Conformal Bi-Slant $\xi^\perp$-Riemannian Submersion: A Note in Contact Geometry

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ABSTRACT: This study explores conformal bi-slant $\xi^\perp$-Riemannian submersion where the total manifold is a contact metric manifold, more specifically a Sasakian manifold. To illustrate this study, few non-trivial examples are discussed. Meanwhile, we addressed the conditions for integrability of anti-invariant and slant distributions and determine the conditions for the map that must be met in order to be totally geodesic. Furthermore, some decomposition theorems for the fibres as well as for the total space are discussed.

Key Words: Conformal bi-slant submersions, conformal submersions, integrability conditions, Sasakian manifold, contact metric manifold.

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1. Introduction

Isometric immersion and Riemannian submersion, which is a double-fold concept of isometric immersion, are the two fundamental maps in Riemannian geometry. Riemannian submersion was induced by B. O’Neill [23] in 1966. Many authors have studied the Riemannian submersion from various perspectives like Riemannian submersion [23], slant submersion ([10], [30]), contact complex submersion [18], semi-invariant submersion [29], h-semi-invariant submersion [25] etc. Riemannian submersions have applications in physics and mathematics, such as in supergravity and superstring theories [20, 22], Kaluza-Klein theory ([19], [22]), and the Yang Mills theory ([8], [34]). Also, Frejlich and Dunn et al. ([11], [14]) obtained submersions of Lie algebra.

Fuglede [15] and Ishihara [21], studied the horizontally conformal maps as special case of Riemannian submersions with $\lambda = 1$. Meanwhile, Akyol and Sahin defined conformal anti-invariant submersions [5], conformal semi-invariant submersions [6], conformal slant submersion [4], and conformal semi-slant submersions [2]. Also, a variety of researchers have examined the geometry of conformal submersions ([17], [23]). Prasad et al. [26] investigated Quasi bi-slant submersion of Kenmotsu manifold whereas Sezin [32] studied bi-slant submersions from contact manifold with taking $\xi$ as horizontal vector field.

As a generalization of conformal slant, conformal [4], conformal semi-slant [3] and conformal hemi-slant submersions [27], we consider conformal bi-slant $\xi^\perp$-Riemannian submersion with total space, a Sasakian manifold, where the vector field $\xi$ is considered in horizontal space of the submersion. The paper has the following structure. Section 2 presents the fundamental information and definitions of contact metric manifolds, particularly Sasakian manifolds with properties relevant to this paper. Section 3 presents all of main results of this study. We define the conformal bi-slant $\xi^\perp$-Riemannian submersion. Along with some basic findings, we explore the condition of integrability for distributions and totally geodesicness. We prove Decomposition theorems for the total space and the fibres as well.

2010 Mathematics Subject Classification: 53C25, 53D10, 32C25.
Submitted October 08, 2022. Published April 27, 2023
Throughout paper, we will use some abbreviations as follows:

- Riemannian Manifold RM
- Riemannian Manifolds RMs
- Sasakian Manifold SM
- Sasakian Manifolds SMs
- conformal bi-slant-$\xi^\perp$ submersion CBSS

2. Preliminaries

A $(2n+1)$-dimensional manifold $M$ which having an almost contact structures $(\phi, \xi, \eta)$, where a $(1,1)$ tensor field $\phi$, a vector field $\xi$ and a 1-form $\eta$ satisfying

\[
\phi^2 = -I + \eta \otimes \xi, \quad \phi \xi = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1
\]

(2.1)

where $I$ is the identity tensor. If $N \oplus d\eta \otimes \xi = 0$, with Nijenhuis tensor $N$ related to $\phi$ then almost contact structure turns into normal. There is also a Riemannian metric $g$ which holds

\[
g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V), \quad \eta(U) = g(U, \xi).
\]

(2.2)

Then $(\phi, \xi, \eta, g)$-structure is called an almost contact metric structure. A normal contact metric structure is called a Sasakian structure, which satisfies

\[
(\nabla U \phi) V = g(U, V)\xi - \eta(V)U
\]

(2.3)

where $\nabla$ is the Levi-Civita connection of $g$. From above formula, we have for Sasakian manifold (SM)

\[
\nabla_U \xi = -\phi U.
\]

(2.4)

Now, we provide a definition for conformal submersion and discuss some useful results that help us to achieve our main results.

**Definition 2.1.** A smooth map $\varphi$ from $(M_1, g_1)$ onto $(M_2, g_2)$ where $M_1$ and $M_2$ are Riemannian manifolds (RMs) with $m_1$ and $m_2$ be the dimensions of manifolds respectively, is called horizontally weakly conformal or semi-conformal at $p \in M_1$ if, either

i. $d\varphi_p = 0$ or

ii. $d\varphi_p$ is surjective and there always have a number $\Omega(p) \neq 0$ satisfying

\[
g_2(d\varphi_p U, d\varphi_p V) = \Omega(p)g_1(U, V)
\]

for $U, V \in \Gamma(ker(d\varphi))$.

In this case, we label a point $p$ satisfying type (i) as a critical point and rank of $d\varphi_p$ is 0 at this point and type (ii) as a regular point at which the rank of $d\varphi_p$ is $m_2$. Also, the number $\Omega(p)$ is called the square dilation. Its square root $\sqrt{\Omega(p)} = \sqrt{\Omega(p)}$ is called the dilation. If the map $\varphi$ is horizontally weakly conformal at each point on $M_1$, it is referred to as horizontally weakly or semi-conformal on $M_1$. If $\varphi$ has no critical point, it is said to be a (horizontally) conformal submersion.

Let $\varphi : M_1 \to M_2$ be a submersion. A vector field $X$ on $M_1$ is called a basic vector field if $X \in \Gamma(ker(d\varphi)^\perp)$ and $\varphi$-related with a vector field $\mathcal{X}$ on $M_2$ i.e $\varphi_*(X(q)) = \mathcal{X}\varphi(q)$ for $q \in M_1$.

