



## On the centralizer of a matrix and wild problem

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ABSTRACT: In this paper, we give a full description of the centralizer of a given square matrix over an arbitrary field and use this result to solve a weaker version of the Wild Problem.

Key Words: Toeplitz matrix, Centralizer, Invariants factors, Sylvester space, Wild problem.

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### 1. Introduction

Throughout this paper,  $K$  denotes an arbitrary field,  $\mathcal{M}_n(K)$  all  $n$  by  $n$  matrices with entries from  $K$ . For a given  $n$  by  $n$  matrix  $A$  over  $K$ , its centralizer  $\mathcal{C}(A)$  is the subalgebra of  $\mathcal{M}_n(K)$  consisting of all matrices  $B$  that commute with  $A$ . For a given  $n$  by  $n$  matrix  $A$  over  $K$  and  $m$  by  $m$  matrix over  $K$ , the Sylvester space  $\mathcal{C}(A, B)$  of  $A$  and  $B$  is the set of all  $n$  by  $m$   $T$ -matrices over  $K$  such that  $AT = TB$  (see [4]).

Many reasons lead us to study the centralizers of matrices: In representation theory, the wild problem is the problem of classifying pairs of matrices up to simultaneous similarity[see [2], [9]]. Knowledge of the explicit description of a matrix's centralizer can help us solve a weaker version of the wild problem. In another way, the centralizer  $\mathcal{C}(A)$  is the set of solutions of the homogenous equation of

$$AX - XA = C \tag{1.1}$$

where  $X \in \mathcal{M}_n(K)$ , Lastly, the representation of the centralizer of a given square matrix,  $\mathcal{C}(A)$  of  $A$  is closely related to the problem of representation of the Sylvester space  $\mathcal{C}(A, B)$  which is the set of solutions of the homogenous matrix equation

$$AX - XB = C \tag{1.2}$$

where  $X \in \mathcal{M}_{n \times m}(K)$ , that play a central role in many areas of applied mathematics and, in particular in systems and control theory. This paper uses a type of Toeplitz matrix and results from the theory of modules to describe the matrix's centralizer.

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## 2. Preliminaries

In this section, we introduce notations and mathematical objects used in this paper.

### 2.1. Notations

The following notations are used throughout this paper.

$K$	an arbitrary field.
$\mathcal{M}_n(K)$ ,	the set of $n$ by $n$ matrices with entries from $K$ .
$K^n$	is identified with $\mathcal{M}_{(n,1)}(K)$
$\mathcal{C}(A)$ ,	the centralize of $A$ .
$\mathcal{C}(A, B)$	the set of all $n$ by $m$ $T$ - matrices over $K$ such that $AT = TB$ .
$diag(A_1, A_2, \dots, A_m)$ ,	block diagonal matrix with matrices $A_1, A_2, \dots, A_m$ on its main diagonal blocks.
$K[B]$ ,	the subspace of $\mathcal{M}_n(K)$ spanned by all powers of $B$ .
$Comp(Q)$ ,	the companion matrix of $Q$ ;
$\bigoplus_{i=1}^r \bigoplus_{j=1}^r \mathcal{C}(A_i, A_j)$ ,	the direct sum of the $\mathcal{C}(A_i, A_j)$ .
$\prod_{i=1}^s \mathcal{C}(A_i)$ ,	the product of the $\mathcal{C}(A_i)$ ;

### 2.2. Basic results from module theory

This section includes some useful results we will use to prove our main result. Let  $A \in \mathcal{M}_n(K)$  be a no zero matrix, and  $M_A$  be the  $K[X]$ -module induced by  $A$ . From the theory of finitely generated torsion module over P.I.D (see [[6],p 215], [[5], §2, p. 556] and [[1], p. 235]), we have the following Theorems:

**Theorem 2.1** *Let  $A \in \mathcal{M}_n(K)$  be a no zero matrix, and  $M_A$  be the  $K[X]$ -module induced by  $A$  then there exists a unique sequence of polynomials  $q_1, \dots, q_r$  such that:*

$$M_A \simeq \frac{K[X]}{(q_1)} \oplus \frac{K[X]}{(q_2)} \oplus \dots \oplus \frac{K[X]}{(q_r)}$$

and

- $q_i$  divides  $q_{i+1}$ , for all  $1 \leq i \leq r-1$
- $q_r = m_A$  the minimal polynomial of  $A$  and  $\prod_{i=1}^r q_i = C_A$  the characteristic polynomial of  $A$ .

