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Study of Maximal Open Sets and Its Images with Ideals

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ABSTRACT: Maximal I-open sets and maximal I^*-open sets have been introduced through this write up with help of the ideal's related two operators local function and its associated set valued set function. The role of these sets in the cofinite sets has been discussed rigorously. Several characterizations, decompositions and examples of these sets have also been discussed. In respect of topological invariance, the homeomorphic images of maximal I-open sets and maximal I^*-open sets and its related structures have also been discussed sternly. The situation of above mentioned sets has also been discussed in the field of compatible ideal.

Key Words: $*$-open set, maximal I-open set, maximal I^*-open set, $()^*$-operator, ψ-operator, homeomorphism.

Contents

1. Introduction and Preliminaries

Recently, Selim and Modak [20] introduced associated set-valued set function (in short associated function) in the literature. The operator ψ [22] on an ideal topological space is an example of an associated function. This associated function ψ has an association with the local function [7] of the ideal topological space and they are related by the following relation $\psi(A) = X \setminus (X \setminus A)^*$ [16,22]. Interior and closure operators in a topological space are also examples of associated functions. On the other hand, in [21], Nakaoka and Oda introduced maximal open sets in topological space and further discussed their various properties in topological spaces and in locally finite spaces. In [27], Rashid and Hussein introduced maximal and minimal regular β-open sets in topological spaces and discussed their related properties.

2. Historical Background

Various types of limit points such as closure points, ω-limit points, ω-accumulation points etc. may be jointly studied with the help of ideals. This study has been introduced by the mathematicians Kuratowski [12] and Vaidyanathswamy [29].

The ideal is a collection I of subsets of a set X which satisfies hereditary and finite additivity properties. If I is an ideal on a topological space (X, τ), then it is called an ideal topological space. Throughout

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this paper, this triplicate (X, τ, I) are denoted as XIT, where the topological space (X, τ) will be denoted as XT, i.e., we have $XIT = (X, \tau, I)$ and $XT = (X, \tau)$. In this work, we use T *space* and space as always a topological space and an ideal topological space respectively, and no separation axioms are assumed unless explicitly stated.

According to the modern notation of local function, we have, $M^*(XIT) = \{p \in X : M \cap V \notin I$ for every $V \in \tau(p)\}$, where $\tau(p) = \{V \in \tau : p \in V\}$, when there is no confusion, we will write M^* or simply M^{*I} or $M^*(I, \tau)$ or M_τ^{*I} and call it the "local function of M". This local function helps to determine some new topologies on X, one of them is $*$-topology. The members of the $*$-topology are called the $*$-open sets. One of the most useful basis of the $*$-topology is, $\beta(I, \tau) = \{M \setminus I_1 : M \in \tau, I_1 \in I\}$ [8].

For the answer of the question, "when $\beta(I, \tau)$ and $*$-topology are equal"? Njåstad [24,25] has given the answer of this question with the help of "Compatibility". The ideal I to be compatible with τ, denoted $I \sim \tau$ if the following holds for every $M \subseteq X$: if for all $p \in M$, there exists $V \in \tau(p)$ where $\tau(p) = \{V \in \tau : p \in V\}$ such that $V \cap M \in I$, then $M \in I$. The operator $\psi : \wp(X) \to \tau$ [7] which has been defined as a associated function [20] of local function $()^*$, that is $\psi(M) = X \setminus (X \setminus M)^*$, we will write simply $\psi_\tau(M)$ or $\psi_\tau^I(M)$. In this context for $I \sim \tau$, $\psi(\psi(M)) = \psi(M)$ [7] and $\psi(M) \setminus M \in I$ for every $M \subseteq X$.

For a $*$-topological space (X, τ^*), we denote 'Cl^*' and 'Int^*' as the closure operator and interior operator respectively. Furthermore, in a space XIT, mathematicians handle two structures on X, thus the condition $\tau \cap I = \{\emptyset\}$ is useful for the study of the same field. This concepts introduced by Newcomb [23] as *τ-boundary*.

In this context e-$\mathcal{I}$-open sets [2,4] and e-$\mathcal{I}$-continuity [2,4] have been defined in terms of regular open sets and the closure operator of the $*$-topology. Also a new flavor of local function has also been defined with the help of regular open sets, and the other notions from the same local function has also been considered by the respective mathematicians.

As an application of local function, I-open set has been defined in literature. For a space XIT : A subset M of X is said to be I-open if $M \subseteq Int(M^*)$ [1,9]. The set of all I-open sets in a space (X, τ, I) is denoted by $IO(XIT)$ or written simply as $IO(X)$, when there is no scope for misunderstanding.

In this paper, we have studied jointly associated functions and maximal-open sets. As an extraction, we have found maximal I-open sets and maximal I^*-open sets in the topological spaces with ideals. These sets play an important role in the study of local functions and set operator ψ. Homeomorphisms in the topological spaces are also played a remarkable role in the study of maximal I-open sets and maximal I^*-open sets.