The formulas provide the two $(1,2)$ tensor fields $T$ and $A$ are

\[
T(E_1, E_2) = T_{E_1}E_2 = \mathcal{H}(\nabla V_{E_1} E_2 + \nabla V_{E_2} E_1, \mathcal{H}E_2
\]

(2.5)
\[ A(E_1, E_2) = A_{E_1} E_2 = \nabla_{\nabla_{\nabla E_1} \mathcal{H} E_2} + \mathcal{H}(\nabla_{\nabla E_1} \nabla E_2) \]  

(2.6)

for \( E_1, E_2 \in \Gamma(TM_1) \).

Note that a Riemannian submersion \( \varphi : M_1 \rightarrow M_2 \) has totally geodesic fibers if and only if \( T \) vanishes identically. From equations 2.5 and 2.6, we can deduce

\[
\begin{align*}
\nabla_{U_1} V_1 &= \mathcal{T}_{U_1} V_1 + \mathcal{V}_{U_1} V_1 \\
\nabla_{U_1} X_1 &= \mathcal{T}_{U_1} X_1 + \mathcal{H}(\nabla_{U_1} X_1) \\
\nabla_{X_1} U_1 &= A_{X_1} U_1 + \mathcal{V}_{X_1} U_1 \\
\nabla_{X_1} Y_1 &= \mathcal{H}(\nabla_{X_1} Y_1) + A_{X_1} Y_1 
\end{align*}
\]

(2.7) for any \( U_1, V_1 \in \text{ker}\varphi \) and \( X_1, Y_1 \in \text{ker}(\text{ker}\varphi)^\perp \).

It is clear that \( T \) and \( A \) are skew-symmetric, i.e.,

\[
g(A_X E_1, E_2) = -g(E_1, A_X E_2), \quad g(T_Y E_1, E_2) = -g(E_1, T_Y E_2)
\]

(2.11)

for all \( E_1, E_2 \in T_p M_1 \). The following results holds for the particular case where \( \varphi \) is horizontally conformal.

**Proposition 2.2.** Let \( \varphi : M_1 \rightarrow M_2 \) be horizontally conformal submersion with dilation \( \lambda \) and \( X, Y \in \Gamma((\text{ker}\varphi)^\perp) \), then

\[
A_X Y = \frac{1}{2}(\nabla(X, Y) - \lambda^2 g_1(X, Y) G_Y(\frac{1}{\lambda^2}))
\]

(2.12)

where gradient of function denoted by \( G \).

The second fundamental form of smooth map \( \varphi \) given by the formula

\[
(\nabla_\varphi)(X, Y) = \nabla_\varphi^\ast Y - \varphi_\ast \nabla X Y
\]

(2.13)

and the map be totally geodesic if \( (\nabla_\varphi)(X, Y) = 0 \) for all \( X, Y \in \Gamma(T_p M_1) \) where \( \nabla \) and \( \nabla_\varphi \) are Levi-Civita and pullback connections.

**Lemma 2.3.** Let \( M_1 \) and \( M_2 \) be RMs and \( \varphi \) be horizontal conformal submersion. Then, for any vector fields \( X_1, Y_1 \in \Gamma((\text{ker}\varphi)^\perp) \) and \( U_1, V_1 \in \Gamma(\text{ker}\varphi) \), we have

\[
\begin{align*}
(i) \quad (\nabla_\varphi)(X_1, Y_1) &= X_1(\ln \lambda) \varphi_\ast(Y_1) + Y_1(\ln \lambda) \varphi_\ast(X_1) - g(X_1, Y_1) \varphi_\ast(\text{grad } \ln \lambda), \\
(ii) \quad (\nabla_\varphi)(U_1, V_1) &= -\varphi_\ast(\mathcal{T}_{U_1} V_1), \\
(iii) \quad (\nabla_\varphi)(X_1, U_1) &= -\varphi_\ast(\nabla_{X_1} U_1) = -\varphi_\ast(A_{X_1} U_1).
\end{align*}
\]

3. Conformal bi-slant \( \xi_- \)-Riemannian submersions

**Definition 3.1.** Consider \( \varphi \) is a conformal submersion from a SM \((M_1, \phi, \xi, \eta, g_1)\) onto RM \((M_2, g_2)\). Then \( \varphi \) is defined a conformal bi-slant \( \xi_- \)-Riemannian submersion (CBSS) if \( D_{\phi_1}^\theta \) and \( D_{\phi_2}^\theta \) are slant distributions with corresponding slant angles \( \bar{\theta}_1 \) and \( \bar{\theta}_2 \), such that \( \text{ker}\varphi_\ast = D_{\phi_1}^\theta \oplus D_{\phi_2}^\theta \). If \( \bar{\theta}_1, \bar{\theta}_2 \) are neither equal to 0 nor \( \frac{\pi}{2} \), then \( \varphi \) is proper.

If \( m \) and \( n \) are dimension of \( D_{\phi_1}^\theta \) and \( D_{\phi_2}^\theta \) respectively, then we can say

i If \( m = 0 \) and \( \bar{\theta}_2 = \frac{\pi}{2} \) then \( \varphi \) is a conformal anti-invariant submersion,

ii If \( m, n \neq 0 \), \( \bar{\theta}_1 = 0 \) and \( \bar{\theta}_2 = \frac{\pi}{2} \) then \( \varphi \) is a conformal semi-invariant submersion.

iii If \( m, n \neq 0 \), \( \bar{\theta}_1 = 0 \) and \( 0 < \bar{\theta}_2 < \frac{\pi}{2} \) then \( \varphi \) is a conformal semi-slant submersion.

iv If \( m, n \neq 0 \), \( \bar{\theta}_1 = \frac{\pi}{2} \) and \( 0 < \bar{\theta}_2 < \frac{\pi}{2} \) then \( \varphi \) is a conformal hemi-slant submersion.
Note that $\mathbb{R}^{2n+1}$ denote a SM with the structure $(\phi, \xi, \eta, g)$ defined as
\[
\phi \left( \sum_{i=1}^{n} \left( X_i \frac{\partial}{\partial x^i} + Y_i \frac{\partial}{\partial y^i} \right) + Z \frac{\partial}{\partial z} \right) = \sum_{i=1}^{n} \left( Y_i \frac{\partial}{\partial x^i} - X_i \frac{\partial}{\partial y^i} \right),
\]
\[
\eta = \frac{1}{2} \left( dz - \sum_{i=1}^{n} y^i dx^i \right), \quad \xi = 2 \frac{\partial}{\partial z}
\]
\[
g = \eta \otimes \eta + \frac{1}{4} \sum_{i=1}^{n} \left( dx^i \otimes dx^i + dy^i \otimes dy^i \right),
\]
where $(x^1, \ldots, x^n, y^1, \ldots, y^n, z)$ are the cartesian coordinates.

