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A New Characterization of Simple K_5-Groups of Type Suzuki Groups $\mathbf{Sz}(\mathbf{2^{2m+1}})$

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ABSTRACT: One of the important problems in finite groups theory, is group characterization by specific property. Properties, such as elements order, the set of elements with the same order, the largest element order and graphs, etc. In this paper, we prove that the simple K_5-groups of type suzuki groups $Sz(2^{2m+1})$, where $q+\sqrt{2q}+1$ is prime number can be uniquely determined by its orders and the largest element orders.

Key Words: Elements order, the largest order of elements, prime graph, Frobenius group.

Contents

1. Introduction

Let G be finite group, the set of prime divisors of $|G|$ is denoted by $\pi(G)$ and the largest element of the set $\pi_e(G)$ of element orders of G is denoted by $k(G)$. Also, a sylow p-subgroup of G is denoted by G_p. The prime graph $\Gamma(G)$ of group G is a graph whose vertex set is $\pi(G)$, and two vertices u and v are adjacent if and only if $uv \in \pi_e(G)$. Moreover, assume that $\Gamma(G)$ has $t(G)$ connected components π_i, for $i = 1, 2, \ldots, t(G)$. In the case where $|G|$ is of even order, we always assume that $2 \in \pi_1$.

One of the important problems in finite groups theory, is group characterization by specific property. Properties, such as elements order, the set of element with the same order and graphs, etc. One of methods is, group characterization by using the order of the group and the largest elements order. In fact, we say the group G is characterizable by the order of the group G and the largest elements ments order, whenever there exist the group H, such that $k(G) = k(H)$ and $|G| = |H|$, then $G \cong H$.

Next, the authors try to characterize some finite simple groups by using less quantities and have successfully for example, characterized sporadic simple groups, $PSL_3(q)$and $PSU_3(q)$ where q is some special power of prime, by using three numbers: the order of group, the largest and the second largest element orders, of which some results can be seen in ([3],[15]) and also in [4], Li-Guan He and Gui-Yun Chen proved a group $L_2(q)$ where $q = p^n < 125$ by largest element order and group order is characterized. Also in [5], Li-Guan He and Gui-Yun proved characterization K_4 -group of type $L_2(p)$ only by using the order of a group and the largest element order, where p is a prime but not $2^n - 1$. Next, in the way, Ebrahimzadeh and etal in ([6],[7],[8],[9],[10],[11],[12],[13]) proved that these groups by using the largest element order and order of the group can be characterized.

For this purpose, in this paper, we characterize simple K_5-group of type $Sz(2^{2m+1})$ via the order of a group and largests element order, where $q+\sqrt{2q}+1$ is prime number. The main theorem of this paper as follows.

Main Theorem. Let G be a group and K_5- suzuki group $Sz(q)$ such that $|G| = |Sz(q)|$ and $k(G) = k(Sz(q))$, where $q = 2^{2m+1}$, $m \geq 1$ and $q+\sqrt{2q}+1$ is a prime number if and if only $G \cong Sz(q)$.

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2. Title Material

In this section, we give some useful lemmas which will be used in the proof of the main theorem.

Lemma 2.1 *[14] Let G be a Frobenius group of even order with kernel K and complement H. Then*

1. *$t(G) = 2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$;*
2. *$|H|$ divides $|K| - 1$;*
3. *K is nilpotent.*

Definition 2.1 A group G is called a 2-Frobenius group if there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernels K/H and H respectively.