3. Maximal I-open sets

In this section, we shall present maximal I-open set and investigate numerous characterizations and features of maximal I-open sets:

Definition 3.1 ([21]) *A proper nonempty open set M in a topological space XT is said to be a maximal open set if and only if every open set which contains M is either X or M.*

Definition 3.2 *A proper nonempty subset M of X in a space XIT is said to be a maximal I-open set if and only if it is an I-open set and every I-open set which contains M is either X or M.*

For existence of maximal I-open set, we intimate following examples:

Example 3.1 *Consider a space XIT where $X = \{e^1, e^2, e^3, e^4\}$, $\tau = \{\emptyset, X, \{e^1, e^2\}, \{e^3\}, \{e^1, e^2, e^3\}\}$ and $I = \{\emptyset, \{e^2\}\}$. Then, $IO(X) = \{\emptyset, X, \{e^1\}, \{e^3\}, \{e^1, e^2\}, \{e^1, e^3\}, \{e^1, e^2, e^3\}, \{e^1, e^3, e^4\}\}$. Therefore $\{e^1, e^2, e^3\}$ and $\{e^1, e^3, e^4\}$ are maximal I-open sets in this space.*

Example 3.2 *Consider XIT be a space where $X = \mathbb{R}$, set of all real numbers, $\tau = \{\emptyset, [a, b], \mathbb{R}\}$ where $a, b \in \mathbb{R}, a \leq b$ and $I = \{\emptyset, \{a\}\}$. Take $\emptyset \neq M \subseteq \mathbb{R} \setminus [a, b]$. Then $M^* = (-\infty, a) \cup (b, \infty)$. This implies $Int(M^*) = \emptyset$ and hence M is not I-open set in this case. If we take $M = \{a\}$, then $M^* = \emptyset$. This implies $Int(M^*) = \emptyset$ and hence M is not I-open set in this case. If we take $\emptyset \neq M \subseteq (a, b]$, then $M^* = \mathbb{R}$ and $Int(M^*) = \mathbb{R}$. This implies $M \subseteq Int(M^*)$ and hence M is an I-open set. If we take $M = \emptyset$, then $M^* = \emptyset$ and $Int(M^*) = \emptyset$. This implies $M \subseteq Int(M^*)$ and hence $M = \emptyset$ is an I-open set. Hence $IO(X) = \{\emptyset\} \bigcup \{M : \emptyset \neq M \subseteq (a, b]\}$. This implies $(a, b]$ is a maximal I-open set.*

By the following, we can say that in a space XIT, an open set (or $*$-open set) need not be a maximal I-open set.

Example 3.3 *In Example 3.2, $\tau^*(I) = \{\emptyset, \mathbb{R}, (a,b], [a,b], \mathbb{R} \setminus \{a\}, \mathbb{R}\}$. Here $[a,b]$ is an open set as well as a $*$-open set but not a maximal I-open set. But in Example 3.1, $\{e^1, e^2, e^3\}$ is an open set, a $*$-open set and a maximal I-open set. Hence an open set (or $*$-open set) need not be a maximal I-open set.*

The set of all maximal I-open sets in a space XIT is denoted by $M_{max}IO(XIT)$ or written simply as $M_{max}IO(X)$ when there is no scope for misunderstanding.
We may deduce from the following examples that maximal openness [21] and maximal I-openess are independent concepts:

Example 3.4 *In Example 3.1, $M_{max}IO(X) = \{\{e^1, e^2, e^3\}, \{e^1, e^3, e^4\}\}$. Here $\{e^1, e^3, e^4\}$ is a maximal I-open set, but $\{e^1, e^3, e^4\}$ is neither a maximal open set nor an open set.*

Example 3.5 *We consider a space XIT, where $X = \{e^1, e^2, e^3, e^4\}$, $\tau = \{\emptyset, X, \{e^1, e^2\}, \{e^3\}, \{e^1, e^2, e^3\}\}$ and $I = \{\emptyset, \{e^1\}, \{e^3\}, \{e^1, e^3\}\}$. Then there is, $IO(X) = \{\emptyset, \{e^2\}, \{e^1, e^2\}\}$. Here $\{e^1, e^2, e^3\}$ is a maximal open set but not a maximal I-open set.*

Let XIT be a space. An XIT will be denoted as $SM_{max}IO(X)$, if it has atleast one maximal I-open set.

