1 \mp \kappa_2}\left(\mathbf{N}_1 \pm \mathbf{N}_2\right) \neq \mathbf{0},$$

where $\delta_{\pm}(s) \neq 0$. So, the matrix of the coefficients (L_{ji}) is

$$B = \left(\begin{array}{ccc} -2\left(r_0-\beta_0\right) & 2\left(r_1-\beta_1\right) & 2\left(r_2-\beta_2\right) \\ 2\beta_0^{'} & -2\beta_1^{'} & -2\beta_2^{'} \end{array} \right).$$

Therefore, the rank$(A)=2$.
(iii) If $\mathfrak{d}_{\mathbf{r}}(s_0)$ has the A_3-singularity at $s_0 \in \mathbb{R}$, then $\mathfrak{d}_{\mathbf{r}}^{'}(s_0)=\mathfrak{d}_{\mathbf{r}}^{''}(s_0)=\mathfrak{d}_{\mathbf{r}}^{'''}(s_0)=0$, and by Corollary 3.2:

$$\mathbf{r}-\beta = \frac{-1}{\kappa_1 \mp \kappa_2}\left(\mathbf{N}_1 \pm \mathbf{N}_2\right) \neq \mathbf{0},$$

where $\delta_{\pm}(s)=0$, and $\delta_{\pm}^{'}(s) \neq 0$. We consider the following matrices:

$$G = \left(\begin{array}{ccc} -2\left(r_0-\beta_0(s_0)\right) & 2\left(r_1-\beta_1(s_0)\right) & 2\left(r_2-\beta_2(s_0)\right) \\ 2\beta_0^{'}(s_0) & -2\beta_1^{'}(s_0) & -2\beta_2^{'}(s_0) \\ 2\beta_0^{''}(s_0) & -2\beta_1^{''}(s_0) & -2\beta_2^{''}(s_0) \end{array} \right),$$

and

$$\widetilde{G} = \left(\begin{array}{ccc} r_0-\beta_0(s_0) & r_1-\beta_1(s_0) & r_2-\beta_2(s_0) \\ \beta_0^{'}(s_0) & \beta_1^{'}(s_0) & \beta_2^{'}(s_0) \\ \beta_0^{''}(s_0) & \beta_1^{''}(s_0) & \beta_2^{''}(s_0) \end{array} \right) = \left(\begin{array}{c} \mathbf{r}-\beta \\ \beta^{'} \\ \beta^{''} \end{array} \right).$$

It is clear that rank(G)=rank($\widetilde{G}$). In fact, the determinate of G at s_0 is

$$\begin{aligned} \det(\widetilde{G}) &= <\beta^{'}\times\beta^{''}, \mathbf{r}-\beta> \\ &= <\beta^{'}\times\beta^{''}, \frac{-1}{\kappa_1 \mp \kappa_2}\left(\mathbf{N}_1 \pm \mathbf{N}_2\right)> \\ &= \frac{\kappa_2 \mp \kappa_1}{\kappa_1 \mp \kappa_2} = \pm 1 \neq 0. \end{aligned}$$

This means that rank(G) =rank($\widetilde{G}$) $=3$. This completes the proof ■.

Proof of Theorem 3.1 By Proposition 3.1, we get the discriminant set of order one is

$$\mathfrak{L}_F = \left\{\mathbf{r}_0 = \beta + u\mathbf{N}_1 \pm u\mathbf{N}_2 | s \in \mathbb{R}\right\},$$

and the discriminant set of order two is

$$\mathfrak{L}_F^2 = \left\{\mathbf{r}_0 = \beta + \frac{-1}{\kappa_1 \mp \kappa_2}\left(\mathbf{N}_1 \pm \mathbf{N}_2\right) | s \in \mathbb{R}\right\}.$$

By Theorem 3.2, and Proposition 3.2, we have the assertions (1), and (2). This completes the proof ■.

Example 3.1. Given the spacelike helix:

$$\alpha(s) = (\frac{\sqrt{3}}{2}\sinh s, \frac{s}{2}, \frac{\sqrt{3}}{2}\cosh s), \ -1 \le s \le 1.$$

It is easy to have that

$$\left.\begin{array}{c} \zeta_1(s) = (\frac{\sqrt{3}}{2}\sinh s, \frac{1}{2}, \frac{\sqrt{3}}{2}\cosh s), \\ \zeta_2(s) = (\sinh s, 0, \cosh s), \\ \zeta_3(s) = (\frac{1}{2}\cosh s, -\frac{\sqrt{3}}{2}, \frac{1}{2}\sinh s), \\ \kappa(s) = \frac{\sqrt{3}}{2}, \text{and } \tau(s) = \frac{1}{2}. \end{array}\right\}$$