Lemma 2.2 *[2] Let G be a 2-Frobenius group of even order. Then*

1. *$t(G) = 2$, $\pi(H) \cup \pi(G/K) = \pi_1$ and $\pi(K/H) = \pi_2$;*
2. *G/K and K/H are cyclic groups satisfying $|G/K|$ divides $|\mathsf{Aut}(K/H)|$.*

Lemma 2.3 *[24] Let G be a finite group with $t(G) \geq 2$. Then one of the following statements holds:*

1. *G is a Frobenius group;*
2. *G is a 2-Frobenius group;*
3. *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, H is a nilpotent group and $|G/K|$ divides $|Out(K/H)|$.*

Lemma 2.4 *Let $S := Sz(q)$ be suzuki simple group, where $q = 2^{2m+1}$ and $m \geq 1$.Then $\pi_e(S)$ consists of exactly all factors of 4, $q-1$, $q+\sqrt{2q}+1$, $q-\sqrt{2q}+1$*

Lemma 2.5 *[25, Theorem B]. Let q, k, l be natural numbers. Then*

1. *$(q^k - 1, q^l - 1) = q^{(k,l)} - 1$.*
2. *$(q^k + 1, q^l + 1) = \begin{cases} q^{(k,l)} + 1 & \text{if both } \frac{k}{(k,l)} \text{ and } \frac{l}{(k,l)} \text{ are odd,} \\ (2, q+1) & \text{otherwise.} \end{cases}$*
3. *$(q^k - 1, q^l + 1) = \begin{cases} q^{(k,l)} + 1 & \text{if } \frac{k}{(k,l)} \text{ is even and } \frac{l}{(k,l)} \text{ is odd,} \\ (2, q+1) & \text{otherwise.} \end{cases}$*

In particular, for every $q \geq 2$ and $k \geq 1$, the inequality $(q^k - 1, q^k + 1) \leq 2$ holds.

Lemma 2.6 *[16] Let G is K_3-group. Then G is isomorphic to one of the following groups: A_5, A_6, $PSL_2(7)$, $PSL_2(8)$, $PSL_2(17)$, $PSL_3(3)$, $PSU_3(3)$ or $PSU_4(2)$.*

Lemma 2.7 *[1] Let G be a simple K_4-group. Then G is isomorphic to one of the following groups:*

[(i)]

1. *$L_2(q)$ where q is a prime power satisfying $q(q^2-1) = (2, q-1)2^a \cdot 3^b \cdot p^c \cdot r^d$ with p, $r > 3$ distinct primes and a, b, c, d natural numbers;*
2. *A_7, A_8, A_9, A_{10}, M_{11}, M_{12}, J_2, $PSL_2(16)$, $PSL_2(25)$, $PSL_2(49)$, $PSL_2(81)$, $PSL_3(4)$, $PSL_3(5)$, $PSL_3(8)$, $PSL_3(17)$, $PSL_4(3)$, $O_5(4)$, $O_5(5)$, $O_5(7)$, $O_5(9)$, $O_7(2)$, $O_8^+(2)$, $G_2(3)$, $PSU_3(4)$, $PSU_3(5)$, $PSU_3(7)$, $PSU_3(8)$, $PSU_3(9)$, $PSU_4(3)$, $PSU_5(2)$, $Sz(8)$, $Sz(32)$, ${}^3D_4(2)$, ${}^2F_4(2)'$.*

Lemma 2.8 *[17] Let G be a simple K_5-group. Then G is isomorphic to one of the following groups:*
[(i)]

1. *$O_5(q)$ where q satisfies $|\pi(q^4-1)|=4$;*
2. *$PSL_2(q)$ where q satisfies $|\pi(q^2-1)|=4$;*
3. *$PSL_3(q)$ where q satisfies $|\pi((q^2-1)(q^3-1))|=4$;*
4. *$PSU_3(q)$ where q satisfies $|\pi((q^2-1)(q^3+1))|=4$;*
5. *$Sz(q)$ where $q=2^{2m+1}$ satisfies $|\pi((q-1)(q^2+1))|=4$;*
6. *$R(q)$ where q is an odd power of 3 and $|\pi(q^2-1)|=3$ and $|\pi(q^2-q+1)|=1$;*
7. *A_{11}, A_{12}, M_{22}, J_3, HS, He, McL, $PSL_4(4)$, $PSL_4(5)$, $PSL_4(7)$, $PSL_5(2)$, $PSL_5(3)$, $PSL_6(2)$, $O_7(3)$, $O_9(2)$, $PSP_6(3)$, $PSP_8(2)$, $PSU_4(4)$, $PSU_4(5)$, $PSU_4(7)$, $PSU_4(9)$, $PSU_5(3)$, $PSU_6(2)$, $O_8^+(3)$, $O_8^-(2)$, ${}^3D_4(3)$, $G_2(4)$, $G_2(5)$, $G_2(7)$, $G_2(9)$.*

