Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–9.
ISSN-0037-8712
doi:10.5269/bspm.65339

Fuzzy Two Point Boundary Value Problem with Linear System

Tahir Ceylan

ABSTRACT: In this manuscript we present two different methods of solving second order linear differential equation that has fuzzy boundary conditions. First, by taking each fuzzy boundary point of the fuzzy boundary value problem (FBVP) as the fuzzy initial point, we will obtain two separate fuzzy initial value problems (FIVPs). Second, we solve each of these FIVPs using the system of differential equations. We provide fuzzy solutions for this system based on an extension of the classical solution via Zadeh's extension principle. We present an example in order to compare the proposed solution.

Key Words: linear system, fuzzy boundary value problem, fuzzy eigenfunction, Zadeh's extension principle.

Contents

1. Introduction

In this paper we consider the fuzzy two point boundary value problem (FTPBVP)

$$L = -\frac{d^2}{dx^2}$$
$$L\widehat{u} = \lambda \widehat{u}, \quad t \in [a, b] \tag{1.1}$$

which satisfies the conditions

$$\widehat{a}_1 \widehat{u}(a) = \widehat{a}_2 \widehat{u}'(a) \tag{1.2}$$

$$\widehat{b}_1 \widehat{u}(b) = \widehat{b}_2 \widehat{u}'(b) \tag{1.3}$$

where $\widehat{a}_1, \widehat{a}_2, \widehat{b}_1, \widehat{b}_2$ non-negative triangular fuzzy numbers, $\lambda > 0$, at least one of the numbers $\widehat{a}_1$ and $\widehat{a}_2$ and at least one of the numbers $\widehat{b}_1$ and $\widehat{b}_2$ are nonzero.

Initial value problems (IVPs) and boundary value problems (BVPs) are two well-known classes of ordinary differential equations [7]. IVPs and BVPs are used as mathematical tools to describe phenomena in several areas of science, such as engineering, physics, medicine etc [5,9,11]. Since some parameters of these problems can be uncertain, many researchers have studied IVPs and BVPs where one or more parameters and/or state variables are given by fuzzy numbers. These problems are called fuzzy Initial value problems (FIVPs) and boundary value problems (FBVPs) [4,6,17,20,21]. Several methods can be employed to solve these problems [1,10,26,13,14] and the solutions can based on Zadeh's extension principle [25].

Submitted October 07, 2022. Published April 20, 2025
2010 *Mathematics Subject Classification:* 34B09; 34L20.

The studies of FTPBVP have been made with the Hukuhara derivative [2,12,15] and generalized Hukuhara derivative [8]. But in some cases the fuzzy solutions with Hukuhara derivative suffer from the defect of having solutions with increasing diameters over time. This means that as time passes, the more fuzzy (uncertain) the process becomes [3,16] and the fuzzy solutions with generalized Hukuhara derivative have some not interval solutions which are associated with the existence of switch points [22]. For this reason, the solution method for this problem has been carried out under limited conditions and limited domain.

The present work focuses on a system of differential equations that describes the problem. More precisely we consider a system of ordinary second order differential equation with initial conditions given by fuzzy numbers. Here, the initial conditions are obtained from the boundary conditions in this (1.1)-(1.3) problem with the help of Titchmarsh's method [23].

This paper is divided as follows. In Section 2 we recall some basic concepts regarding fuzzy set theory and classic solution to the problem. In Section 3 we provide fuzzy solutions for FTPBVP given by second order linear system and in Section 4 we present an example of an FTPBVP in order to illustrate the main contributions of this study.

2. Preliminaries

2.1. Solution for a crisp boundary value problem

Let' s consider the fuzzy problem (1.1)-(1.3) as the crisp problem. Then we shall make use of solutions of (1.1) defined by initial conditions instead of boundary conditions in a manner similar to Titchmarsh [23].

We define two solutions $\Phi_\lambda(t)$ and $\Psi_\lambda(t)$ of the equation (1.1) as follows. Let $\Phi_\lambda(t) = \Phi(t,\lambda)$ be the solution of equation (1.1) on $[a,b]$, which satisfies the initial conditions

$$\begin{pmatrix} u(a) \\ u'(a) \end{pmatrix} = \begin{pmatrix} a_2 \\ a_1 \end{pmatrix} \tag{2.1}$$

and $\Psi_\lambda(t) = \Psi(t,\lambda)$ be the solution of equation (1.1) on $[a,b]$, which satisfies the initial conditions

$$\begin{pmatrix} u(b) \\ u'(b) \end{pmatrix} = \begin{pmatrix} b_2 \\ b_1 \end{pmatrix}. \tag{2.2}$$

For each fixed t these functions and derivatives are entire in λ.