Now, taking into account the definition above, we can provide the following examples:

**Example 3.2.** Let $F : (\mathbb{R}^3, g_{\mathbb{R}^3}) \rightarrow (\mathbb{R}^5, g_{\mathbb{R}^5})$ be a conformal submersion defined by
\[
F(x_1, x_2, x_3, x_4, y_1, y_2, y_3) = \pi^6 \left( \cos \alpha x_1 - \sin \alpha x_3, \frac{x_2 + x_4}{\sqrt{2}}, \sin \beta y_3 + \cos \beta y_4, y_1, z \right)
\]
then
\[
\ker F^* = < V_1 = \sin \alpha \partial x_1 + \cos \alpha \partial x_3, V_2 = \frac{1}{\sqrt{2}} (\partial x_2 - \partial x_4) >,
\]
\[
V_3 = \cos \beta \partial y_3 - \sin \beta \partial y_4, V_4 = \partial y_2 > \quad \text{and}
\]
\[
(\ker F^*)^\perp = < H_1 = \cos \alpha \partial x_1 - \sin \alpha \partial x_3, H_2 = \frac{1}{\sqrt{2}} (\partial x_2 + \partial x_4) >
\]
\[
H_3 = \sin \beta \partial y_3 + \cos \beta \partial y_4, H_4 = \partial y_1, H_5 = \xi = \partial z >
\]
Thus, the submersion $F$ is conformal bi-slant $\xi^\perp$-Riemannian submersion with $D_1 = \{V_1, V_3\}$ with slant angle $\bar{\theta}_1$ such that $\cos \bar{\theta}_1 = \sin(\beta - \alpha)$ and $D_2 = \{V_2, V_4\}$ with the slant angle $\bar{\theta}_2 = \frac{\pi}{4}$.

**Example 3.3.** Let $F : (\mathbb{R}^3, g_{\mathbb{R}^3}) \rightarrow (\mathbb{R}^5, g_{\mathbb{R}^5})$ such that
\[
(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, z) = e^{5} \left( \frac{x_1 + \sqrt{3}x_2}{2}, \sin \alpha x_3 + \cos \alpha x_4, y_1, y_2, z \right).
\]
Then it follows that
\[
D_1 = < V_1 = \frac{1}{2} (\sqrt{3} \partial x_1 - \partial x_2) , V_3 = \partial y_3 >,
\]
\[
D_2 = \text{span} \{ V_2 = \cos \alpha \partial x_3 - \sin \alpha \partial x_4, V_4 = \partial y_4 \} \quad \text{and}
\]
\[
(\ker F^*)^\perp = < H_1 = \frac{1}{2} (\partial x_1 + \sqrt{3} \partial x_2), H_2 = \sin \alpha \partial x_3 + \cos \alpha \partial x_4,
\]
\[
H_3 = \partial y_1, H_4 = \partial y_2, H_5 = \xi = \partial z >.
\]
Hence, $F$ is a conformal bi-slant $\xi^\perp$-Riemannian submersion with the slant angles $\bar{\theta}_1 = \frac{\pi}{3}$ and $\bar{\theta}_2 = \alpha$, respectively and $\lambda = e^{\sqrt{3}}$.

Let us consider $\varphi$ is CBSS from SM $(M_1, \phi, \xi, \eta, g_1)$ onto a RM $(M_2, g_2)$, with taking $U \in \ker \varphi^*$, we can write
\[
U = P_1 U + P_2 U
\] (3.1)
where $P_1 \mathcal{U} \in \Gamma(D_1^\phi)$ and $P_2 \mathcal{U} \in \Gamma(D_2^\phi)$.

Also, for $\mathcal{U} \in \Gamma(ker\varphi_\ast)$

$$\phi \mathcal{U} = \delta \mathcal{U} + \zeta \mathcal{U} \quad (3.2)$$

where $\delta \mathcal{U} \in \Gamma(ker\varphi_\ast)$ and $\zeta \mathcal{U} \in \Gamma((ker\varphi_\ast)^\perp)$. For $X \in \Gamma((ker\varphi_\ast)^\perp)$, we have

$$\phi X = tX + fX \quad (3.3)$$

where $tX \in \Gamma(ker\varphi_\ast)$ and $fX \in \Gamma((ker\varphi_\ast)^\perp)$.

The horizontal distribution $(ker\varphi_\ast)^\perp$ is decomposed as

$$(ker\varphi_\ast)^\perp = \zeta D_1^\phi \oplus \zeta D_2^\phi \oplus \mu \quad (3.4)$$

where $\mu = \phi \mu \oplus \xi$ such that $\mu$ is distribution which is complementary to $\zeta D_1^\phi \oplus \zeta D_2^\phi$ in $(ker\varphi_\ast)^\perp$.

**Theorem 3.4.** Let $\varphi : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ be CBSS from SM onto a RM with slant angles $\bar{\theta}_1$ and $\bar{\theta}_2$. Then we have

$$\delta^2 U_1 = - \cos^2 \bar{\theta}_1 U_1 \quad (3.5)$$

$$\delta^2 U_2 = - \cos^2 \bar{\theta}_2 U_2 \quad (3.6)$$

for $U_1 \in \Gamma(D_1^\phi)$ and $U_2 \in \Gamma(D_2^\phi)$.