Taking $\theta_0 = 0$ we have $\theta(s) = -\frac{1}{2}s$. Using the Eq. (2.11), we obtain

$$\kappa_1(s) = \cosh\frac{s}{2}, \text{ and } \kappa_2(s) = -\sinh\frac{s}{2}.$$

Hence, the geometric invariant is

$$\delta_{\pm}(s) = \pm\frac{1}{2}.$$

The transformation matrix can be expressed as:

$$\begin{pmatrix} \mathbf{T}_1 \\ \mathbf{N}_1 \\ \mathbf{N}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\frac{s}{2} & -\sinh\frac{s}{2} \\ 0 & -\sinh\frac{s}{2} & \cosh\frac{s}{2} \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix}.$$

Subsequently, we have

$$\begin{aligned} \mathbf{N}_1 &= \begin{pmatrix} N_{11} \\ N_{12} \\ N_{13} \end{pmatrix} = \begin{pmatrix} \cosh\frac{s}{2}\sinh s - \frac{1}{2}\sinh\frac{s}{2}\cosh s \\ \frac{\sqrt{3}}{2}\sinh\frac{s}{2} \\ \cosh\frac{s}{2}\cosh s - \frac{1}{2}\sinh\frac{s}{2}\sinh s \end{pmatrix}, \\ \mathbf{N}_2 &= \begin{pmatrix} N_{21} \\ N_{22} \\ N_{23} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\cosh\frac{s}{2}\cosh s - \sinh\frac{s}{2}\sinh s \\ -\frac{\sqrt{3}}{2}\cosh\frac{s}{2} \\ \frac{1}{2}\cosh\frac{s}{2}\sinh s - \sinh\frac{s}{2}\cosh s \end{pmatrix}. \end{aligned}$$

Therefore, the lightlike sweeping surface can be written as (Figure 4)

$$M^{\pm} : \mathbf{R}(s,u) = \left(\frac{\sqrt{3}}{2}\sinh s, \frac{s}{2}, \frac{\sqrt{3}}{2}\cosh s\right) + u\left[\begin{pmatrix} N_{11} \\ N_{12} \\ N_{13} \end{pmatrix} \mp \begin{pmatrix} N_{21} \\ N_{22} \\ N_{23} \end{pmatrix}\right].$$

The striction curve is (Figure 5)

$$\gamma^{\pm}(s) = \left(\frac{\sqrt{3}}{2}\sinh s, \frac{s}{2}, \frac{\sqrt{3}}{2}\cosh s\right) + \frac{1}{\sinh\frac{s}{2} \mp \cosh\frac{s}{2}}\left[\begin{pmatrix} N_{11} \\ N_{12} \\ N_{13} \end{pmatrix} \mp \begin{pmatrix} N_{21} \\ N_{22} \\ N_{23} \end{pmatrix}\right].$$

3.3. Singularities of developable surfaces

Developable surfaces can be considered as particular events of ruled surfaces. Such surfaces are frequently used, for example, in the airplane wings, manufacture of automobile body parts, and ship hulls. Therefore, we analyze the case that the profile curve $\mathbf{r}(u) = (0, u, u)$ degenerates into a spacelike/timelike line. Then, we have the following spacelike developable surface

$$M : \mathbf{Q}(s,u) = \beta(s) + u\mathbf{N}_2(s), \ u \in \mathbb{R}. \tag{3.7}$$

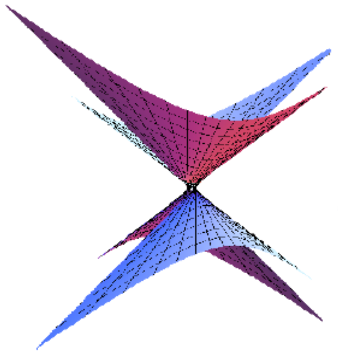

Figure 4: $M^{\pm}$ with spacelike helix singularity curve

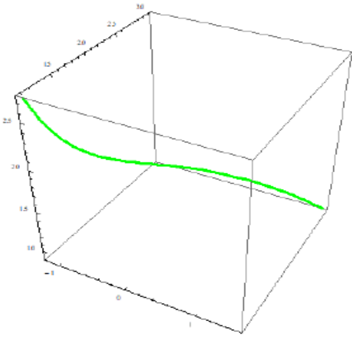

Figure 5: $\gamma^{\pm}$ of the singular points

Similarly, we have the following timelike developable surface

$$M^{\perp}:\mathbf{Q}^{\perp}(s,u)=\beta(s)+u\mathbf{N}_1(s), u\in\mathbb{R}. \tag{3.8}$$