Example 3.6 *Consider the example $X = \{e^1, e^2\}$, $\tau = \{\emptyset, X, \{e^1\}\}$ and $I = \{\emptyset, \{e^1\}\}$. Then, $IO(X) = \{\emptyset\}$. Therefore there is no member in this space XIT which belongs to $M_{max}IO(X)$.*

Theorem 3.1 ([9]) *Arbitrary union of I-open sets in a space XIT is also an I-open set.*

Lemma 3.1 *Consider a space $SM_{max}IO(X)$. Then:*

1. *for $M \in M_{max}IO(X)$ and $O \in IO(X)$, either $M \cup O = X$ or $O \subseteq M$.*
2. *for $M, N \in M_{max}IO(X)$, either $M \cup N = X$ or $M = N$.*

Proof: *(1)* Let O be an I-open set such that $M \cup O \neq X$. Since M is a maximal I-open set and $M \subseteq M \cup O$, then we have either $M \cup O = M$ or $M \cup O = X$. But $M \cup O \neq X$, then $M \cup O = M$ and hence $O \subseteq M$.
(2) If $M \cup N \neq X$, then $M \subseteq N$ and $N \subseteq M$ by (1) of Lemma 3.1. Thus $M = N$. □

Note 1 *From Lemma 3.1* (2)*, in a space $SM_{max}IO(X)$, union of two members of $M_{max}IO(X)$ is either a member of $M_{max}IO(X)$ or equals X .*

Note 2 *Let $SM_{max}IO(X)$ be a space, intersection of two members of $M_{max}IO(X)$ may not be a member of $M_{max}IO(X)$ again.*
In Example 3.1, $\{e^1, e^2, e^3\}$ and $\{e^1, e^3, e^4\}$ are maximal I-open sets. But $\{e^1, e^2, e^3\} \cap \{e^1, e^3, e^4\} = \{e^1, e^3\}$ is not a maximal I-open set.

Theorem 3.2 *Let $SM_{max}IO(X)$ be a space and let M_1, M_2 and M_3 be three members of $M_{max}IO(X)$ with $M_1 \neq M_2$. If $M_1 \cap M_2 \subseteq M_3$, either $M_1 = M_3$ or $M_2 = M_3$.*

Proof: We have,

$$\begin{aligned}
M_1 \cap M_3 &= M_1 \cap (M_3 \cap X) \\
&= M_1 \cap (M_3 \cap (M_1 \cup M_2))(\textit{by Lemma } 3.1 \ (1)) \\
&= M_1 \cap ((M_3 \cap M_1) \cup (M_3 \cap M_2)) \\
&= (M_1 \cap M_3) \cup (M_1 \cap M_2 \cap M_3) \\
&= (M_1 \cap M_3) \cup (M_1 \cap M_2)[\textit{since } M_1 \cap M_2 \subseteq M_3] \\
&= M_1 \cap (M_2 \cup M_3).
\end{aligned}$$

Thus if $M_3 \neq M_2$, then $M_2 \cup M_3 = X$ and hence $M_1 \cap M_3 = M_1$. This implies $M_1 \subseteq M_3$. Since M_1 and M_3 are maximal I-open sets, then $M_1 = M_3$. □

Following is the contrapositive way of the Theorem 3.2:

In Example 3.1, if we consider $M_1 = \{e^1, e^2, e^3\}$ and $M_2 = \{e^1, e^3, e^4\}$. Then, $M_1, M_2 \in M_{max}IO(X)$. Here, $M_1 \cap M_2 = \{e^1, e^3\} = M_3$ (say) is not a maximal I-open set and also neither $M_1 \neq M_3$ nor $M_2 \neq M_3$.

Theorem 3.3 *Let $SM_{max}IO(X)$ be a space and let M_1, M_2 and M_3 be three members of $M_{max}IO(X)$ such that they are different from each other. Then $M_1 \cap M_2 \nsubseteq M_1 \cap M_3$.*

Proof: If possible, let $M_1 \cap M_2 \subseteq M_1 \cap M_3$, then

$$(M_1 \cap M_2) \cup (M_2 \cap M_3) \subseteq (M_1 \cap M_3) \cup (M_2 \cap M_3).$$

This implies $M_2 \cap (M_1 \cup M_3) \subseteq (M_1 \cup M_2) \cap M_3$. Since $M_1 \cup M_3 = X = M_1 \cup M_2$, then $M_2 \subseteq M_3$ and hence $M_2 = M_3$ that contradicts our given condition. Thus, $M_1 \cap M_2 \nsubseteq M_1 \cap M_3$. □

Now we consider the concept of I-neighbourhood:
Let XIT be a space and $p \in X$. A subset $S \subseteq X$ is called an I-neighbourhood of p if there exists $A \in IO(X)$ such that $p \in A \subseteq S$.

Clearly, every I-open sets is an I-neighbourhood of each of its points. The collection of all I-neighbourhoods of a point $p \in X$ of a space XIT is simply denoted by $I\mathcal{N}(p)$.