**Theorem 3.5.** Let $\varphi$ be the CBSS from the SM $(M_1, \phi, \xi, \eta, g_1)$ onto a RM $(M_2, g_2)$ with slant angles $\bar{\theta}_1$ and $\bar{\theta}_2$. Then

(i) the distribution $D_1^\phi$ is integrable if and only if

$$\lambda^{-2} g_2((\nabla \varphi_\ast)(U, \zeta V), \varphi_\ast \zeta W) = g_1(T_U \zeta \delta U - T_U \zeta \delta V, \eta W)$$

$$+ g_1(T_U \zeta V - T_U \zeta U, \delta W)$$

$$+ \lambda^{-2} g_2((\nabla \varphi_\ast)(V, \zeta U), \varphi_\ast \zeta W) \quad (\text{ii})$$

(ii) the distribution $D_2^\phi$ is integrable if and only if

$$\lambda^{-2} g_2((\nabla \varphi_\ast)(U, \zeta V), \varphi_\ast \zeta V) = g_1(T_Z \zeta \delta W - T_U \zeta \delta Z < U)$$

$$+ g_1(T_U \zeta Z - T_Z \zeta W, \delta U)$$

$$+ \lambda^{-2} g_2((\nabla \varphi_\ast)(Z, \zeta W), \varphi_\ast \zeta U) \quad \text{for } U, V \in \Gamma(D_1^\phi) \text{ and } Z, W \in \Gamma(D_2^\phi).$$

**Proof.** (i). For any vector fields $U, V \in \Gamma(D_1^\phi)$ and $W \in \Gamma(D_2^\phi)$ and on using equations (2.2), (2.3) and from (3.2), we have

$$g_1([U, V], W) = g_1(\nabla V \delta^2 U, W - g_1(\nabla \phi \delta^2 V, W)$$

$$- g_1(\nabla \phi \delta V, \phi W) + g_1(\nabla \phi \delta U, \phi W)$$

$$+ g_1(\nabla \phi \zeta V, \phi W) - g_1(\nabla \phi \zeta U, \phi W).$$

Considering Theorem 3.4, we have

$$\sin^2 \bar{\theta}_1 g_1([U, V], W) = - g_1(\nabla \phi \delta V, \phi W) + g_1(\nabla \phi \zeta U, \phi W)$$

$$+ g_1(\nabla \phi \zeta V, \phi W) - g_1(\nabla \phi \delta U, \phi W).$$

On using equation (2.8), we obtained

$$\sin^2 \bar{\theta}_1 g_1([U, V], W) = g_1(\nabla \phi \delta U - T_U \zeta \delta V, W)$$

$$- g_1(\nabla \phi \zeta V - T_V \zeta \delta U, \delta W)$$

$$+ g_1(\nabla \phi \zeta V - T_V \zeta \zeta U, \zeta W).$$
Now considering Lemma 2.3 and equation (2.13), we yields
\[
\sin^2 \bar{\theta}_1 g_1([U, V], W) = g_1(T_U \zeta U - T_U \zeta V, W)
- g_1(T_U \zeta V - T_U \zeta U, \delta W)
- \lambda^{-2} g_2((\nabla \varphi_*)(U, \zeta V), \varphi_\zeta W)
+ \lambda^{-2} g_2((\nabla \varphi_*)(V, \zeta U), \varphi_\zeta W)
\]

For part (ii) the calculation is same as (i). \(\square\)

**Theorem 3.6.** Let \(\varphi\) be the CBSS from the SM \((M_1, \phi, \xi, \eta, g_1)\) onto a RM \((M_2, g_2)\) with slant angles \(\bar{\theta}_1\) and \(\bar{\theta}_2\). Then the distribution \(D_1^\theta\) defines totally geodesic foliation if and only if
\[
\lambda^{-2} g_2((\nabla \varphi_*)(\zeta V, U), \varphi_\zeta Z) = g_1(T_U \zeta V, \delta Z) - g_1(T_U \zeta \delta V, Z) \quad (3.7)
\]
and
\[
\lambda^{-2} g_2((\nabla \varphi_*)(\zeta V, U), \varphi_\zeta V) = \sin^2 \bar{\theta}_1 g_1([U, X], V) + g_1(A_X \zeta \delta U, V)
+ g_1(G(\ln \lambda), X) g_1(\zeta U, \zeta V)
+ g_1(G(\ln \lambda), \zeta U) g_1(X, \zeta V)
- g_1(G(\ln \lambda), \zeta V) g_1(X, \zeta U)
- g_1(A_X \zeta U, \delta V) \quad (3.8)
\]
for \(U, V \in \Gamma(D_1^\theta)\), \(Z \in \Gamma(D_2^\theta)\) and \(X \in ((\ker \varphi_*)^\perp)\).

**Proof.** for \(U, V \in \Gamma(D_1^\theta)\) and \(Z \in \Gamma(D_2^\theta)\) with using equation (2.2), (2.3) and (3.2), we have
\[
g_1([U, V], Z) = g_1(\nabla_U \zeta V, \delta Z) - g_1(\nabla_U \zeta \delta V, Z) - g_1(\nabla_U \delta^2 V, Z).
\]
From Theorem 3.4, we can write
\[
\sin^2 \bar{\theta}_1 g_1(\nabla_U V, Z) = -g_1(\nabla_U \zeta \delta V, Z) + g_1(\nabla_U \zeta V, \delta Z)
\]
On using (2.8), we have
\[
\sin^2 \bar{\theta}_1 g_1(\nabla_U V, Z) = g_1(T_U \zeta V, \delta Z) - g_1(T_U \zeta \delta V, Z) + g_1(T_U \zeta \zeta V, \zeta Z).
\]
Considering equation (2.13) and Lemma 2.3, we obtain
\[
\sin^2 \bar{\theta}_1 g_1(\nabla_U V, Z) = g_1(T_U \zeta V, \delta Z) - g_1(T_U \zeta \delta V, Z) - \lambda^{-2} g_2((\nabla \varphi_*)(\zeta V, U), \varphi_\zeta Z)
\]
which is the equation first in Theorem 3.6.