Clearly, $\mathbf{P}(s,0)=\alpha(s)$ (resp. $\mathbf{P}^{\perp}(s,0)=\alpha(s)$), $0\leq s\leq L$, that is, the surface M (resp. $M^{\perp}$) interpolate the curve $\alpha(s)$. We can also calculate that

$$M:\mathbf{Q}_s\times\mathbf{Q}_u=(1-u\kappa_2)\,\mathbf{N}_1(s),$$

and

$$M^{\perp}:\mathbf{Q}_s^{\perp}\times\mathbf{Q}_u^{\perp}=(1+u\kappa_1)\,\mathbf{N}_1(s).$$

Then we have M (resp. $M^{\perp}$) is non-singular at (s_0,u_0) iff $1-u_0\kappa_2(s_0)\neq 0$ (resp. $(1+u_0\kappa_1(s_0)\neq 0)$. Hence, we can classify the singularities of developable surface M by using κ_2.

Theorem 3.3. Let M be the spacelike developablein Eq. (3.9). Then
(1) M is locally diffeomorphic to cuspidal edge at (s_0,u_0) iff $\kappa_2(s_0)=0$, and $\kappa_2'(s_0)\neq 0$;
(2) M is locally diffeomorphic to swallowtail at (s_0,u_0) iff $\kappa_2(s_0)\neq 0$, and $\frac{\kappa_2'(s_0)}{\kappa_2^2(s_0)}\neq 0$.

Proof. If there exists a parameter s_0 such that $\kappa_2(s_0)=0$, and $u_0'=\frac{\kappa_2'(s_0)}{\kappa_2^2(s_0)}\neq 0$ ($\kappa_2'(s_0)\neq 0$), then M is locally diffeomorphic to Cuspidal edge at (s_0,u_0). So, assertion (1) holds. Also, if there exists a parameter s_0 such that $u_0=\frac{1}{\kappa_2(s_0)}\neq 0$, $u_0'=\frac{\kappa_2'(s_0)}{\kappa_2^2(s_0)}=0$, and $\left(\frac{1}{\kappa_2(s_0)}\right)''\neq 0$, then M is locally diffeomorphic to Swallowtail at (s_0,u_0), assertion (2) holds ■.

Example 3.2. By making using of Example 3.1, we have the following:
(1) If $s_0=0$, then $\kappa_2(s_0)=0$, $\kappa_2'(s_0)\neq 0$. The timelike developable surface

$$M:\mathbf{Q}(s,u)=\left(\frac{\sqrt{3}}{2}\sinh s,\frac{s}{2},\frac{\sqrt{3}}{2}\cosh s\right)+u\begin{pmatrix}\frac{1}{2}\cosh\frac{s}{2}\cosh s-\sinh\frac{s}{2}\sinh s\\ -\frac{\sqrt{3}}{2}\cosh\frac{s}{2}\\ \frac{1}{2}\cosh\frac{s}{2}\sinh s-\sinh\frac{s}{2}\cosh s\end{pmatrix}, u\in\mathbb{R}.$$

is locally diffeomorphic to the cuspidal edge, see Figure 6.
(2) If $s_0=0$, then $\kappa_1(s_0)\neq 0$, $\kappa_1'(s_0)=0$. The spacelike developable surface

$$M^{\perp}:\mathbf{Q}^{\perp}(s,u)=\left(\frac{\sqrt{3}}{2}\sinh s,\frac{s}{2},\frac{\sqrt{3}}{2}\cosh s\right)+u\begin{pmatrix}\cosh\frac{s}{2}\sinh s-\frac{1}{2}\sinh\frac{s}{2}\cosh s\\ \frac{\sqrt{3}}{2}\sinh\frac{s}{2}\\ \cosh\frac{s}{2}\cosh s-\frac{1}{2}\sinh\frac{s}{2}\sinh s\end{pmatrix}, u\in\mathbb{R},$$

Figure 6: Timelike developable surface

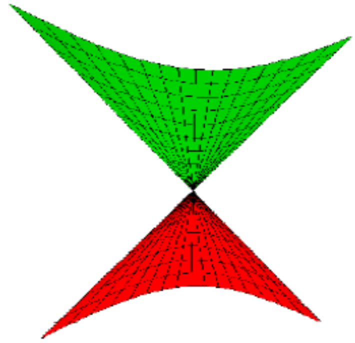

Figure 7: Spacelike developable surface

is locally diffeomorphic to Swallowtail, see Figure 7

4. Conclusion

This paper investigate the properties of lightlike sweeping surface by setting up an orthonormal RMF to each point of the spine curve and applying the moving frame method. Consequently, we have solved the problem of requiring the surface that is sweeping surface and at the same time developable surface. Meanwhile, examples illustrates the application of the obtained formula are introduced. There are several opportunities for further work. An analogue of the problem addressed in this paper may be consider for 3-surfaces in 4-space. We will study this problem in the future.

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