The ascending sequence of polynomials  $q_1, \dots, q_r$  are unique up similarity and called the invariant factors of  $A$ .

A matrix  $A$  with entries from a field  $K$  is said to be in a rational canonical form if there exists an ordered set  $\{f_1, f_2, \dots, f_r\}$  of polynomials in  $K[X]$  such that  $f_i$  divides  $f_{i+1}$  for all  $i \in \{1, \dots, r-1\}$  and

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & A_r \end{bmatrix}, \text{ where } A_i \text{ is the companion matrix of } f_i, \text{ and each } 0 \text{ is zero matrix of}$$

appropriate order.

**Theorem 2.2 (Rational canonical form)** *Every square matrix with entries in a field  $K$  is similar to a unique matrix in rational canonical form.*

**2.3. The  $m^{\text{th}}$ -Toeplitz matrix,  $m \in \mathbb{N}^*$**

**Definition 2.1** A square matrix of order  $n$  is called Upper Toeplitz if it is of the form.

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ 0 & a_1 & a_2 & & a_{n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & \cdots & & a_1 & a_2 \\ 0 & \cdots & 0 & 0 & a_1 \end{bmatrix}_{n \times n}$$

**Definition 2.2** Let  $A \in \mathcal{M}_n(K)$ ,  $u \in K^n$  and  $m \in \mathbb{N}^*$ . The  $n \times m$  rectangular matrix  $u \otimes_m A$  whose columns correspond to  $u, Au, \dots, A^{m-1}u$  is called the  $m^{\text{th}}$  Toeplitz matrix associated to  $A$  and  $u$ . That is  $u \otimes_m A := [u|Au|\dots|A^{m-1}u]$ .

**Proposition 2.1** If  $N_n$  is the  $n \times n$  basic nilpotent Jordan matrix( see[[7](#)],p.38) then  $u \otimes_n N_n^\top$  is similar to the usual upper Toeplitz matrix.

**Proof:** Let  $u = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$  and  $N_n = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & \cdots & & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix}_{n \times n}$

Then  $N_n^\top = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & \vdots \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

and by straightforward calculations we get

$$u \otimes_n N_n^\top = \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 \\ a_2 & a_1 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots \\ a_n & a_{n-1} & \cdots & a_2 & a_1 \end{bmatrix}_{n \times n}$$

which is similar to  $\begin{bmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ 0 & a_1 & a_2 & & a_{n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & \cdots & & a_1 & a_2 \\ 0 & \cdots & 0 & 0 & a_1 \end{bmatrix}_{n \times n}$  as it is its transpose. □

**Theorem 2.3** Let  $A \in \mathcal{M}_n(K)$ . If  $A$  is a companion matrix then  $u \otimes_n A$  is a polynomial in  $A$  for every  $u \in K^n$ .

To prove the above Theorem, we need the following Lemma

**Lemma 2.1** ([\[1\]](#),**Proposition 4.41,p 237**) Let  $A \in M_n(K)$  be a companion matrix, and let  $T : K^n \rightarrow K^n$  be defined by multiplication by  $A$ , i.e.,  $T(v) = Av$  where  $K^n$  is identified with  $M_{n,1}(K)$ . Then there exists a vector  $e \in K^n$  such that  $\mathcal{B} = (e, Te, \dots, T^{n-1}e)$  is an ordered basis of  $K^n$ .

**Proof of the Theorem** Let  $A \in \mathcal{M}_n(K)$ . If  $A \in \mathcal{M}_n(K)$  is a companion matrix, by Lemma 2.1 there exists a linear transformation  $T : K^n \rightarrow K^n$  and  $e \in K^n$  such that the matrix  $[f]_{\mathcal{B}} = A$  where  $\mathcal{B} = (e_1, \dots, e_n)$  is an ordered basis of  $V$  such that  $e_i = T^{n-i}(e)$ , for all  $1 \leq i \leq n$ .