These two solutions $\Phi_\lambda(t)$ and $\Psi_\lambda(t)$ of the equation (1.1) are obtained by using the solution methodology of Systems Of Differential Equations .

Based on classical differential equations theory, a second-order linear and homogeneous system can be transformed into a set of two first-order linear equations, using appropriated state variable substitutions. Thus, this type of linear system can be represented in matrix form using the substitution $u_1 = u(t)$, $u_2 = u'(t)$ given by [19]

$$u'(t) = Ku(t) \tag{2.3}$$

where $u = [u_1 \ \ u_2]^T$, $u' = [u_1' \ \ u_2']^T$ and K is a 2×2 matrix which includes all of the constant values λ. The general solution of the system 2.3 is given by [19]

$$u(t) = c_1 u_1(t) + c_2 u_2(t)$$

for all $t \in [t_0, T]$, where u_1, u_2 are two linearly independent vectorial solutions of the system 2.3 and c_1, c_2 are real constants. Let $v_{1,2} = a \pm ib$ be the eigenvectors associated with the eigenvalues $r_{1,2} = \alpha \pm i\beta$ of the real matrix K . Thus the independent solutions u_1, u_2 are give as follows:

$$\begin{aligned} u_1(t) &= e^{\alpha t}\cos(\beta t)\, a - e^{\alpha t}\sin(\beta t)\, b \\ u_2(t) &= e^{\alpha t}\sin(\beta t)\, a + e^{\alpha t}\cos(\beta t)\, b \end{aligned} \tag{2.4}$$

where $r_1 = a + i\beta$, $r_2 = a - i\beta$. Therefore the general solution is given by

$$u(t) = U(t)\, c \tag{2.5}$$

for all $t \in [t_0, T]$, where $c = [c_1 \quad c_2]$ and $U(t) = [u_1(t) \quad u_2(t)]$ which is called as fundamental matrix. Considering the initial condition $u(t_0) = u_0$, for the system of (2.3), we have an IVP. If the matrix $U(t_0) = M$ is non-singular [24] then the constant vector c can be uniquely determined as follows

$$c = U(t_0)^{-1} u_0 = M^{-1} u_0.$$

Therefore, we can rewrite the general solution (2.5), in terms of the initial conditions

$$u(t) = U(t) M^{-1} u_0 \tag{2.6}$$

Finally we recall that the required solution is obtained in the first line of the vectorial solution $u(t)$, respectively.

So the solutions $\Phi_\lambda(t)$ and $\Psi_\lambda(t)$ obtained from the above solution method put

$$W(\lambda) = \Phi_\lambda(t)\, \Psi'_\lambda(t) - \Phi'_\lambda(t)\, \Psi_\lambda(t) \tag{2.7}$$

which is independent of $x \in [a, b]$.

Lemma 2.1 *If $\lambda = \lambda_0$ is an eigenvalue, then $\Phi(t, \lambda_0)$ and $\Psi(t, \lambda_0)$ are linearly independent and eigenfunctions corresponding to this eigenvalue [23].*

Theorem 2.1 *The eigenvalues of the problem (1.1)-(1.3) are the zeros of the function $W(\lambda)$ [23].*

In section 3, we will use in a manner similar method of Titchmarsh to define a fuzzy solution for second order two point boundary values problems with fuzzy boundary values.

Before we introduce our approach to solve a FBVP, it is necessary first to review some concepts of fuzzy sets theory.