3. Proof of the Main Theorem

In this section, we prove the main theorem. For this purpose, we denote the simple K_5-groups of type suzuki groups $Sz(2^{2m+1})$ and also the prime number $q+\sqrt{2q}+1$ by B and p, respectively. For proof of the main theorem, we denote the several lemmas, where that by proving this lemmas, the main theorem be proved. Also we know that $|B|=q^2(q^2+1)(q-1)$ and $k(B)=q+\sqrt{2q}+1$, where $q=2^{2m+1}$. We note that $|G|=|B|=q^2(q^2+1)(q-1)$ and $k(G)=k(B)=q+\sqrt{2q}+1$.

Lemma 3.1 *p is an isolated vertex of $\Gamma(G)$.*

Proof: We, prove that p is an isolated vertex of $\Gamma(G)$. On opposite, assume that it is not. So, there is the natural number t belong to $\pi_e(G)$ such that $t\neq p$ and $tp\in\pi_e(G)$. Thus we deduce that $tp\geq 2p\geq 2(q+\sqrt{2q}+1)>(q+\sqrt{2q}+1)$ and hence $k(G)>q+\sqrt{2q}+1$ which is impossible. So, we conclude that p is an isolated vertex of $\Gamma(G)$ and $t(G)\geq 2$. Now lemma 2.3 implies that G satisfies one of the following: □

Lemma 3.2 *The group G is not a Frobenius group.*

Proof: Let G be a Frobenius group with kernel K and complement H. Then by lemma 2.1, $t(G)=2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$. Since, p is an isolated vertex of $\Gamma(G)$, we have (i) $|H|=|G|/p$ and $|K|=p$, or (ii) $|H|=p$ and $|K|=|G|/p$. Assume that $|H|=|G|/p$ and $|K|=p$. Then lemma 2.1 implies that $|G|/p$ divides $p-1$, which is impossible. So, the case $|H|=p$ and $|K|=|G|/p$ can be occurred. Lemma 2.1 implies that p divides $\frac{|G|}{p}-1$. In the other words, $q+\sqrt{2q}+1 \mid \frac{q^2(q^2+1)(q-1)}{q+\sqrt{2q}+1}-1$. Hence, $q+\sqrt{2q}+1\mid q^2(q-\sqrt{2q}+1)(q-1)-1$. So we deduce $q+\sqrt{2q}+1\mid(q+\sqrt{2q}+1)(q^3-2\sqrt{2q}q^2+3q^2-4q+4\sqrt{2q}-4)+3$. Thus $q+\sqrt{2q}+1\mid 3$, which is a contradiction. Therefore, G is not a Frobenius group.