On the other hand, \(U, V \in \Gamma(D_1)\) and \(X \in \Gamma((\ker \varphi_*)^\perp)\) with using (2.2), (2.3) and (3.2), we can write
\[
g_1(\nabla_U V, X) = -g_1([U, X], V) + g_1(\varphi_\nabla X \delta U, V) - g_1(\nabla_X \zeta U, \varphi V).
\]
Considering Theorem 3.4, we obtained
\[
\sin^2 \bar{\theta}_1 g_1(\nabla_U V, X) = -g_1([U, V], X) + g_1(\nabla_X \zeta \delta U, V) - g_1(\nabla_X \zeta U, \varphi V).
\]
On using equation (2.10), we have
\[
\sin^2 \bar{\theta}_1 g_1(\nabla_U V, X) = \sin^2 \bar{\theta}_1 g_1([U, X], V) + g_1(A_X \zeta \delta U, V)
- g_1(A_X \zeta U, \delta V) - \lambda^{-2} g_2(\varphi_\nabla X \zeta U, \varphi_\zeta U).
\]
Using Lemma 2.3, we yields

\[
\sin^2 \bar{\theta}_1 g_1((\nabla \bar{U} V, X) = \sin^2 \bar{\theta}_1 g_1([U, X], V) + g_1(A_X \zeta U, V)
- \lambda^{-2} g_2(\nabla^{\bar{X}} \phi, \zeta U, \varphi_* \zeta V)
+ g_1(G(\ln \lambda), X) g_1(\zeta U, \zeta V)
+ g_1(G(\ln \lambda), \zeta U) g_1(X, \zeta V)
- g_1(G(\ln \lambda), \zeta V) g_1(X, \zeta U)
- g_1(A_X \zeta U, \delta V)
\]

This completes the proof of the Theorem.

**Theorem 3.7.** Let \( \varphi \) be the CBSS from the SM \((M_1, \phi, \xi, \eta, g_1)\) onto a RM \((M_2, g_2)\) with slant angles \( \bar{\theta}_1 \) and \( \bar{\theta}_2\). Then the distribution \( D^\varphi_2 \) defines totally geodesic foliation if and only if

\[
\lambda^{-2} g_2((\nabla \varphi_*)(\zeta Z, W), \varphi_* \zeta U) = -g_1(T_w \zeta \delta Z, U) + g_1(T_w \zeta Z, \delta U)
\]

(3.9)

and

\[
\lambda^{-2} g_2(\nabla^{\bar{X}} \varphi, \zeta W, \varphi_* \zeta Z) = \sin^2 \bar{\theta}_1 g_1([W, X], Z) + g_1(A_X \zeta \delta W, Z)
+ g_1(G(\ln \lambda), X) g_1(\zeta W, \zeta Z)
+ g_1(G(\ln \lambda), \zeta W) g_1(X, \zeta Z)
- g_1(G(\ln \lambda), \zeta Z) g_1(X, \zeta W)
- g_1(A_X \zeta W, \delta Z)
\]

(3.10)

for \( Z, W \in \Gamma(D^\varphi_2), U \in \Gamma(D^\varphi_1) \) and \( X \in ((\ker \varphi_*)^\perp) \).

**Proof.** The proof of this Theorem is same as Theorem 3.6. \( \square \)

**Theorem 3.8.** Let \( \varphi \) be the CBSS from the SM \((M_1, \phi, \xi, \eta, g_1)\) onto a RM \((M_2, g_2)\) with slant angles \( \bar{\theta}_1 \) and \( \bar{\theta}_2\). Then vertical distribution \((\ker \varphi_*)^\perp\) is locally Riemannian product \(M_1 D^\varphi_1 \times M_2 D^\varphi_2\) if and only if equation (3.7)-(3.10) holds where \( M_1 D^\varphi_1 \) and \( M_2 D^\varphi_2 \) are integral manifolds of distributions \( D^\varphi_1 \) and \( D^\varphi_2 \) respectively.

**Theorem 3.9.** Let \( \varphi \) be the CBSS from the SM \((M_1, \phi, \xi, \eta, g_1)\) onto a RM \((M_2, g_2)\) with slant angles \( \bar{\theta}_1 \) and \( \bar{\theta}_2\). Then horizontal distribution \((\ker \varphi_*)^\perp\) defines totally geodesic foliation if and only if

\[
\lambda^{-2} g_2(\nabla^{\bar{X}} \varphi, \zeta U, \varphi_* f Y) = -g_1(A_X \zeta U, t Y') - \eta(Y) g_1(\phi X, U)
+ \lambda^{-2} g_2(G(\ln \lambda), X) g_1(\zeta U, f Y)
+ g_1(G(\ln \lambda), \zeta U) g_1(X, f Y)
- g_1(G(\ln \lambda), f Y) g_1(X, \zeta U)
+ g_1(G(\ln \lambda), X) g_1(\zeta \delta U, Y)
+ g_1(G(\ln \lambda), \zeta \delta U) g_1(X, f Y)
- g_1(G(\ln \lambda), Y) g_1(X, \zeta \delta U)
- \lambda^{-2} g_2(\nabla^{\bar{X}} \varphi, \zeta \delta U, \varphi_* Y).
\]

(3.11)
\[ \lambda^{-2} g_2(\nabla^\phi X, \zeta V, \varphi_* f Y) = -g_1(A_X \zeta V, t Y) - \eta(Y) g_1(\phi X, V) + \lambda^{-2} g_2(G(\ln \lambda), X) g_1(\zeta V, f Y) + g_1(G(\ln \lambda), \zeta V) g_1(X, f Y) - g_1(G(\ln \lambda), f Y) g_1(X, \zeta V) + g_1(G(\ln \lambda), X) g_1(\zeta \delta V, Y) + g_1(G(\ln \lambda), \zeta \delta V) g_1(X, f Y) - g_1(G(\ln \lambda), Y) g_1(X, \zeta \delta V) - \lambda^{-2} g_2(\nabla^\phi X, \zeta \delta V, \varphi_* Y). \] (3.12)

for \( X, Y \in \Gamma((\ker \varphi_*)^\perp) \) and \( V \in \Gamma(D_2^0) \).