2.2. Basic concepts of fuzzy sets

Definition 2.1 *Let E be a universal set. A fuzzy subset $\widehat{A}$ of E is given by its membership function $\mu_{\widehat{A}} : E \to [0, 1]$, where $\mu_{\widehat{A}}(t)$ represents the degree to which $t \in E$ belongs to $\widehat{A}$. We denote the class of the fuzzy subsets of E by the symbol $F(E)$ [20].*

Definition 2.2 *The $\alpha-$ level of a fuzzy set $\widehat{A} \subseteq E$, denoted by $\left[\widehat{A}\right]^\alpha$, is defined as*

$$\left[\widehat{A}\right]^\alpha = \left\{x \in E : \widehat{A}(t) \geq \alpha\right\},$$

for all $\alpha \in (0, 1]$. If E is also topological space, then the $0-$ level is defined as the closure of the support of $\widehat{A}$,that is, $\left[\widehat{A}\right]^0 = \overline{\left\{x \in E : \widehat{A}(t) > 0\right\}}$. The $1-$ level of a fuzzy subset $\widehat{A}$ is also called as core of $\widehat{A}$ and denoted by $\left[\widehat{A}\right]^1 = core\left(\widehat{A}\right)$ [18].

Definition 2.3 *A fuzzy subset $\widehat{u}$ on $\mathbb{R}$ is called a fuzzy real number (fuzzy interval), whose $\alpha-$ cut set is denoted by $[\widehat{u}]^\alpha$, i.e., $[\widehat{u}]^\alpha = \{x : \widehat{u}(t) \geq 0\}$, if it satisfies two axioms:*

i. *There exists $r \in \mathbb{R}$ such that $\widehat{u}(r) = 1$,*

ii. *For all $0 < \alpha \leq 1$, there exist real numbers $-\infty < u^-_\alpha \leq u^+_\alpha < +\infty$ such that $[\widehat{u}]^\alpha$ is equal to the closed interval $[u^-_\alpha, u^+_\alpha]$ [21].*

The set of all fuzzy real numbers (fuzzy intervals) is denoted by $\mathbb{R}_F$.$F_K(\mathbb{R})$, the family of fuzzy sets of $\mathbb{R}$ whose $\alpha-$ cuts are nonempty compact convex subsets of $\mathbb{R}$. If $\widehat{u} \in \mathbb{R}_F$ and $\widehat{u}(t)$ whenever $t < 0$, then $\widehat{u} = 0$ is called a non-negative fuzzy real number and $\mathbb{R}^+_F$ denotes the set of all non-negative fuzzy real numbers. For all $\widehat{u} \in \mathbb{R}^+_F$ and each $\alpha \in (0, 1]$, real number u^-_α is positive.

Definition 2.4 *An arbitrary fuzzy number $\widehat{u}$ in the parametric form is represented by an ordered pair of functions $[u_\alpha^-, u_\alpha^+]$, $0 \leq \alpha \leq 1$, which satisfy the following requirements*

i. *u_α^- is bounded non-decreasing left continuous function on $(0,1]$ and right- continuous for $\alpha = 0$,*

ii. *u_α^+ is bounded non- increasing left continuous function on $(0,1]$ and right- continuous for $\alpha = 0$,*

iii. *$u_\alpha^- \leq u_\alpha^+$, $0 < \alpha \leq 1$ [1].*

Definition 2.5 *A fuzzy number $\widehat{A}$ is said to be triangular if the parametric representation of its $\alpha-$ level is of the form $\left[\widehat{A}\right]^\alpha = [(a_2 - a_1)\alpha + a_1, a_3 - (a_3 - a_2)\alpha]$, for all $\alpha \in [0,1]$, where $\left[\widehat{A}\right]^0 = [a_1, a_3]$ and core $\left(\widehat{A}\right) = a_2$. A triangular fuzzy number is denoted by the triple (a_1, a_2, a_3) [16].*

The Zadeh's extension principle is a mathematical method to extend classical functions to deal with fuzzy sets as input arguments [25] . For multiple fuzzy variables as arguments, the Zadeh's extension principle is defined as follows.

Definition 2.6 *Let $f : X_1 \times \cdots \times X_n \to Z$ a classical function and let $A_i \in_F (X_i)$, for $i = 1, ..., n$. The Zadeh's extension $\widehat{f}$ of f, applied to $(A_1, ..., A_n)$, is the fuzzy set $\widehat{f}\ (A_1, ..., A_n)$ of Z, whose membership function is defined by*

$$\widehat{f}(A_1, ..., A_n)(z) = \underset{(x_1,...,x_n)\in f^{-1}(z)}{sup} min\{A_1(x_1), ..., A_n(x_n)\},$$

where $f^{-1}(z) = \{(x_1, ..., x_n) \in X_1 \times \cdots \times X_n : f(x_1, ..., x_n) = z\}$.