□

Lemma 3.3 *The group G is not a 2- Frobenius group.*

Proof: We prove that G is not a 2-Frobenius group. On opposite we assume G be a 2-Frobenius group. Then by lemma 2.2, G has a normal series $1\trianglelefteq H\trianglelefteq K\trianglelefteq G$ such that G/H and K are Frobenius groups with kernels K/H and H respectively. Also, we have $t(G)=2$, $\pi(G/K)\cup\pi(H)=\pi_1$ and $\pi(K/H)=\pi_2$ and $|G/K|$ divides $|Aut(K/H)|$. Since p is an isolated vertex of $\Gamma(G)$, we deduce that $\pi_2=\{p\}$ and $|K/H|=p$. If $p=q+\sqrt{2q}+1$, then by lemma 2.5 $(q-1;p-1)=1$ and since $|G/K|$ divided $p-1$, we deduce that $q-1$ divides $|H|$. So $K/H\rtimes H_{q-1}$ is a Frobenius group with kenel H_{q-1}. Hence lemma 2.1(b) implies that $p\mid q-2$. We know that p is a odd prime. Thus $p\mid(q-2)/2$. Hence $2p\mid(q-2)$. So $2(q+\sqrt{2q}+1)\leq(q-2)$ which is a contradiction. So G is not a 2-Frobenius group. □

Lemma 3.4 *The group G is isomorphic to B.*

Proof: By up lemmas and lemma 2.1, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups K/H is a non-abelian simple group. Since K/H is a non- abelian simple group and K/H is a K_5- group, so $\pi(K/H) < 6$. As a result K/H is isomorphic to K_n-group, where $n = 3, 4, 6$. For this purpose, first we consider the following isomorphic. □