Proposition 3.1 *Every I-neighbourhood is an I-open set in a space XIT.*

Proof: Let $\mathcal{W}$ be an I-neighbourhood. Then for all $p \in \mathcal{W}$, there exists $O \in IO(X)$ such that $p \in O \subseteq \mathcal{W}$. Also $\mathcal{W} = \bigcup_{p \in \mathcal{W}} \{p\} \subseteq \bigcup_{p \in O \in IO(X)} O \subseteq \mathcal{W}$. This implies $\mathcal{W} = \bigcup O$ where $p \in O \in IO(X)$. Then, by Theorem 3.1, $\mathcal{W}$ is an I-open set. □

Proposition 3.2 *Let $SM_{max}IO(X)$ be a space. If $M \in M_{max}IO(X)$ and $p \in M$, then $M = \bigcup\{N(p) : N(p) \in I\mathcal{N}(p)$ and $N(p) \cup M \neq X\}$.*

Proof: If $N(p) \in I\mathcal{N}(p)$ and $N(p) \cup M \neq X$, then by Lemma 3.1 (1), $N(p) \subseteq M$. Hence $M \subseteq \bigcup\{N(p) : N(p) \in I\mathcal{N}(p)$ such that $N(p) \cup M \neq X\} \subseteq M$. This implies $M = \bigcup\{N(p) : N(p) \in I\mathcal{N}(p)$ and $N(p) \cup M \neq X\}$. Hence the result. □

For exceptional circumstances, we now check the presence of proper maximal I-open sets. For this we consider cofinite I-open sets that means those I-open sets whose complement is a finite subset.

Theorem 3.4 *If O is a proper nonempty cofinite I-open subset in a space XIT, then, there exists atleast one (cofinite) maximal I-open set M such that $O \subseteq M$.*

Proof: If O is a maximal I-open set, we take $M = O$. If not, then there exists an I-open set M_1(cofinite) such that $O \subsetneqq M_1 \neq X$. If M_1 is a maximal I-open set, we take $M = M_1$. If not, then there exists an I-open set M_2 (cofinite) such that $O \subsetneqq M_1 \subsetneqq M_2 \neq X$. We have a sequence of I-open sets satisfying, $O \subsetneqq M_1 \subsetneqq M_2 \subsetneqq M_3 \subsetneqq M_4 ... \subsetneqq M_k ...$, if we continue this approach. Since O is a cofinite, then the aforementioned process occurs only finitely and consequently we have a maximal I-open set $M = M_n$ for some $n \in \mathbb{N}$. Hence the result. □

Theorem 3.5 *In a space $SM_{max}IO(X)$, if $M \in M_{max}IO(X)$ and p is a member of $X \setminus M$, then $X \setminus M \subseteq W$ for any $W \in I\mathcal{N}(p)$*

Proof: Since $p \in X \setminus M$, then for any $W \in I\mathcal{N}(p)$, $W \nsubseteq M$. Thus $M \cup W = X$, by Lemma 3.1 (1) and hence $X \setminus M \subseteq W$. □

Corollary 3.1 *Let $SM_{max}IO(X)$ be a space and suppose $M \in M_{max}IO(X)$. Then one of the followings are true:*

1. *for any $p \in X \setminus M$ and for each $W \in I\mathcal{N}(p)$, $W = X$.*
2. *there exists an I-open set W such that $X \setminus M \subseteq W$ and $W \subsetneqq X$.*

Proof: If *(1)* is not true, then there exists an element p of $X \setminus M$ and $W \in I\mathcal{N}(p)$ such that $W \subsetneq X$. Then, by Theorem 3.5, we have $X \setminus M \subseteq W$. □

Corollary 3.2 *Let $SM_{max}IO(X)$ be a space. If $M \in M_{max}IO(X)$, then one of the followings is true:*

1. *for any $p \in X \setminus M$ and for each $W \in I\mathcal{N}(p)$, $X \setminus M \subsetneqq W$.*
2. *there exists an I-open set W such that $X \setminus M = W \neq X$.*

Proof: Assume that *(2)* does not hold. Then, by Theorem 3.5, for each $p \in X \setminus M$ and for any $W \in I\mathcal{N}(p)$, $X \setminus M \subset W$. Hence we have, $X \setminus M \subsetneqq W$. □

Now, we are giving supporting examples to discuss that in a space XIT every e-$\mathcal{I}$-open set [4] may not be a maximal I-open set where as a maximal I-open set may be an e-$\mathcal{I}$-open set:

Example 3.7 *Consider a space XIT where $X = \{e^1, e^2, e^3, e^4\}$, $\tau = \{\emptyset, X, \{e^1\}, \{e^2, e^4\}, \{e^1, e^2, e^4\}\}$ and $I = \{\emptyset, \{e^3\}, \{e^4\}, \{e^3, \{e^4\}\}$. Take, $A = \{e^1, e^3\}$. Then, A is an e-$\mathcal{I}$-open set but not a maximal I-open set.*

Example 3.8 *In Example 3.1, $\{e^1, e^3, e^4\}$ is a maximal I-open set as well as an e-$\mathcal{I}$-open set.*

4. Maximal I^*-open sets

In this part, we shall introduce maximal I^*-open set and study several characterizations and properties of the maximal I^*-open sets:

Definition 4.1 *A nonempty $*$-open set M in a space XIT is said to be a maximal I^*-open set if and only if every $*$-open set which contains $\psi(M)$ is either $\psi(M)$ or X.*