We can apply the Zadeh's extension principle to define the standard arithmetic for fuzzy numbers [25]. Let $[\widehat{u}]^\alpha = [u_\alpha^-, u_\alpha^+]$ and $[\widehat{v}]^\alpha = [v_\alpha^-, v_\alpha^+]$. For all $\alpha \in [0,1]$ and $\lambda \in \mathbb{R}$, we have

$$\begin{aligned} [\widehat{u} \oplus \widehat{v}]^\alpha &= [\widehat{u}]^\alpha + [\widehat{v}]^\alpha = \{x + y : x \in [\widehat{u}]^\alpha, y \in [\widehat{v}]^\alpha\}, \\ [\lambda \odot \widehat{u}]^\alpha &= \lambda \odot [\widehat{u}]^\alpha = \{\lambda x : x \in [\widehat{u}]^\alpha\}. \end{aligned}$$

3. Solution Method of the FTPBVP

In this section we concern with how to solve the FTPBVP. To do this firstly we get two FIVPs using (1.2) and (1.3) fuzzy boundary conditions from method of Titchmarsh [23]. Then we solve this two FIVPs from method of Systems of ordinary differential equations. Here λ is crisp number and $\lambda = p^2$, $p > 0$.

We consider two FIVPs involving a crisp differential equation but with fuzzy initial values:

$$\begin{cases} u'' + \lambda u = 0 \\ u(a) = \widehat{a}_2 \\ u'(a) = \widehat{a}_1 \end{cases} \tag{3.1}$$

and

$$\begin{cases} u'' + \lambda u = 0 \\ u(b) = \widehat{b}_2 \\ u'(b) = \widehat{b}_1 \end{cases} \tag{3.2}$$

where $\widehat{a}_1 = (a_0, a_1, a_2)$, $\widehat{a}_2 = (a_1, a_2, a_3)$, $\widehat{b}_1 = (b_0, b_1, b_2)$,$\widehat{b}_2 = (b_1, b_2, b_3)$.

We define two solutions $\widehat{\Phi}_\lambda(t) = \widehat{\Phi}(t, \lambda)$ and $\widehat{\Psi}_\lambda(t) = \widehat{\Psi}(t, \lambda)$ of the FIVPs (3.1) and (3.2), respectively. To get these solutions, we solve this problems separately via linear system that represent fuzzy problems. Here, it is important to recall that we have assumed that the initial conditions may be uncertain to provide a more realistic representation of the problem. So we consider the following FIVP

$$\begin{cases} u'(t) = Ku(t) \\ u(0) = \widehat{u}_0 \end{cases} \tag{3.3}$$

where $\widehat{u}_0 = [u_0 \ \ u_1]^T$ is the vector of initial fuzzy conditions.

Let $\Phi(t,\lambda)$ and $\Psi(t,\lambda)$ be the reel solution of the associated IVP, given in (2.4), where Φ_0, Ψ_0 is the initial condition. Firstly,we obtain the solution $\Phi(t,\lambda)$. Then we obtain $\Psi(t,\lambda)$ by similar methods. Let U be an open set in $\mathbb{R}^2$ such that there exists a solution $\Phi(.,\Phi_0)$ of (3.3) with $\Phi_0 \in U$ in the interval $[t_0,T]$, and for all $t \in [t_0,T]$, $\Phi(t,.)$ *is* continuous on U [27]. For each t, let be the operator $S_t : U \to \mathbb{R}^2$ given by $S_t(\Phi_0) = \Phi(t,\Phi_0)$ and let J Zadeh's extension principle of $\widehat{a}_2,\ \widehat{a}_1 \in \mathbb{R}_F$. Then we obtain the following solution called the Zadeh's fuzzy solution [27]

$$\begin{aligned}\widehat{\Phi}(t) &= (S_t)_{(J)}\left(\widehat{\Phi}_0\right), \\ &= n_1(t)\,\widehat{a}_1 + n_2(t)\,\widehat{a}_2 \end{aligned} \tag{3.4}$$

for all $t \in [t_0,T]$ and $N(t) = \Phi(t)\,M^{-1} = [n_1(t) \quad n_2(t)]$ where $n_1(t)$, $n_2(t)$ are column vectors of the matrix $N(t)$.