Step 1. We prove that $K/H \ncong K_3$-groups. On opposite, we assume $K/H \cong K_3$- groups, then K/H is isomorphics to groups A_5, A_6, $PSL_2(7)$, $PSL_2(8)$, $PSL_2(17)$, $PSL_3(3)$, $PSU_3(3)$ or $PSU_4(2)$. Hence, we consider the following.
(i). If $K/H \cong A_n$, where $n = 5; 6$, then by [18], $k(A_5) = 5$. Next, we know $k(G) = q+\sqrt{2q}+1$, on the other hand $|K/H|$ divided $|G|$. For this purpose, we consider $q+\sqrt{2q}+1 = 5$. As a result $2^{2m+1}+2^{m+1}+1 = 5$, so $2^{2m+1} + 2^{m+1} - 4 = 0$. Now we assume $2^m = x$, then we deduce $2x^2 + 2x - 4 = 0$, as a result $x = 1$ or $m = 0$. Since $|A_n| \nmid |Sz(2)|$, so is a contradiction.
(ii). $K/H \cong PSL_2(q)$, where $q = 7; 8; 17$. Then by [18] $k(PSL_2(q)) = 7; 9; 17$, respectively. For this purpose, we consider $q + \sqrt{2q} + 1 = 7; 9; 17$, we can see easily this equations has not any solution in natural number $\mathbb{N}$.
(iii). $K/H \ncong PSL_3(3); PSU_3(3); PSU4(2)$, where by [18], $k(PSL_3(3) = 13$, $k(PSU_3(3) = 12$, $k(PSU_4(2) = 12$. Similarily we consider $q + \sqrt{2q} + 1 = 12; 13$, then we can see easily is a contradiction. Hence $K/H \ncong K_3$- groups.
Step 2. We prove that $K/H \ncong K_4$-groups. On opposite, we assume $K/H \cong K_4$- groups, so K/H is isomorphics to groups:
(1) $PSL_2(q')$ where q' is a prime power satisfying $q'(q'^2 - 1) = (2; q' - 1)2^a.3^b.p^c.r^d$ with $p, r > 3$ distint primes and a, b, c, d natural numbers; (2) A_7, A_8, A_9, A_{10}, M_{11}, M_{12}, J_2, $PSL_2(16)$, $PSL_2(25)$, $PSL_2(49)$, $PSL_2(81)$, $PSL_3(4)$, $PSL_3(5)$, $PSL_3(8)$, $PSL_3(17)$, $PSL_4(3)$, $O_5(4)$, $O_5(5)$, $O_5(7)$, $O_5(9)$, $O_7(2)$, $O_8^+(2)$, $G_2(3)$, $PSU_3(4)$, $PSU_3(5)$, $PSU_3(7)$, $PSU_3(8)$, $PSU_3(9)$, $PSU_4(3)$, $PSU_5(2)$, $Sz(8)$, $Sz(32)$, ${}^3D_4(2)$, ${}^2F_4(2)'$. First, we consider the groups of the case (1). Hence we consider the following isomorphic.
(i). If $K/H \cong PSL_2(q')$, where q' is a prime power satisfying $q'(q'^2 - 1) = (2; q' - 1)2^a.3^b.p^c.r^d$ with $p, r > 3$ distint primes and a; b; c; d natural numbers. For this purpose, since q' is a prime power, so $|PSL_2(q')| = \frac{q'(q'^2-1)}{2}$ and also by [18], $k(PSL_2(q')) = q'$. Hence we consider $q + \sqrt{2q} + 1 = q'$, then $2^{2m+1} + 2^{m+1} + 1 = q'$. As a result $2^{m+1}(2^m + 1) = q'$, which is a contradiction, because $q' = p'^m$.
(ii). Suppose that $K/H \cong A_n$, where $n = 7; 8; 9; 10$, so by [18], $k(A_n) = 7; 15; 21; 21$. For this purpose, we consider $q + \sqrt{2q} + 1 = 7; 15; 21$. For example if $q + \sqrt{2q} + 1 = 7$, then we see that the equation $2^{2m+1} + 2^{m+1} - 6 = 0$, has not any solution in natural number $\mathbb{N}$, so this is a contradiction. Now, for the other groups we have a contradiction.
(*).$K/H \ncong M_n$, where $n = 11; 12$ and by [18], $k(M_n) = 11$. Because the equation $q + \sqrt{2q} + 1 = 11$, has not any solution in natural number $\mathbb{N}$.
(**). $K/H \cong PSL_2(q')$, where $q' = 16; 25; 49; 81; 243$ and by [18], $k(PSL_2(q')) = 17; 13; 25; 41; 61$ respectively. Because the equation $q + \sqrt{2q} + 1 = 17; 13; 25; 41; 61$, have not any solution in natural number $\mathbb{N}$. For the other groups we have a contradiction, similarily. Hence $K/H \ncong K_4$-groups.
Step3. Weprove that $K/H \ncong K_5$-groups. On opposite, we assume $K/H \cong K_5$- groups, then K/H is isomorphics to groups of lemma 2.8. How ever first, we can see easily that K/H is not isomorphics to groups of case (2). Because we know that by [18], $k(A_n) = 21; 35$ where $n = 11; 12$, then the equation $q + \sqrt{2q} + 1 = 21; 35$, have not any solution in natural number $\mathbb{N}$. The other groups is a contradiction, similarily proof. Now, we consider the groups of case (1). Hence, we have a following isomorphics.
(i). Suppose that $K/H \cong PSL_2(q')$, where q' satisfies $|\pi(q'^2 - 1)| = 4$. Now, by step 2 (i) we have a contradiction.
(ii). Suppose that $K/H \cong PSL_3(q')$, where q' satisfies $|\pi(q^2 - 1)(q^3 - 1)| = 4$. Now, by [18] $k(PSL_3(q') = \frac{q'^2+q'+1}{(3,q'-1)}$.On the other hand, we know $|PSL_3(q') \mid |G|$, so $\frac{q'^3(q'^3-1)(q'^2-1)}{(3,q'-1)} \mid q^2(q^2+1)(q-1)$. For this purpose, we consider $q+\sqrt{2q}+1 = \frac{q'^2+q'+1}{(3,q'-1)}$. First, if $(3, q'-1) = 1$, then $q+\sqrt{2q}+1 = q'^2+q'+1$. It follows that $2^{m+1}(2^m + 1) = q'(q' + 1)$. Since $(q', q' + 1) = 1$ so $2^m + 1 = q'$, $2^{m+1} = q' + 1$. Now if