X is always a maximal I^*-open set. This maximal I^*-open set is called improper maximal I^*-open set. For existence of maximal I^*-open set, we intimate following examples:

Example 4.1 *Let $X = \{e^1, e^2, e^3, \}$, $\tau = \{\emptyset, X, \{e^1\}, \{e^2\}, \{e^1, e^2\}\}$ and $I = \{\emptyset, \{e^1\}\}$. Therefore all the $*$-open sets are $\emptyset, X, \{e^1\}, \{e^2\}, \{e^1, e^2\}, \{e^2, e^3\}$. Take $M = \{e^1, e^2\}$. Then $\psi(M) = X \setminus (X \setminus M)^* = X \setminus \{e^3\}^* = X \setminus \{e^3\} = \{e^1, e^2\}$. Here the non empty $*$-open sets which contain $\psi(M)$ is either $\psi(M)$ or X. Therefore $M = \{e^1, e^2\}$ is a maximal I^*-open set.*

Example 4.2 *Consider XIT be a space where $X = \mathbb{N}$, set of all natural numbers, $\tau = \{\emptyset, X, \{1\}, \{1, 2\}, \{1, 2, 3\}, ..., \{1, 2, 3, ..., n\}, ...\}$ and $I = \{\emptyset, \{1\}\}$. Therefore $\tau^*(I) = \{\emptyset, X, \{1\}, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}, \{2, 3, 4\}..., \{1, 2, 3, ..., n\}, ...\}$. Clearly, there is no finite proper maximal I^*-open set.*

Example 4.3 *Consider XIT be a space where $X = \mathbb{R}$, set of all real numbers, $\tau = \{\emptyset, [a, b], \mathbb{R}\}$ where $a, b \in \mathbb{R}, a \leq b$ and $I = \{\emptyset, \{a\}\}$. Therefore $\tau^*(I) = \{\emptyset, \mathbb{R}, (a, b], [a, b], \mathbb{R} \setminus \{a\}, \mathbb{R}\}$. Take $M = (a, b]$. Then $\psi(M) = \mathbb{R} \setminus \{(-\infty, a] \cup (b, \infty)\}^* = \mathbb{R} \setminus ((-\infty, a]^* \cup (b, \infty)^*)$. Now $(-\infty, a]^* = \{x \in \mathbb{R} : (-\infty, a] \cap U \notin I$ for all $U \in \tau(x)\} = (-\infty, a) \cup (b, \infty)$ where $\tau(x) = \{U \in \tau : x \in U\}$. Similarly, $(b, \infty)^* = (-\infty, a) \cup (b, \infty)$. This implies $(-\infty, a]^* \cup (b, \infty)^* = (-\infty, a) \cup (b, \infty)$ and hence $\psi(M) = \mathbb{R} \setminus ((-\infty, a) \cup (b, \infty)) = [a, b]$. Here the non empty $*$-open sets which contain $\psi(M)$ is either $\psi(M)$ or $\mathbb{R}$. Therefore $M = (a, b]$ is a maximal I^*-open set.*

Definition 4.2 *Let XIT be a space. Then XIT is called a space with proper maximal I^*-open set(s) (simply $M_{max}I^*O(XIT)$ or $M_{max}I^*O(X)$ when there is no scope for misunderstanding), if there exists a proper maximal I^*-open set in XIT.*

Example 4.4 *Consider a space XIT where $X = \{e^1, e^2, e^3\}$, $\tau = \{\emptyset, X, \{e^1, e^3\}\}$ and $I = \{\emptyset\}$. Then $\tau^*(I) = \{\emptyset, X, \{e^1, e^3\}\}$. Take $M = \{e^1, e^3\}$. Then $\psi(M) = \{e^1\}$. Thus M is not a maximal I^*-open set. Hence there is no proper maximal I^*-open set in this space XIT.*

By the following, we can say that in a space XIT, an open set (or $*$-open set) need not be a maximal I^*-open set.

Example 4.5 *In Example 4.2, $\{1, 2\}$ is an open set as well as a $*$-open set but not a maximal I^*-open set. In Example 4.1, $M = \{e^1, e^2\}$ is an open set, a $*$-open set and a maximal I^*-open set. Hence an open set (or $*$-open set) need not be a proper maximal I^*-open set.*

We may deduce from the following examples that maximal openness, maximal I-openess and maximal I^*-openess are independent concepts:

Example 4.6 *In examples 3.4 and 3.5, we have already proved that maximal openness and maximal I-openess are independent to each other.*

Example 4.7 *In Example 3.1, $\tau^*(I) = \{\emptyset, X, \{e^1, e^2\}, \{e^3\}, \{e^1, e^2, e^3\}, \{e^1\}, \{e^1, e^3\}, \{e^1, e^3, e^4\}\}$. Take $M = \{e^1, e^3\}$. Then $\psi(M) = X \setminus \{e^2, e^4\}^* = X$. This implies M is a maximal I^*-open set but M is neither a maximal I-open set nor an open set.*