From the above theory, the second order ODE in (3.1) can be rewritten in matrix form using the substitution $\Phi_1 = \Phi(t)$ and $\Phi_2 = \Phi'(t)$ which gives

$$\begin{pmatrix} \Phi_1' \\ \Phi_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\lambda & 0 \end{pmatrix}\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \tag{3.5}$$

The independent vector solutions of (3.5) are also established from (2.4) with the eigenvalues $\lambda_{1,2} = \pm\sqrt{\lambda}$ and their associated eigenvectors $u_{1,2} = \pm i\begin{bmatrix}0 & \sqrt{\lambda}\end{bmatrix}^T$. Then, the fundamental matrix of the system given by (3.5) is

$$\widehat{\Phi}(t) = \begin{pmatrix} \cos(\sqrt{\lambda}t) & \sin(\sqrt{\lambda}t) \\ -\sqrt{\lambda}\sin(\sqrt{\lambda}t) & \sqrt{\lambda}\cos(\sqrt{\lambda}t) \end{pmatrix}. \tag{3.6}$$

Now let us consider $t_0 = a$ and the fuzzy initial conditions given by the fuzzy numbers $u(a) = \widehat{a}_2 = (a_1;a_2;a_3)$ and $u'(a) = \widehat{a}_1 = (a_0;a_1;a_2)$ .Then using (3.6), the fuzzy solution $\widehat{\Phi}(t)$ is obtained from (3.4) such that

$$\widehat{\Phi}_\lambda(t) = n_1(t)\,\widehat{a}_2 + n_2(t)\,\widehat{a}_1 = \cos(pt)\,\widehat{a}_2 + \frac{1}{p}\sin(pt)\,\widehat{a}_1. \tag{3.7}$$

Similarly we find $\widehat{\Psi}(t)$ such that

$$\widehat{\Psi}_\lambda(x) = \cos(pt - pb)\,\widehat{b}_2 + \frac{1}{p}\sin(pt - pb)\,\widehat{b}_1. \tag{3.8}$$

Then (3.7) and (3.8) have a unique solution $\widehat{\Phi}_\lambda(t)$ and $\widehat{\Psi}_\lambda(t)$ respectively from [23].

So, putting (3.7) and (3.8) solutions the above equation in (2.3) ,we get crisp Wronskian function as

$$W(\lambda) = \left(a_2 b_2 p + \frac{a_1 b_1}{p}\right)\sin(pb) + (a_1 b_2 - a_2 b_1)\cos(pb). \tag{3.9}$$

Definition 3.1 *The values of the parameter λ are called the eigenvalues of (1.1)-(1.3) if the equation (1.1) has the nontrivial solutions satisfying (1.2-1.3). These corresponding solutions are called fuzzy eigenfunctions of (1.1)-(1.3) fuzzy problem [12].*

Theorem 3.1 *The roots of equation (3.9) coincide with the eigenvalues of the fuzzy boundary value problem (1.1)-(1.3) [12].*

4. Example

Consider the two point fuzzy boundary value problem

$$-\widehat{u}'' = \lambda\widehat{u} \tag{4.1}$$

$$\widehat{2}\widehat{u}(0) = \widehat{1}\widehat{u}'(0) \tag{4.2}$$

$$\widehat{4}\widehat{u}\,(1) = \widehat{3}\widehat{u}'\,(1) \tag{4.3}$$

where $\widehat{1} = (0, 1, 2)$, $\widehat{2} = (1, 2, 3)$, $\widehat{3} = (2, 3, 4)$, $\widehat{4} = (3, 4, 5)$, and $\lambda = p^2$, $p > 0$.

From (4.1)-(4.3) problem, we get two FIVPs involving a crisp differential equation (4.1) but with fuzzy initial values as follows:

$$\Phi'' + p^2\Phi = 0, \quad \Phi\,(0) = \widehat{1}, \quad \Phi'\,(0) = \widehat{2} \tag{4.4}$$

and

$$\Psi'' + p^2\Psi = 0, \quad \Psi\,(1) = \widehat{3}, \quad \Psi'\,(1) = \widehat{4} \tag{4.5}$$

From the theory described in Section 3, the second order ODE in (4.1) can be rewritten in matrix form using the substitution $\Phi_1 = \Phi\,(t)$ and $\Phi_2 = \Phi'\,(t)$ which gives

$$\begin{pmatrix} \Phi_1' \\ \Phi_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\lambda & 0 \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}. \tag{4.6}$$