Let  $u = \sum_{i=1}^n a_i e_i \in K^n$ , then  $u = \sum_{i=1}^n a_i T^{n-i}(e)$ , so  $u = P(T)(e)$  where  $P = \sum_{i=1}^n a_i X^{n-i}$ .

Let  $T' : K^n \rightarrow K^n$  be the linear transformation defined by  $T'(e_i) = T^{n-i}(u)$ , for all  $1 \leq i \leq n$ . Notice that  $u \otimes_n A = [T']_{\mathcal{B}}$ .

**Claim 1**  $T' = P(T)$ .

Indeed, for all  $1 \leq i \leq n$  we have

$$\begin{aligned} T'(e_i) &= T^{n-i}(u) \\ &= T^{n-i}(P(T)(e)) \\ &= P(T)(T^{n-i}(e)) \\ &= P(T)(e_i) \end{aligned}$$

Hence  $u \otimes_n A = P(A)$ .

Let  $V$  be a  $K$ -vector subspace of the vector space  $K^n$  and  $A \in \mathcal{M}_n(K)$ . Let  $V \otimes_m A$  be the set of all  $m^{\text{th}}$  Toeplitz matrices i.e.,  $V \otimes_m A := \{u \otimes_m A \in \mathcal{M}_{(n,m)}(K) \mid u \in V\}$ .

### 3. Main Results

**Theorem 3.1** Let  $A \in \mathcal{M}_n(K)$  and  $B \in \mathcal{M}_m(K)$ . If  $B = \text{Comp}(Q)$  where  $Q \in K[X]$  is a monic polynomial, then

$$\mathcal{C}(A, B) = \text{Ker}(Q(A)) \otimes_m A$$

In other words  $M \in \mathcal{C}(A, B)$  if and only if there exists  $u \in \text{Ker}(Q(A))$  such that  $M = u \otimes_m A$ .

**Theorem 3.2** Let  $A \in \mathcal{M}_n(K)$  be a nonzero matrix and  $q_1, \dots, q_r$  its invariant factors. Set  $A_i = \text{Comp}(q_i)$  and  $d_i = \deg q_i$  for  $1 \leq i \leq r$ .

Then

$$\mathcal{C}(A) \simeq \bigoplus_{i=1}^r \bigoplus_{j=1}^r \mathcal{C}(A_i, A_j).$$

where

$$\mathcal{C}(A_i, A_j) = \begin{cases} \text{Ker} q_j(A_i) \otimes_{d_j} A_i & \text{if } i \neq j \\ K[A_i] & \text{if } i = j \end{cases}$$

To prove Theorem 3.1 and Theorem 3.2 we need the following Lemmas:

**Lemma 3.1** Let  $A \in \mathcal{M}_n(K)$  and  $B \in \mathcal{M}_m(K)$ . If  $B = \text{Comp}(Q)$  where  $Q \in K[X]$  is a monic polynomial, then

$$\text{Ker}(Q(A)) \otimes_m A \subseteq \mathcal{C}(A, B)$$

**Proof:** Set  $B = \text{Comp}(Q)$  where  $Q = X^m + \sum_{i=1}^m a_{i-1} X^{i-1}$  and  $V = K^m$ , then there exists an ordered basis  $\mathcal{B}_1 = (e_1, \dots, e_m)$  of  $V$  and a linear transformation  $f : V \rightarrow V$  such

$$\begin{cases} f(e_i) &= e_{i+1} \text{ if } i < m \\ f(e_m) &= -\sum_{j=1}^m a_{j-1} e_j \end{cases}$$

Set  $W = K^n$ , there exists an ordered basis  $\mathcal{B}_2$  of  $W$  and a linear transformation  $g : W \rightarrow W$  such that the matrix  $[g]_{\mathcal{B}_2} = A$ . Let  $u \in V$  and define the linear transformation  $h : W \rightarrow V$  by  $h(e_i) = g^{i-1}(u)$ , for  $1 \leq i \leq m$ . Notice that the matrix  $[h]_{\mathcal{B}_1, \mathcal{B}_2} = u \otimes_m A$ , hence

$$u \otimes_m A \in \mathcal{C}(A, B) \text{ if and only if } g \circ h = h \circ f.$$