Note 3 *In examples 3.2 and 3.3, $\{b\}$ is an I-open set but not a $*$-open set. So every I-open set need not be a $*$-open set and hence we conclude that every maximal I-open set need not be a maximal I^*-open set.*

Lemma 4.1 *Let XIT be a space. Then union of any two maximal I^*-open sets is again a maximal I^*-open set.*

Proof: Let M and N be two maximal I^*-open sets. We have to prove that $M \cup N$ is a maximal I^*-open set. If possible suppose $M \cup N$ is not a maximal I^*-open set, then there exists a nonempty proper $*$-open set M_1 such that $\psi(M \cup N) \subsetneqq M_1$. This implies $\psi(M) \cup \psi(N) \subsetneqq M_1$ [7] and hence $\psi(M) \subsetneqq M_1$ and $\psi(N) \subsetneqq M_1$ which lead a contradiction to the fact that M and N are maximal I^*-open sets. Hence our assumption is wrong. Thus union of any two maximal I^*-open sets is again a maximal I^*-open set. □

Note 4 *In a space XIT, intersection of two members of $M_{max}I^*O(X)$ need not be a member of $M_{max}I^*O(X)$ again.*
Consider the space XIT where $X = \{e^1, e^2\}$, $\tau = \{\emptyset, X, \{e^1\}\}$ and $I = \{\emptyset, \{e^1\}\}$. Then $\tau^*(I) = \{\emptyset, X, \{e^1\}, \{e^2\}\}$. Here $M = \{e^1\}$ and $N = \{e^2\}$ are members of $M_{max}I^*O(X)$ but $M \cap N = \emptyset$ is not a member of $M_{max}I^*O(X)$.

Lemma 4.2 *Let XIT be a space. Then any proper maximal I^*-open set M and for any $*$-open set O, either $\psi(M) \cup O = X$ or $O \subseteq \psi(M)$.*

Corollary 4.1 *Let XIT be a space. Then, any proper maximal I^*-open set M and for any open set O, either $\psi(M) \cup O = X$ or $O \subseteq \psi(M)$.*

Proof: For any $*$-open set O, $\psi(M) \subseteq \psi(M) \cup O$. Since $M \in M_{max}I^*(X)$ and $\psi(M)$ contained in $*$-open set $\psi(M) \cup O$, then either $\psi(M) \cup O = X$ or $\psi(M) \cup O = \psi(M)$. This implies either $\psi(M) \cup O = X$ or $O \subseteq \psi(M)$. □

Definition 4.3 *Let XIT be a space and $x \in X$. A subset $S \subseteq X$ is called a $*$-open neighbourhood of x if there exists $A \in \tau^*(I)$ such that $x \in A \subseteq S$.*

Clearly, every $*$-open sets is an $*$-open neighbourhood of each of its points. The collection of all $*$-open neighbourhoods of a point $x \in X$ of a space XIT is simply denoted by $\mathcal{N}^*(x)$.

Proposition 4.1 *In a space XIT, every $*$-open neighbourhood is an $*$-open set.*

Proposition 4.2 *Let $M \in M_{max}I^*O(X)$ and $p \in M$.Then for any $W \in \mathcal{N}^*(x)$, either $\psi(M) \cup W = X$ or $W \subseteq \psi(M)$.*

Proof: Proof is obvious by Lemma 4.2 and hence omitted. □

Lemma 4.3 *Let XIT be a space. Then for any two $M, N \in M_{max}I^*O(X)$, either $\psi(M) \cup \psi(N) = X$ or $\psi(M) = \psi(N)$ when $\tau \sim I$.*

Proof: Since $M, N \in M_{max}I^*O(X)$, then by Lemma 4.2, either $\psi(M) \cup N = X$ or $N \subseteq \psi(M)$ and either $\psi(N) \cup M = X$ or $M \subseteq \psi(N)$. These implies $\psi(M) \cup \psi(N) = X$ or $N \subseteq \psi(M)$ and $M \subseteq \psi(N)$. Since $\tau \sim I$, then $N \subseteq \psi(M)$ and $M \subseteq \psi(N)$ implies $\psi(N) \subseteq \psi(\psi(M)) = \psi(M)$ and $\psi(M) \subseteq \psi(\psi(N)) = \psi(N)$. Combining $\psi(M) = \psi(N)$. Hence either $\psi(M) \cup \psi(N) = X$ or $\psi(M) = \psi(N)$. □

Theorem 4.1 *Let XIT be a space with $\tau \sim I$. Let $M_1, M_2, M_3 \in M_{max}I^*O(X)$ such that $\psi(M_1) \neq \psi(M_2)$. If $\psi(M_1) \cap \psi(M_2) \subseteq \psi(M_3)$, then either $\psi(M_1) = \psi(M_3)$ or $\psi(M_2) = \psi(M_3)$.*