The independent vector solutions of (4.6) are also established from (2.4) with the eigenvalues $\lambda_{1,2} = \pm\sqrt{\lambda}$ and their associated eigenvectors $u_{1,2} = \pm i \begin{bmatrix} 0 & \sqrt{\lambda} \end{bmatrix}^T$. Then, the fundamental matrix of the system given by (4.6) is

$$(\Phi\,(t)) = \begin{pmatrix} \cos(\sqrt{\lambda}t) & \sin(\sqrt{\lambda}t) \\ -\sqrt{\lambda}\sin(\sqrt{\lambda}t) & \sqrt{\lambda}\cos(\sqrt{\lambda}t) \end{pmatrix}. \tag{4.7}$$

Now let us consider $t_0 = 0$ and the fuzzy initial conditions given by the fuzzy numbers $\Phi(0) = \widehat{1} = (0; 1; 2)$, and $\Phi'(0) = \widehat{2} = (1; 2; 3)$ .Then using (4.7), the fuzzy solution $\widehat{\Phi}\,(t)$ is obtained from (3.4) such that

$$\widehat{\Phi}_\lambda\,(t) = n_1\,(x)\,\widehat{a}_2 + n_2\,(x)\,\widehat{a}_1 = \cos\,(pt)\,\widehat{a}_2 + \frac{1}{p}\sin\,(pt)\,\widehat{a}_1. \tag{4.8}$$

Similarly we find $\widehat{\Psi}\,(t)$ such that

$$\widehat{\Psi}_\lambda\,(t) = \cos\,(pt - pb)\,\widehat{b}_2 + \frac{1}{p}\sin\,(pt - pb)\,\widehat{b}_1. \tag{4.9}$$

Then putting the solutions (4.8) and (4.9) the above equation (2.3), we get Wronskian function as

$$W\,(\lambda) = \left(3p + \frac{8}{p}\right)\sin\,(p) + (6 - 4)\cos\,(p)\,. \tag{4.10}$$

From Theorem 2.1, equation (4.1-4.3) has a nontrivial solution if and only if $W\,(\lambda) = 0$.

For the purposes of this example we found the first four numerically and then we will use the approximation of the remaining eigenvalues. We can see from Figure 1 that the graphs intersect at infinitely many points $p_n \approx n\pi \quad (n = 1, 2, 3...)$, where the error in this approximation approaches zero as n$\to\infty$. Given this estimate, we can use Matlab program to compute p_n more accurately. Those values are given Table 1.

Table 1: Eigenvalues of the fuzzy problem

	p_n	λ_n
$n = 1$	2.9709	8.8262
$n = 2$	6.1827	38.2257
$n = 3$	9.3557	87.5291
$n = 4$	12.514	156.6
$n \approx$	$n\pi$	$(n\pi)^2$

From the equations (4.8) and (4.9)

$$\widehat{\Phi}_{\lambda_n}\,(x) = \cos\,(p_n x)\,(0, 1, 2) + \frac{1}{p_n}\sin\,(p_n x) \tag{4.11}$$

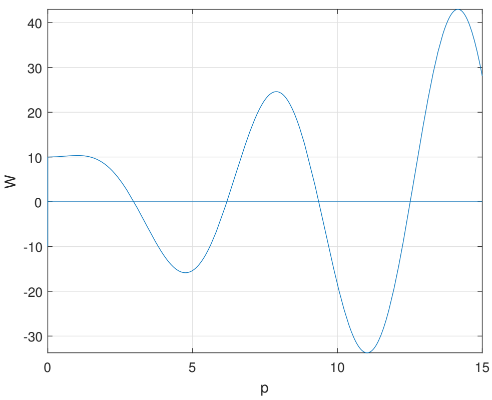

Figure 1: The function $W(\lambda) = \left(3p + \frac{8}{p}\right)\sin(p) + (4-6)\cos(p)$

and

$$\widehat{\Psi}_{\lambda_n}(x) = \cos(p_n x - p_n b)(2,3,4) + \frac{1}{p}\sin(p_n x - p_n b) \tag{4.12}$$

are fuzzy eigenfunctions associated with λ_n.