**Claim 2**

$$g \circ h = h \circ f \text{ if } u \in \text{Ker}(Q(g)).$$

Indeed for all  $1 \leq i < m$ ,

$$\begin{aligned} (g \circ h)(e_i) &= g(g^{i-1}(u)) \\ &= g^i(u) \\ &= h(e_{i+1}) \\ &= (h \circ f)(e_i) \end{aligned}$$

For  $i = m$ ,

$$\begin{aligned} (g \circ h)(e_m) &= g(g^{m-1}(u)) \\ &= g^m(u) \end{aligned}$$

As  $f(e_m) = -\sum_{i=1}^m a_{i-1}e_i$  then

$$\begin{aligned} (h \circ f)(e_m) &= -\sum_{i=1}^m a_{i-1}h(e_i) \\ &= -\sum_{i=1}^m a_{i-1}g^{i-1}(u) \end{aligned}$$

So if  $u \in \text{Ker}(Q(A))$  i.e.,  $u \in \text{Ker}(Q(g))$  then  $g^m(u) + \sum_{i=1}^m a_{i-1}g^{i-1}(u) = 0$  i.e.,  $g^m(u) = -\sum_{i=1}^m a_{i-1}g^{i-1}(u)$  and hence  $(g \circ h)(e_m) = (h \circ f)(e_m)$ .

In conclusion if  $u \in \text{Ker}(Q(A))$  then  $g \circ h = h \circ f$  i.e.,  $u \otimes_m A \in \mathcal{C}(A, B)$ . □

**Lemma 3.2** *Let  $A \in \mathcal{M}_n(K)$  and  $B \in \mathcal{M}_m(K)$  be nonzero matrices. If  $B = \text{Comp}(Q)$  where  $Q \in K[X]$  is a monic polynomial Then*

$$\dim_K \mathcal{C}(A, B) = \sum_{i=1}^r \deg(\gcd(q_i, Q))$$

. where  $q_1, \dots, q_r$  are the invariant factors of  $A$ .

**Proof:** From [[1], Theorem 5.15, p. 336]

$$\dim_K \mathcal{C}(A, B) = \sum_{i,j} \deg(\gcd(q_i(A), q_j(B)))$$

where and  $q_i(A)$  (resp.  $q_i(B)$ ) are the invariant factors of  $A$  (resp. of  $B$ ). In particular, when  $B = \text{Comp}(Q)$ , it has a unique invariant factor, namely  $.Q$  □

**Lemma 3.3** [see [3]] *Let  $A \in \mathcal{M}_n(K)$  be a non zero matrix and  $q_1, \dots, q_r$  its invariant factors. Then*

$$\dim_K(\text{Ker}P(A)) = \sum_{i=1}^r \deg(\gcd(P, q_i))$$

for any  $P \in K[X]$ .

**Lemma 3.4** *Let  $A \in \mathcal{M}_n(K)$  and  $V$  a subspace of  $K^n$ . Then  $V \otimes_m A$  is a vector subspace of the  $K$  vector space  $\mathcal{M}_{(n,m)}(K)$  and*

$$\dim_K(V \otimes_m A) = \dim_K V$$

**Proof:** The map  $\psi : V \rightarrow V \otimes_m A$ ,  $u \mapsto u \otimes_m A$  is a one to one linear transformation.  $\square$

**Lemma 3.5** *Let  $A \in \mathcal{M}_n(K)$  and  $B \in \mathcal{M}_m(K)$ . If  $P \in \mathcal{M}_n(K)$  and  $Q \in \mathcal{M}_m(K)$  are two invertible matrices such that  $A' = PAP^{-1}$  and  $B' = QBQ^{-1}$ . Then  $\mathcal{C}(A, B) \simeq \mathcal{C}(A', B')$ .*