Proof: We have,

$$\begin{aligned}
\psi(M_1) \cap \psi(M_3) &= \psi(M_1) \cap (\psi(M_3) \cap X) \\
&= \psi(M_1) \cap (\psi(M_3) \cap (\psi(M_1) \cup \psi(M_2)))(\textit{by Lemma } 4.3) \\
&= \psi(M_1) \cap [((\psi(M_3) \cap \psi(M_1)) \cup (\psi(M_3) \cap \psi(M_2)))] \\
&= (\psi(M_1) \cap \psi(M_3)) \cup (\psi(M_1) \cap \psi(M_2) \cap \psi(M_3)) \\
&= (\psi(M_1) \cap \psi(M_3)) \cup (\psi(M_1) \cap \psi(M_2))[\textit{since } \psi(M_1) \cap \psi(M_2) \subseteq \psi(M_3)] \\
&= \psi(M_1) \cap (\psi(M_2) \cup \psi(M_3)).
\end{aligned}$$

Thus if $\psi(M_2) \neq \psi(M_3)$, then $\psi(M_2) \cup \psi(M_3) = X$ and hence $\psi(M_1) \cap \psi(M_3) = \psi(M_1)$. This implies $\psi(M_1) \subseteq \psi(M_3)$. Since M_1 and M_3 are maximal I^*-open sets, then $\psi(M_1) = \psi(M_3)$. □

Theorem 4.2 *Let XIT be a space with $\tau \sim I$. Let $M_1, M_2, M_3 \in M_{max}I^*O(X)$ in which $\psi(M_1)$, $\psi(M_2)$ and $\psi(M_3)$ are different from each other, then $\psi(M_1) \cap \psi(M_2) \nsubseteq \psi(M_1) \cap \psi(M_3)$.*

Proof: If possible, let $\psi(M_1) \cap \psi(M_2) \subseteq \psi(M_1) \cap \psi(M_3)$, then

$$(\psi(M_1) \cap \psi(M_2)) \cup (\psi(M_2) \cap \psi(M_3)) \subseteq (\psi(M_1) \cap \psi(M_3)) \cup (\psi(M_2) \cap \psi(M_3)).$$

This implies $\psi(M_2) \cap (\psi(M_1) \cup \psi(M_3)) \subseteq (\psi(M_1) \cup \psi(M_2)) \cap \psi(M_3)$. Since $\psi(M_1)$, $\psi(M_2)$ and $\psi(M_3)$ are different from each other, then $\psi(M_1) \cup \psi(M_3) = X = \psi(M_1) \cup \psi(M_2)$ by Lemma 4.3. This implies $\psi(M_2) \subseteq \psi(M_3)$ and hence $\psi(M_2) = \psi(M_3)$ that contradicts our given condition. Thus $\psi(M_1) \cap \psi(M_2) \nsubseteq \psi(M_1) \cap \psi(M_3)$. □

We now check for the presence of maximal I^*-open sets in unusual instances. For this we consider cofinite I^*-open sets that means those I^*-open sets whose complement is a finite subset.

Theorem 4.3 *Let XIT be a space with $\tau \sim I$. If O is a proper nonempty cofinite $*$-open subset. Then, there exists atleast one (cofinite) proper maximal I^*-open set M such that $\psi(O) \subseteq \psi(M)$.*

Proof: If O is a maximal I^*-open set, we take $M = O$. If not, then there exists a $*$-open set M_1(cofinite) such that $\psi(O) \subsetneqq M_1 \neq X$. If M_1 is a maximal I^*-open set, we take $M = M_1$. Since $\tau \sim I$, then $\psi(O) \subsetneqq M$ implies $\psi(O) = \psi(\psi(O)) \subseteq \psi(M)$. If not , then there exists an $*$-open set M_2 (cofinite) such that $\psi(O) \subseteq \psi(M_1) \subsetneqq M_2 \neq X$. Since $\tau \sim I$, then $\psi(O) \subseteq \psi(M_1) \subseteq \psi(M_2)$. Continuing this process, we have a sequence of $*$-open sets satisfying, $\psi(O) \subseteq \psi(M_1) \subseteq \psi(M_2) \subseteq \psi(M_3) \subseteq \psi(M_4) ... \subseteq \psi(M_k)...$. Since O is a cofinite, then $\psi(O)$ is also cofinite. Thus the aforementioned process occurs only finitely and consequently we have a maximal I^*-open set $M = M_n$ for some $n \in \mathbb{N}$. Hence the result. □

Proposition 4.3 *Let XIT be a space. Let $M \in M_{max}I^*O(X)$ and $x \in \psi(M)$, then $\psi(M) = \bigcup\{W : W \in \mathcal{N}^*(x)$ and $W \cup \psi(M) \neq X\}$.*