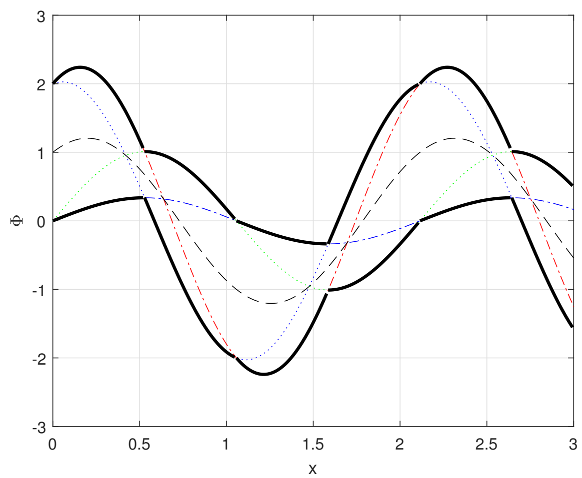

Figure 2: The fuzzy solution $\widehat{\Phi}_{\lambda_1}(x)$ for Example. The dashed black line represents the crisp solution. The black lines represents the upper and lower boundaries of the band

In particular, we select $p_1 = 2.9709$ in Table 1. If we substitute this value respectively in (4.11) and (4.12), we have the following equations and figures.

$$\widehat{\Phi}_{\lambda_1}(x) = \cos(2.9709x)(0,1,2) + \frac{1}{2.9709}\sin(2.9709x)(1,2,3) \tag{4.13}$$

and

$$\widehat{\Psi}_{\lambda_1}(x) = \cos(2.9709x - 2.9709)(2,3,4) + \frac{1}{2.9709}\sin(2.9709x - 2.9709)(4,5,6) \tag{4.14}$$

The fuzzy solutions (4.13) and (4.14) form a band in the xy-plane Figure 2 and Figure 3. The functions cos and sin functions take both positive and negative values in the interval $0 < x < 1$. So, $\overline{1}$ and $\underline{1}$, $\overline{2}$

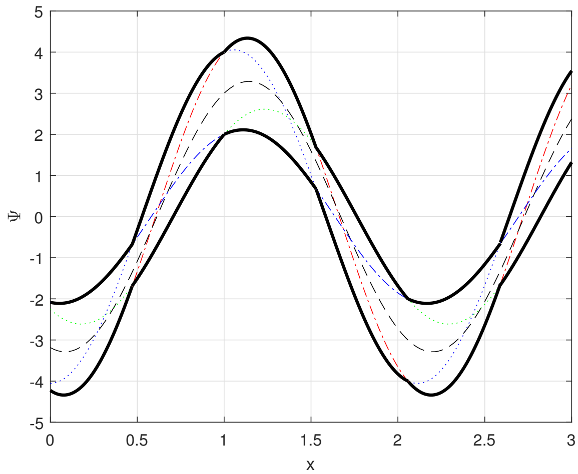

Figure 3: The fuzzy solution $\widehat{\Psi}_{\lambda_1}(x)$ for Example. The dashed black line represents the crisp solution. The black lines represents the upper and lower boundaries of the band

and $\underline{2}$, $\overline{3}$ and $\underline{3}$, $\overline{4}$ and $\underline{4}$ by turns dominate in generating the upper and lower boundary values of the band. So similarly for all values, the eigenfunctions (4.8) and (4.9) form a band in the xy-plane. That is, the eigenfunctions (4.8) and (4.9) are fuzzy eigenfunctions of the problem (4.1) - (4.3).

5. conclusion

In this study we have researched the FTPBVP by using two different methods as Titchmarsh and Linear System. From the method of Titchmarsh we get the FIVPs with the boundary condition at each point. Then we solved this FIVPs with linear system of differential equations as a fuzzy set of real functions. In future work we plan to make the parameter in the boundary conditions to the fuzzy eigenvalue problem. Then we will apply this method to the Sturm-Liouville problem.