**Proof:** Indeed, If  $A' = PAP^{-1}$ ,  $B' = QBQ^{-1}$  and  $M \in \mathcal{C}(A, B)$ , we have

$$\begin{aligned} T \in \mathcal{C}(A, B) &\text{ iff } AM = MB \\ &\text{ iff } PAM = PMB \\ &\text{ iff } PAMQ^{-1} = PMBQ^{-1} \\ &\text{ iff } PAP^{-1}PMQ^{-1} = PMQ^{-1}QBQ^{-1} \\ &\text{ iff } A'M' = M'B' \\ &\text{ iff } M' \in \mathcal{C}(A', B'). \end{aligned}$$

with  $M' = PMQ^{-1}$ . Let  $\phi_{P,Q^{-1}} : \mathcal{C}(A, B) \rightarrow \mathcal{C}(A', B')$ , such that  $\phi_{P,Q^{-1}}(M) = M'$ . We see that  $\phi_{PQ^{-1}}$  is linear and that

$$\phi_{P^{-1},Q} \circ \phi_{P,Q^{-1}} = \text{id}$$

and

$$\phi_{P,Q^{-1}} \circ \phi_{P^{-1},Q} = \text{id}.$$

So  $\phi_{P,Q^{-1}}$  is an isomorphism. This shows that  $\mathcal{C}(A, B) \simeq \mathcal{C}(A', B')$ .  $\square$

**Lemma 3.6** *Let  $A \in \mathcal{M}_n(K)$  and  $B \in \mathcal{M}_n(K)$ . If  $A$  and  $B$  are similar such that  $A = P^{-1}BP$ , then*

$$\mathcal{C}(A) = P^{-1}\mathcal{C}(B)P$$

**Proof:** Consequence of Lemma 3.5.  $\square$

**Lemma 3.7** *Let  $A \in \mathcal{M}_n(K)$ . If  $A$  is a block diagonal matrix i.e.,  $A = \text{diag}(A_1, A_2, \dots, A_m)$  where  $A_i$  are  $n_i \times n_i$  square matrices such that  $n_1 + \dots + n_r = n$ . Then the centralizer of the matrix  $A$  is*

$$\mathcal{C}(A) \simeq \bigoplus_{i=1}^r \bigoplus_{j=1}^r \mathcal{C}(A_i, A_j).$$

**Proof:** Let  $T \in \mathcal{M}_n(K)$ . Write  $T = (T_{ij})$  as a  $r \times r$  block matrix with the same block structure as  $A$ . If  $T \in \mathcal{C}(A)$  then  $A_i T_{ij} = T_{ij} A_j$  hence  $T_{ij} \in \mathcal{C}(A_i, A_j)$ . Conversely if  $T = (T_{ij})$  with  $T_{ij} \in \mathcal{C}(A_i, A_j)$ . Then  $AT = (A_i T_{ij})$  and  $TA = (A_j T_{ij})$ , since  $T_{ij} \in \mathcal{C}(A_i, A_j)$  then  $T \in \mathcal{C}(A)$ .  $\square$

### Proof of Theorem 2.3

By Lemma 3.2, Lemma 3.1, Lemma 3.3 and Lemma 3.4 we get the desired result.

### Proof of Theorem 3.1

Indeed, by theorem 2.1  $A$  is similar to a diagonal matrix  $\text{diag}(\text{Comp}(q_1), \dots, \text{Comp}(q_r))$  and by Lemma 3.1 we have  $\mathcal{C}(A) \simeq \bigoplus_{i=1}^r \bigoplus_{j=1}^r \mathcal{C}(A_i, A_j)$ . where  $A_i = \text{Comp}(q_i)$ . Hence  $\mathcal{C}(A_i, A_i) = \mathcal{C}(A_i) = K[A_i]$  if  $i = j$ . If  $i \neq j$  and by applying the Lemma 2.1 we get  $\mathcal{C}(A_i, A_j) = \text{Ker}q_j(A_i) \otimes_{d_j} A_i$ .