$2^m+1=q'$,then $2^{m+1}(2^m+1)=2^m+1(2^m+2)$. As a result $2^{m+1}=2^m+2$, then $m=1$, $q'=3$. Since $|PSL_3(3)|\nmid|Sz(8)|$, so this is a contradiction. The case $2^{m+1}=q'+1$, is a contradiction, similarly. Now if $(3;q'-1)=3$, then we consider $q+\sqrt{2q}+1=\frac{q'^2+q'+1}{(3,q'-1)}$ it follows that $3(2^{2m+1})+3(2^{m+1})=(q'-1)(q'+2)$. On the other hand $(q'-1,q'+2)=1,3$, so we have the following cases.
(i) $3\mid q'-1$, $q'+1\mid 2^{2m+1}+2^{m+1}$
or
(ii) $q'-1\mid 2^{2m+1}+2^{m+1}$, $3\mid q'+2$. We can see easily two cases are a contradiction.
(iii)). Suppose that $K/H\cong PSU_3(q')$, where q' satisfies $|\pi(q^2-1)(q^3+1)|=4$. Now, by [18] $k(PSU_3(q')=\frac{q'^2+q'}{(3,q'+1)}$. On the other hand, we know $|PSU_3(q')\mid|G|$, so $\frac{q'^3(q'^3+1)(q'^2-1)}{(3,q'-1)}\mid q^2(q^2+1)(q-1)$. For this purpose, we consider $q+\sqrt{2q}+1=\frac{q'^2+q'}{(3,q+1)}$. First, if $(3,q'+1)=1$, then $q+\sqrt{2q}+1=q'^2+q'$. Thus $q+\sqrt{2q}=q'^2+q'-1$ as $2^{m+1}(2^m+1)=(\frac{q'-\sqrt{5}}{2})(\frac{q'+\sqrt{5}}{2})$. So we deduce $2^{m+1}=\frac{q'+\sqrt{5}}{2}$ and $2^m+1=\frac{q'-\sqrt{5}}{2}$. We can see easily these equations have not any solution in natural number $\mathbb{N}$. Now if $(3;q'+1)=3$, then we consider $q+\sqrt{2q}+1=\frac{q'^2+q'}{3}$. Similarly last proof we have a contradiction.
(iv). Suppose that $K/H\cong R(q')$, where $q'=3^{2m'+1}$ satisfies $|\pi(q^2-1)|=3$ and $|\pi(q^2-q+1)|=1$. Now, by [18] $k(R(q'))=q'+\sqrt{3q'}+1$. On the other hand, we know $|R(q')\mid|G|$, so $q'^3(q'^3+1)(q'-1)\mid q^2(q^2+1)(q-1)$. For this purpose, we consider $q+\sqrt{2q}+1=q'+\sqrt{3q'}+1$. Hence $2^{2m+1}+2^{m+1}+1=3^{2m'+1}+3^{m'+1}$. So $2^{m+1}(2^m+1)=3^{m'+1}(3^{m'}+1)$. On the other hand $(2^{m+1},2^m+1)=1$, so $2^{m+1}=3^{m'+1}$, $2^m+1=3^{m'}+1$. As a result we see that every two the equations are impossible. It follows that $K/H\cong Sz(q')$, where $q'=2^{2m'+1}$, $m'\geq 1$. We know that $H\trianglelefteq K\trianglelefteq G$, so $2^{2m'+1}\leq 2^{2m+1}$ it follows that $m'\leq m$. On the other hand, p is an isolated vertex of $\Gamma(G)$ as a result $p\mid|K/H|$. Hence $q-\sqrt{2q}\leq p\leq q+\sqrt{2q}+1=k(K/H)$. As a result $m\leq m'$, $Sz(q')=S$. Now, since $|K/H|=|S|$ and $1\trianglelefteq H\trianglelefteq K\trianglelefteq G$, it follows that $H=1$ and $G=K\cong S$. The proof be compeleted.

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