Proof. for \( X, Y \in \Gamma((\ker \varphi_*)^\perp) \) and \( U \in \Gamma(D_1^0) \) with using (2.2), (2.3) and (3.2), we have

\[ g_1(\nabla_X Y, U) = g_1(\nabla_X \phi \delta U, Y) - g_1(\nabla_X \zeta U, Y) - \eta(Y) g_1(\phi X, U). \]

Taking account the fact from Theorem 3.4, we can write

\[ \sin^2 \theta_1 g_1(\nabla_X Y, U) = -g_1(\nabla_X \zeta \delta U, Y) - g_1(\nabla_X \zeta U, Y) - \eta(Y) g_1(\phi X, U). \]

From (2.10), we can obtained

\[ \sin^2 \theta_1 g_1(\nabla_X Y, U) = -g_1(A_X \zeta U, t Y) - \eta(Y) g_1(\phi X, U) - \lambda^{-2} g_2(\varphi_* \nabla_X \zeta U, \varphi_* f Y) - \lambda^{-2} g_2(\varphi_* \nabla_X \zeta \delta U, \varphi_* Y). \]

Considering equation Lemma 2.3, we have

\[ \sin^2 \theta_1 g_1(\nabla_X Y, U) = -g_1(A_X \zeta U, t Y) - \eta(Y) g_1(\phi X, U) - \lambda^{-2} g_2(\nabla^\phi \zeta U, \varphi_* f Y) + \lambda^{-2} g_2(G(\ln \lambda), X) g_1(\zeta U, f Y) + g_1(G(\ln \lambda), \zeta U) g_1(X, f Y) - g_1(G(\ln \lambda), f Y) g_1(X, \zeta U) + g_1(G(\ln \lambda), X) g_1(\zeta \delta U, Y) + g_1(G(\ln \lambda), \zeta \delta U) g_1(X, f Y) - g_1(G(\ln \lambda), Y) g_1(X, \zeta \delta U) - \lambda^{-2} g_2(\nabla^\phi X, \zeta \delta U, \varphi_* Y). \]

Similarly, for \( X, Y \in \Gamma((\ker \varphi_*)^\perp) \) and \( V \in \Gamma(D_2) \), we have

\[ \sin^2 \theta_2 g_1(\nabla_X Y, V) = -g_1(A_X \zeta V, t Y) - \eta(Y) g_1(\phi X, V) - \lambda^{-2} g_2(\nabla^\phi \zeta V, \varphi_* f Y) + \lambda^{-2} g_2(G(\ln \lambda), X) g_1(\zeta V, f Y) + g_1(G(\ln \lambda), \zeta V) g_1(X, f Y) - g_1(G(\ln \lambda), f Y) g_1(X, \zeta V) + g_1(G(\ln \lambda), X) g_1(\zeta \delta V, Y) + g_1(G(\ln \lambda), \zeta \delta V) g_1(X, f Y) - g_1(G(\ln \lambda), Y) g_1(X, \zeta \delta V) - \lambda^{-2} g_2(\nabla^\phi X, \zeta \delta V, \varphi_* Y). \]
Theorem 3.10. Let $\varphi$ be the CBSS from the SM $(M_1, \phi, \xi, \eta, g_1)$ onto a RM $(M_2, g_2)$ with slant angles $\vartheta_1$ and $\vartheta_2$. Then vertical distribution $(\ker \varphi_\ast)$ defines totally geodesic foliation if and only if

$$
\lambda^{-2} g_2(\nabla_{\xi} \varphi_\ast \zeta U, \varphi_\ast \zeta V) = (\cos^2 \vartheta_1 - \cos^2 \vartheta_2) g_1(\nabla_{\xi} P_2 U, V) + g_1(A_{\delta} \varphi_\ast \zeta U) - g_1(A_{\xi} \varphi_\ast \zeta \delta U) + g_1(G(\ln \lambda), \varphi_\ast \zeta U) + g_1(G(\ln \lambda), \varphi_\ast \zeta U) - g_1(G(\ln \lambda), \varphi_\ast \zeta U) - g_1(G(\ln \lambda), \varphi_\ast \zeta \delta U) + g_1(G(\ln \lambda), \varphi_\ast \zeta \delta U) - g_1(G(\ln \lambda), \varphi_\ast \zeta \delta U) - g_1(G(\ln \lambda), \varphi_\ast \zeta \delta U),
$$

(3.13)

for $U, V \in \Gamma(\ker \varphi_\ast)$ and $X \in \Gamma((\ker \varphi_\ast)^\perp)$.

Proof. On taking $U, V \in \Gamma(\ker \varphi_\ast)$ and $X \in \Gamma((\ker \varphi_\ast)^\perp)$ with using (2.2), (2.3) and (3.2), we have

$$
g_1(\nabla_{\xi} V, X) = -g_1([U, X], V) + g_1(\nabla_{\xi} \varphi_\ast \zeta U, V) - g_1(\nabla_{\xi} \varphi_\ast \zeta U, V).
$$

On using decomposition (3.1) and Theorem 3.4, we obtained

$$
g_1(\nabla_{\xi} V, X) = -g_1([U, X], V) - \cos^2 \vartheta_1 g_1(\nabla_{\xi} P_1 U, V)
$$

$$
- \cos^2 \vartheta_2 g_1(\nabla_{\xi} P_2 U, V) + g_1(\nabla_{\xi} \varphi_\ast \zeta \delta U, V)
$$

$$
- g_1(\nabla_{\xi} \varphi_\ast \zeta U, V) - g_1(\nabla_{\xi} \varphi_\ast \zeta \delta U).
$$

With taking account the fact of equation (2.10), we can write

$$
\sin^2 \vartheta_1 g_1(\nabla_{\xi} V, X) = (\cos^2 \vartheta_1 - \cos^2 \vartheta_2) g_1(\nabla_{\xi} P_2 U, V)
$$

$$
- \sin^2 \vartheta_1 g_1([U, X], V) + g_1(A_{\xi} \varphi_\ast \zeta U, V)
$$

$$
- g_1(A_{\xi} \varphi_\ast \zeta \delta U, V) - g_1(\nabla_{\xi} \varphi_\ast \zeta U, V).
$$

Using equation (2.13), we yields

$$
\sin^2 \vartheta_1 g_1(\nabla_{\xi} V, X) = (\cos^2 \vartheta_1 - \cos^2 \vartheta_2) g_1(\nabla_{\xi} P_2 U, V)
$$

$$
- \sin^2 \vartheta_1 g_1([U, X], V) + g_1(A_{\xi} \varphi_\ast \zeta U, V)
$$

$$
+ \lambda^{-2} g_2((\nabla_{\varphi_\ast}) (X, \varphi_\ast \zeta U, \varphi_\ast \zeta U))
$$

$$
- \lambda^{-2} g_2((\nabla_{\varphi_\ast}) (X, \varphi_\ast \zeta U, \varphi_\ast \zeta U)).
$$