**Corollary 3.1** *Let  $N_n$  (resp.  $N_m$ ) be the  $n \times n$  (resp.  $m \times m$ ) the basic nilpotent Jordan matrix.*

- If  $n \leq m$  then

$$\mathcal{C}(N_n, N_m) \simeq \left\{ X \in \mathcal{M}_{(n,m)}(K) \mid X = \begin{bmatrix} T_{n \times n} & 0_{n \times (m-n)} \end{bmatrix} \right\}$$

- If  $m \leq n$  then

$$\mathcal{C}(N_n, N_m) \simeq \left\{ X \in \mathcal{M}_{(n,m)}(K) \mid X = \begin{bmatrix} 0_{(n-m) \times m} \\ T_{m \times m} \end{bmatrix} \right\}$$

Where  $T_{s \times s}$ , for a natural number  $s$ , is an upper Toeplitz matrix of order  $s$ , over the field  $K$ .

**Proof:**

As  $N_n$  is similar to  $N_n^\top$  and  $N_m$  is similar to  $N_m^\top$  then  $\mathcal{C}(N_n, N_m) \simeq \mathcal{C}(N_n^\top, N_m^\top)$  and since  $N_m^\top = \text{Comp}(X^m)$  then

$$\begin{aligned} \mathcal{C}(N_n^\top, N_m^\top) &= \mathcal{C}(N_n^\top, \text{Comp}(X^m)) \\ &= \text{Ker}((N_n^\top)^m) \otimes_m N_n^\top \text{ (by Theorem 3.1)} \end{aligned}$$

Hence,

- If  $n \leq m$  then  $(N_n^\top)^m = 0_n$  then  $\text{Ker}((N_n^\top)^m) = K^n$  hence,

$$\begin{aligned} \mathcal{C}(N_n^\top, N_m^\top) &= K^n \otimes_m N_n^\top \\ &= \left\{ X \in \mathcal{M}_{(n,m)}(K) \mid X = \begin{bmatrix} T_{n \times n} & 0_{n \times (m-n)} \end{bmatrix} \right\} \end{aligned}$$

- If  $m \leq n$  then by Lemma 3.4

$$\begin{aligned} \dim_K \text{Ker}((N_n^\top)^m) &= \deg(\gcd(X^n, X^m)) \\ &= \deg(X^m) \\ &= m \end{aligned}$$

So  $\text{Ker}((N_n^\top)^m) \simeq K^m$  hence,

$$\begin{aligned} \mathcal{C}(N_n^\top, N_m^\top) &\simeq K^m \otimes_m N_n^\top \\ &= \left\{ X \in \mathcal{M}_{(n,m)}(K) \mid X = \begin{bmatrix} 0_{(n-m) \times m} \\ T_{m \times m} \end{bmatrix} \right\} \end{aligned}$$

Where  $T_{s \times s}$ , for a natural number  $s$ , is an upper Toeplitz matrix of order  $s$ , over the field  $K$ . □

The Corollary 3.1 confirms the P. Singla result in [[8], lemma 2.2.3, p13].

**Corollary 3.2**  $\mathcal{C}(N_n) \simeq \{X \in \mathcal{M}_n(K) \mid X = T_{n \times n}\}$  Where  $T_{n \times n}$  is an upper Toeplitz Matrix of order  $n$ , over the field  $K$ .

## 4. Applications

### 4.1. Computational example

Let  $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 1 & -3 \\ -1 & 1 & 5 \end{bmatrix}$  and  $K = \mathbb{R}$ . The invariant factors of  $A$  are  $q_1 = X - 2$  and  $q_2 = X^2 - 5X + 6$ . Set  $A_1 = \text{Comp}(q_1) = [2]$  and  $A_2 = \text{Comp}(q_2) = \begin{bmatrix} 0 & -6 \\ 1 & 5 \end{bmatrix}$  Then  $A$  is similar to the matrix  $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & -6 \\ 0 & 1 & 5 \end{bmatrix}$  Hence,

$$\begin{aligned} \mathcal{C}(A) &\simeq \mathcal{C}(B) \\ &= \left\{ \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \mid T_{ij} \in \mathcal{C}(A_i, A_j) \right\} \end{aligned}$$