References

1. P. Diamond and P. Kloeden, *Metric spaces of fuzzy sets World Scientific*. Singapore: World Scientific, 1994.
2. H. G. Citil, *Comparisons of the exact and the approximate solutions of second-order fuzzy linear boundary value problems*. Miskolc Mathematical Notes, 20 (2), 823–837, (2019).
3. O. Kaleva, *Fuzzy differential equations*. Fuzzy sets and systems, 24 (3), 301–317, (1987).
4. T. Allahviranloo and K. Khalilpour, *A numerical method for two-point fuzzy boundary value problems*. World Applied Sciences Journal, 13 (10), 2137–2147, (2011).
5. G. Anastassiou, *Handbook of analytic - computational methods in applied mathematics*. BocaRaton: Chapman and Hall/CRC, 2000.
6. B. Bede and L. Stefanini, *Generalized differentiability of fuzzy-valued functions*. Fuzzy Sets and Systems, 230, 119–141, (2013).
7. W. E. Boyce and R. C. DiPrima, *Elementary Differential Equations and Boundary Value Problems*. John Wiley and Sons, 2009.
8. T. Ceylan and N. Altınışık,*Eigenvalue problem with fuzzy coefficients of boundary conditions*. Scholars Journal of Physics, Mathematics and Statistics, 5 (2), 187–192, (2018).
9. L. C. de Barros, R. C. Bassanezi, and W. A. Lodwick, *A first Course in Fuzzy Logic, Fuzzy Dynamical Systems , and Biomathematics: Theory and applications*. London: Springer Cham, 2017.
10. N. Gasilov, S. E. Amrahov, and A. G. Fatullayev, *Solution of linear differential equations with fuzzy boundary values*. Fuzzy Sets and Systems, 257, 169-183, (2013).
11. L. T. Gomes, L. C. de Barros, and B. Bede, *Fuzzy Differential Equations in Various Approaches*. London: Springer Cham, 2015.

12. H. G. Citil and N. Altınışık, *On the eigenvalues and the eigenfunctions of the sturm-liouville fuzzy boundary value problem*. J. Math. Comput. Sci., 7 (4), 786–805, (2017).

13. H. G. Citil, *The Problem with Fuzzy Eigenvalue Parameter in One of the Boundary Conditions*. An International Journal of Optimization and Control: Theories and Applications, 10(2), 159-165, (2020).

14. H. G. Citil, *Comparison Results of Linear Differential Equations with Fuzzy Boundary Values*, Journal of Science and Arts, 1(42), 33-48, (2018).

15. H. G. Citil and N. Altınışık,*The examination of eigenvalues and eigenfunctions of the sturm-liouville fuzzy problem according to boundary conditions*. International Journal of Mathematical Combinatorics, 1, 51–60, (2018).

16. A. Kandel and W. J. Byatt, *Fuzzy differential equations, Proceedings of the International Conference on Cybernetics and Society*. Tokyo, Japan, 1213–1216, (1978).

17. A. Khastan and J. J. Nieto, *A boundary value problem for second order fuzzy differential equations*. Fuzzy sets and systems, 72 (9-10), 3583–3593, (2010).

18. G. J. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic*. New Jersey: Prentice Hall, 1995.

19. R. K. Nagle, E. B. Saff, and A. D. Snider, *Fundamentals of Differential Equations and Boundary Value Problems*. Pearson, 2017.

20. M. L. Puri and D. A. Ralescu,*Differentials of fuzzy functions, Journal of Mathematical Analysis and Applications*. 91 (2), 552–558, (1983).

21. I. Sadeqi, M. Moradlou, and M. Salehi, *On approximate cauchy equation in felbin's type fuzzy normed linear spaces*. Iranian Journal of Fuzzy Systems, 10 (3), 2013, (2013).

22. L. Stefanini and B. Bede, *Generalized hukuhara differentiability of interval-valued functions and interval differential equations*. Nonlinear Analysis: Theory, Methods and Applications, 71 (3-4), 1311–1328, (2009).

23. E. C. Titchmarsh, *Eigenfunction expansions associated with second-order differential equations I*. 2nd edn. London: Oxford University Press, 1962.

24. D. S. Watkins, *Fundamentals of Matrix Computations*. New York: John Wiley and Sons, 2004.

25. L. A. Zadeh, *The concept of a linguistic variable and its application to approximate reasoning-iii*. Information Sciences, 8 (3), 199–249, (1975).

26. M. Hukuhara, *Integration des applications mesurables dont la valeur est un compact convexe, Funkcialaj Ekvacioj*. 10 (3), 205-223, (1967).

27. M. T. Mizukoshi and L. C. de Barros and Y. Chalco-Cano and H Roman-Flores and R. C. Bassanezi, *Fuzzy differential equations and the extension principle*. Information Sciences, 177, 3627-3635, (2007).

Tahir Ceylan,
Department of Mathematics,
Sinop University,
Turkey.
E-mail address: tceylan@sinop.edu.tr