Considering Lemma 2.3, have

$$
\sin^2 \vartheta_1 g_1(\nabla_{\xi} V, X) = (\cos^2 \vartheta_1 - \cos^2 \vartheta_2) g_1(\nabla_{\xi} P_2 U, V)
$$

$$
- \sin^2 \vartheta_1 g_1([U, X], V)
$$

$$
+ g_1(A_{\xi} \varphi_\ast \zeta U, V) - g_1(A_{\xi} \varphi_\ast \zeta U, V)
$$

$$
+ g_1(G(\ln \lambda), \varphi_\ast \zeta U, \varphi_\ast \zeta U)
$$

$$
+ g_1(G(\ln \lambda), \varphi_\ast \zeta U, \varphi_\ast \zeta U)
$$

$$
- g_1(G(\ln \lambda), \varphi_\ast \zeta U, \varphi_\ast \zeta U)
$$

$$
- \lambda^{-2} g_2((\nabla_{\varphi_\ast}) (X, \varphi_\ast \zeta U, \varphi_\ast \zeta U)).
$$

This completes the proof of the Theorem.\(\square\)

We can provide these decomposition Theorems by taking into consideration the prior Theorems:
Theorem 3.11. Let \( \varphi \) be the CBSS from the SM \((M_1, \phi, \xi, \eta, g_1)\) onto a RM \((M_2, g_2)\) with slant angles \(\bar{\theta}_1\) and \(\bar{\theta}_2\). Then the total space \(M_1 \oplus \times M_1 \otimes \times M_1(\ker \varphi_*)^{-}\) is locally product if and only if equation (3.7)-(3.12) are holds where \(M_1 \oplus \times D_2^\theta_1, D_2^\theta_2\) and \((\ker \varphi_*)^{-}\) respectively.

Theorem 3.12. Let \( \varphi \) be the CBSS from the SM \((M_1, \phi, \xi, \eta, g_1)\) onto a RM \((M_2, g_2)\) with slant angles \(\bar{\theta}_1\) and \(\bar{\theta}_2\). Then the total space \(M_1 \ker \varphi_* \times M_1(\ker \varphi_*)^{-}\) is locally product if and only if equation (3.11)-(3.13) are holds where \(M_1 \ker \varphi_*\) and \(M_1(\ker \varphi_*)^{-}\) are integral manifolds of the distributions \((\ker \varphi_*)^{-}\) and \((\ker \varphi_*)^{-}\) respectively.

Theorem 3.13. Let \( \varphi \) be the CBSS from the SM \((M_1, \phi, \xi, \eta, g_1)\) onto a RM \((M_2, g_2)\) with slant angles \(\bar{\theta}_1\) and \(\bar{\theta}_2\). Then \( \varphi \) is totally geodesic map if and only if

\[
\begin{align*}
\lambda^{-2} g_2(\nabla_{\nabla_V \varphi_*} \xi \delta V, \varphi_* X) &= (\cos^2 \bar{\theta}_1 - \cos^2 \bar{\theta}_2) g_1(\nabla_V P_2 V, X) \\
&- g_1(\nabla_U \xi V, X) \\
&+ g_1(\nabla_M \xi V, X) g_1(\xi \delta V, X) \\
&- g_1(\nabla_M \xi V, Y) g_1(\xi \delta V, X) \\
&- g_1(\nabla_M \xi V, X) g_1(\xi \delta V, Y) \\
&+ g_1(\nabla_M \xi V, Y) g_1(\xi \delta V, Y) \\
&- \lambda^{-2} g_2(\nabla_{\nabla_V \varphi_*} \xi \delta V, \varphi_* X)
\end{align*}
\]

\[
\begin{align*}
\lambda^{-2} g_2(\nabla_{\nabla_V \varphi_*} \xi \delta U, \varphi_* Y) &= (\cos^2 \bar{\theta}_2 - \cos^2 \bar{\theta}_1) g_1(\nabla_X P_1 U, Y) \\
&+ g_1(\nabla_U \xi U, \xi \delta V, \varphi_* Y) \\
&+ g_1(\nabla_M \xi U, \xi \delta V, \varphi_* Y) \\
&- g_1(\nabla_M \xi U, \xi \delta V, \varphi_* Y) \\
&+ g_1(\nabla_M \xi U, \xi \delta V, \varphi_* Y) \\
&+ g_1(\nabla_M \xi U, \xi \delta V, \varphi_* Y) \\
&+ g_1(\nabla_M \xi U, \xi \delta V, \varphi_* Y) \\
&+ g_2(\nabla_{\nabla_V \varphi_*} \xi \delta U, \varphi_* Y)
\end{align*}
\]

for \( U, V \in \Gamma(\ker \varphi_*) \) and \( X, Y \in \Gamma((\ker \varphi_*)^{-}) \).

Proof. For \( U, V \in \Gamma(\ker \varphi_*) \) and \( X \in \Gamma((\ker \varphi_*)^{-}) \) with using equation (2.13), we can write

\[
\lambda^{-2} g_2((\nabla \varphi_*)(U, V), \varphi_* X) = g_1(\nabla_U V, X) \tag{3.14}
\]

From equations (2.2), (2.3) and (3.2), we have

\[
g_1(\nabla_U V, X) = g_1(\nabla_U \xi V, \phi X) - g_1(\nabla_U \phi \delta V, X) \\
+ g_1(\nabla_U \delta V, \eta(X) + g_1(\nabla_U, \delta V) \eta(X)
\]