$$\begin{aligned}
\mathcal{C}(A_1, A_1) &= \mathbb{R}[A_1] & \mathcal{C}(A_1, A_2) &= \text{Ker}q_2(A_1) \otimes_2 A_1 \\
&= \{f(A_1) \mid f \in \mathbb{R}[X], \text{ with } \text{degree}(f) < 1 \mid\} & &= \{[u, A_1]u \mid u \in \text{Ker}q_2(A_1)\} \\
&= \{[a] \mid a \in \mathbb{R}\} & &= \{[\alpha, 2\alpha] \mid \alpha \in \mathbb{R}\} \text{ Since}
\end{aligned}$$

$$q_2(A_1) = [0] \text{ and } \text{Ker}(q_2(A_1)) = \mathbb{R}$$

$$\begin{aligned}
\mathcal{C}(A_2, A_1) &= \text{Ker}q_1(A_2) \otimes_1 A_2 \\
&= \left\{ [u] \mid u \in \text{span} \left( \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right) \right\} & q_1(A_2) &= \begin{bmatrix} -2 & -6 \\ 1 & 3 \end{bmatrix} \text{ and } \text{Ker}(q_1(A_2)) = \text{span} \left( \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right) \\
&= \left\{ \begin{bmatrix} -3\beta \\ \beta \end{bmatrix} \mid \beta \in \mathbb{R} \right\}
\end{aligned}$$

$$\begin{aligned}
\mathcal{C}(A_2, A_2) &= \mathbb{R}[A_2] \\
&= \{f(A_2) \mid f \in \mathbb{R}[X], \text{ with } \text{degree}(f) < 2\} \\
&= \{bA_2 + cI_2 \mid b, c \in \mathbb{R}\} & \text{Hence} \\
&= \left\{ \begin{bmatrix} c & -6b \\ b & 5b + c \end{bmatrix} \mid a, b \in \mathbb{R} \right\}
\end{aligned}$$

$$\mathcal{C}(B) = \left\{ \begin{bmatrix} a & \alpha & 2\alpha \\ -3\beta & c & -6b \\ \beta & b & 5b + c \end{bmatrix} \mid a, b, c, \alpha, \beta \in \mathbb{R} \right\}$$

we have  $A = P^{-1}BP$  where  $P = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 0 \\ -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{2} \end{bmatrix}$ , then by Lemma 3.6

$$\mathcal{C}(A) = P^{-1}\mathcal{C}(B)P$$

$$\begin{aligned}
\mathcal{C}(A) &= \left\{ P^{-1} \begin{bmatrix} a & \alpha & 2\alpha \\ -3\beta & c & -6b \\ \beta & b & 5b + c \end{bmatrix} P \mid a, b, c, \alpha, \beta \in \mathbb{R} \right\} \\
&= \left\{ \begin{bmatrix} b + c - 3\beta & b & 3b + 3\beta \\ 3a - 5b - 3c + 2\alpha + 6\beta & b + c - \alpha & -3a + 3b + 3c - 3\alpha - 6\beta \\ \frac{-3a + 3b + 3c - 2\alpha - 9\beta}{3} & \frac{3b + \alpha}{3} & a + 3b + \alpha + 3\beta \end{bmatrix} \mid a, b, c, \alpha, \beta \in \mathbb{R} \right\}
\end{aligned}$$

One can check that

$$\begin{aligned}
AC &= CA \\
&= \begin{bmatrix} -b + c - 6\beta & 5b + c & 15b + 3c + 6\beta \\ \frac{6a - 7b - 5c + 4\alpha + 12\beta}{3} & -b + c - 2\alpha & -6a - 3b + 3c - 6\alpha - 12\beta \\ \frac{-6a - 3b + 3c - 4\alpha - 18\beta}{3} & \frac{15b + 3c + 2\alpha}{3} & 2a + 15b + 3c + 2\alpha + 6\beta \end{bmatrix}
\end{aligned}$$

$$\text{where } C = \begin{bmatrix} b + c - 3\beta & b & 3b + 3\beta \\ 3a - 5b - 3c + 2\alpha + 6\beta & b + c - \alpha & -3a + 3b + 3c - 3\alpha - 6\beta \\ \frac{-3a + 3b + 3c - 2\alpha - 9\beta}{3} & \frac{3b + \alpha}{3} & a + 3b + \alpha + 3\beta \end{bmatrix}$$

## 4.2. The wild problem

In linear algebra and representation theory, a problem is wild if it involves the problem of classifying pairs of square matrices up to simultaneous similarity.