Considering Theorem 3.4 and decomposition (3.1), we obtained

\[
\begin{align*}
\sin^2 \bar{\theta}_1 g_1(\nabla_U V, X) \\
= (\cos^2 \bar{\theta}_2 - \cos^2 \bar{\theta}_1) g_1(\nabla_U P_2 V, X) \\
+ g_1(\nabla_U \xi \delta V, X) + g_1(\nabla_U \xi \phi V).
\end{align*}
\]
From equation (3) and (3.14), we get
\[
\sin^2 \vartheta_1 \lambda^{-2} g_2((\nabla \varphi_*)(U, V), \varphi_* X)
\]
\[
= (\cos^2 \vartheta_2 - \cos^2 \vartheta_1) g_1(\nabla U P_2 V, X)
\]
\[
+ g_1(\nabla U \zeta \delta V, X) + g_1(\nabla U \zeta \phi X).
\]
On using equations (2.7) and (2.8), we have
\[
\sin^2 \vartheta_1 \lambda^{-2} g_2((\nabla \varphi_*)(U, V), \varphi_* X)
\]
\[
= (\cos^2 \vartheta_2 - \cos^2 \vartheta_1) g_1(\nabla U P_2 V, X)
\]
\[
+ g_1(\nabla \zeta \delta \nabla U, X) + g_1(\nabla \zeta \phi, tX)
\]
\[
+ g_1(\nabla \zeta \phi X, fX).
\]
From equation (2.13) and Lemma 2.3, we get
\[
\sin^2 \vartheta_1 \lambda^{-2} g_2((\nabla \varphi_*)(U, V), \varphi_* X)
\]
\[
= \lambda^{-2} g_2(\nabla (\ln \lambda) \varphi_* \xi \delta V, \varphi_* X)
\]
\[
+ \lambda^{-2} g_2(\nabla \xi \varphi_* \xi \delta V, \varphi_* X)
\]
\[
+ \lambda^{-2} g_2(\nabla (\ln \lambda) \varphi_* \xi \delta V, \varphi_* X)
\]
\[
+ \lambda^{-2} g_2(\nabla \xi \varphi_* \xi \delta V, \varphi_* fX).
\]
On the other hand, for \(U \in \Gamma(ker \varphi_*)\) and \(X, Y \in \Gamma((ker \varphi_*)^\perp)\), we get
\[
\lambda^{-2} g_2((\nabla \varphi_*)(X, U), \varphi_* Y) = g_1(\nabla X U, Y). \tag{3.15}
\]
On using (2.2), (2.3) and (3.2), we have
\[
g_1(\nabla X U, Y) = -g_1(\nabla X \phi \delta U, Y) + g_1(\nabla X \zeta \phi Y) + g_1(tX, U) \eta(Y)
\]
With the help of decomposition (3.8) and Theorem 3.4, we obtain
\[
\sin^2 \vartheta_1 g_1(\nabla X U, Y)
\]
\[
= (\cos^2 \vartheta_2 - \cos^2 \vartheta_1) g_1(\nabla X P_1 U, Y)
\]
\[
+ g_1(\nabla X \zeta U, \phi Y) + g_1(tX, U) \eta(Y)
\]
\[
- g_1(\nabla X \zeta \delta U, Y). \tag{3.16}
\]
From (3.15) and (3.16), we can write
\[
\sin^2 \vartheta_1 \lambda^{-2} g_2((\nabla \varphi_*)(X, U), \varphi_* Y)
\]
\[
= (\cos^2 \vartheta_2 - \cos^2 \vartheta_1) g_1(\nabla X P_1 U, Y)
\]
\[
+ g_1(\nabla X \zeta U, tY) + g_1(tX, U) \eta(Y)
\]
\[
- g_1(\nabla X \zeta \delta U, Y) + g_1(\nabla X \zeta U, Y).
\]
From equation (2.9) and (2.10), we have
\[
\sin^2 \vartheta_1 \lambda^{-2} g_2((\nabla \varphi_*)(X, U), \varphi_* Y)
\]
\[
= (\cos^2 \vartheta_2 - \cos^2 \vartheta_1) g_1(A X P_1 U, Y)
\]
\[
+ g_1(AX \zeta U, tY) + g_1(tX, U) \eta(Y)
\]
\[
- g_1(\nabla X \zeta \delta U, Y) + g_1(\nabla X \zeta U, Y).
2.3 with equation (2.13), we yields
\[
\sin^2 \theta_1 \lambda^{-2} g_2((\nabla \varphi_\ast)(X, U), \varphi_\ast Y)
= (\cos^2 \theta_2 - \cos^2 \theta_1) g_1(A \xi P_1 U, Y)
+ g_1(A \xi \zeta U, t Y) + g_1(t X, Y) \eta(Y)
- g_1(G(\ln \lambda), X) g_1(\zeta U, Y)
- g_1(G(\ln \lambda), \zeta \delta U) g_1(X, Y)
+ g_1(G(\ln \lambda), Y) g_1(X, \zeta U)
+ g_1(G(\ln \lambda), X) g_1(\zeta U, f Y)
+ g_1(G(\ln \lambda), f Y) g_1(X, \zeta U)
- \lambda^{-2} g_2(\nabla \xi \varphi_\ast \zeta U)
- \lambda^{-2} g_2(\nabla \xi \varphi_\ast \zeta U, \varphi_\ast f Y).
\]

Finally we show that \( \lambda \) is constant on \( \Gamma(D_1) \). For \( U_1, U_2 \in \Gamma(D_1) \) and from Lemma 2.3, we obtain
\[
(\nabla \varphi_\ast)(\zeta U_1, \zeta U_2) = \zeta U_1(\ln \lambda) \varphi_\ast \zeta U_2 + \zeta U_2(\ln \lambda) \varphi_\ast \zeta U_1
- g_1(\zeta U_1, \zeta U_2) \varphi_\ast (G(\ln \lambda)).
\]
Replacing \( U_2 \) by \( U_1 \) in above equation, we get
\[
(\nabla \varphi_\ast)(\zeta U_1, \zeta U_1) = 2\zeta U_1(\ln \lambda) \varphi_\ast \zeta U_1
- g_1(\zeta U_1, \zeta U_1) \varphi_\ast (G(\ln \lambda)).
\] (3.17)
Taking inner product with \( \varphi_\ast \zeta U_1 \) in (3.17), we can write
\[
2g_1(G(\ln \lambda), \zeta U_1) g_2(\varphi_\ast \zeta U_1, \varphi_\ast \zeta U_1)
- g_1(\zeta U_1, \zeta U_1) g_2(\varphi_\ast G(\ln \lambda), \varphi_\ast \zeta U_1) = 0,
\]
which shows that \( \lambda \) is constant on \( \Gamma(D_1^0) \). Similarly, we can show that \( \lambda \) is constant on \( \Gamma(D_2^0) \) and \( \Gamma(\mu) \). This completes the proof of the Theorem. □

**Funding:** This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

**Conflicts of Interest:** The authors declare no conflict of interest.

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