**Definition 4.1** Two pairs  $(A_1, A_2)$  and  $(B_1, B_2)$  of matrices of  $\mathcal{M}_n(K)$  are said to be simultaneously similar when there exists  $P \in GL_n(K)$  such that

$$PA_1P^{-1} = B_1 \text{ and } PA_2P^{-1} = B_2$$

**Lemma 4.1** *Let  $A, B \in \mathcal{M}_n(K)$  such that  $PAP^{-1} = B$ ,  $P \in GL_n(K)$ . Then  $\forall P_1 \in GL_n(K)$*

$$P_1AP_1^{-1} = B \text{ if and only if } P_1 \in PC(A)$$

**Proof:** If  $PAP^{-1} = B$  then  $\forall P_1 \in GL_n(K)$

$$\begin{aligned} P_1AP_1^{-1} = B & \text{ if and only if } & P_1A = BP_1 \\ & \text{ if and only if } & P_1A = PAP^{-1}P_1 \\ & \text{ if and only if } & (P^{-1}P_1)A = A(P^{-1}P_1) \\ & \text{ if and only if } & P^{-1}P_1 \in \mathcal{C}(A) \\ & \text{ if and only if } & P_1 \in PC(A) \end{aligned}$$

□

**Proposition 4.1** *Let  $(A_1, A_2)$  and  $(B_1, B_2)$  be two pairsof matrices of  $\mathcal{M}_n(K)$ . If  $PA_1P^{-1} = B_1$  and  $QA_2Q^{-1} = B_2$  where  $P, Q \in GL_n(K)$  Then  $(A_1, A_2)$  and  $(B_1, B_2)$  are simultaneously similar if and only if*

$$PC(A_1) \cap QC(A_2) \neq \emptyset$$

**Proof:** A consequence of the lemma above.

□

**Remark 4.1** *As  $PC(A_1)$  and  $QC(A_2)$  are  $k$ -vector spaces, Then, the problem of simultaneous similarity is reduced to determine if the intersection of two vector spaces is empty or not.*

## 5. Conclusion

In this paper, we propose a method for computing the centralizer of a given matrix over a field  $K$ . This method is based on the invariant factors of the matrix and  $m^{th}$ -Toeplitz matrices, where  $m \in \mathbb{N}^*$ . Additionally, we provide a weak solution to the wild similarity problem.

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## References

1. W. A. Adkins and S. H. Weintraub, *Algebra: An Approach Via Module Theory*, vol. 136, Springer Science & Business Media, 2012.
2. Belitskii, Genrich R., and Vladimir V. Sergeichuk. *Complexity of matrix problems*. Linear Algebra and its applications 361 (2003): 203-222.
3. S. Bouarga and M. E. Charkani, *On primary decomposition and polynomial of matrices*, International Journal of Algebra, Vol. 8, 2014, no. 4, 149 - 157.
4. M. Charkani and S. Bouarga, *Polynomial solution of sylvester matrix equation*, International Journal of Algebra 9(2015), 185-194.
5. S. Lang. *Algebrat*, Graduate Texts in Mathematics, New York, USA (2002), 572-585.
6. R. Lal, *Algebra 2: Linear Algebra, Galois Theory, Representation Theory, Group Extensions and Schur Multiplier*, Springer, 2017.
7. K. O'Meara, J. Clark and C. Vinsonhaler, *Advanced Topics in Linear Algebra: Weaving Matrix Problems Through The Weyr Form*, Oxford University Press, 2011.
8. P. Singla, *On representations of general linear groups over principal ideal local rings of length two*, J. Algebra. 324(2010), no. 9, 2543-2563.
9. Wang, Tianhao. *Explicit description of centralizers for a matrix*. arXiv preprint arXiv:1910.13666 (2019).

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