Proof: If $W \in \mathcal{N}^*(x)$ and $W \cup \psi(M) \neq X$, then by Proposition 4.2, $W \subseteq \psi(M)$. Hence $\psi(M) \subseteq \bigcup\{W : W \in \mathcal{N}^*(x)$ and $W \cup \psi(M) \neq X\} \subseteq \psi(M)$. This implies $\psi(M) = \bigcup\{W : W \in \mathcal{N}^*(x)$ and $W \cup \psi(M) \neq X\}$. Hence the result. □

Theorem 4.4 *Let $M \in M_{max}I^*O(X)$ and $x \in X \setminus \psi(M)$. Then $X \setminus \psi(M) \subseteq W$ for any $W \in \mathcal{N}^*(x)$*

Proof: Since $x \in X \setminus \psi(M)$, then for any $W \in \mathcal{N}^*(x)$, we have $W \nsubseteq \psi(M)$. Thus $W \cup \psi(M) = X$ by Lemma 4.2 and hence $X \setminus \psi(M) \subseteq W$. □

Corollary 4.2 *Let $M \in M_{max}I^*O(X)$, then one of the following is true:*

1. *for any $x \in X \setminus \psi(M)$ and for each $W \in \mathcal{N}^*(x)$, $W = X$.*
2. *there exists a $*$-open set W such that $X \setminus \psi(M) \subseteq W$ and $W \subsetneq X$.*

Proof: If *(1)* is not true, then there exists an element x of $X \setminus \psi(M)$ and $W \in \mathcal{N}^*(x)$ such that $W \subsetneqq X$. Then by Theorem 4.4, we have $X \setminus \psi(M) \subseteq W$. □

Corollary 4.3 *If $M \in M_{max}I^*O(X)$, then one of the following is true:*

1. *for any $x \in X \setminus \psi(M)$ and for each $W \in \mathcal{N}^*(x)$, $X \setminus \psi(M) \subsetneqq W$.*
2. *there exists a $*$-open set W such that $X \setminus \psi(M) = W \neq X$.*

Proof: Assume that *(2)* does not hold. Then by Theorem 4.4, for each $x \in X \setminus \psi(M)$ and for any $W \in \mathcal{N}^*(x)$, $X \setminus \psi(M) \subseteq W$. Hence we have, $X \setminus \psi(M) \subsetneqq W$. □

Theorem 4.5 *Let $M \in M_{max}I^*O(X)$. Then either $Cl^*(\psi(M)) = X$ or $Cl^*(\psi(M)) = \psi(M)$.*

Proof: Since M is a proper maximal I^*-open set, only the following cases *(1)* and *(2)* occured by Corollary 4.3.
(1) For each $x \in X \setminus \psi(M)$ and each $W \in \mathcal{N}^*(x)$, we have $X \setminus \psi(M) \subsetneqq W$. Let x be any element of $X \setminus \psi(M)$ and W be any $*$-open neighbourhood of x. Since $X \setminus \psi(M) \subsetneqq W$, we have $W \cap \psi(M) \neq \emptyset$ for any $*$-open neighbourhood of x. Hence $x \in Cl^*(\psi(M))$. Thus $X \setminus \psi(M) \subseteq Cl^*(\psi(M))$. Since $X = \psi(M) \cup (X \setminus \psi(M))$, then $X \subseteq \psi(M) \cup Cl^*(\psi(M)) = Cl^*(\psi(M)) \subseteq X$. Hence $X = Cl^*(\psi(M))$.
(2) There exists a $*$-open set W such that $X \setminus \psi(M) = W \neq X$. Since $X \setminus \psi(M) = W$ is a $*$-open set, then $\psi(M)$ is a $*$-closed set. Therefore $Cl^*(\psi(M)) = \psi(M)$. □

Theorem 4.6 *Let $M \in M_{max}I^*O(X)$. Then either $Int^*(X \setminus \psi(M)) = X \setminus \psi(M)$ or $Int^*(X \setminus \psi(M)) = \emptyset$.*

Proof: By Corollary 4.3, we have either,
(1) $Int^*(X \setminus \psi(M)) = \emptyset$ or
(2) there exists a $*$-open set W such that $X \setminus \psi(M) = W \neq X$. This implies, $Int^*(X \setminus \psi(M)) = Int^*(W) = W = X \setminus \psi(M)$. □

Theorem 4.7 *Let $M \in M_{max}I^*O(X)$ and S be any nonempty subset of $X \setminus \psi(M)$. Then $Cl^*(S) = X \setminus \psi(M)$.*

Proof: Since $\emptyset \neq S \subseteq X \setminus \psi(M)$, we have $W \cap S \neq \emptyset$ for any element x of $X \setminus \psi(M)$ and any $*$-open neighbourhood W of x by Theorem 4.4. This implies $x \in Cl^*(S)$ and hence $X \setminus \psi(M) \subseteq Cl^*(S)$. Since $X \setminus \psi(M)$ is a $*$-closed set and $S \subseteq X \setminus \psi(M)$, then $Cl^*(S) \subseteq Cl^*(X \setminus \psi(M)) = X \setminus \psi(M)$. Hence $Cl^*(S) = X \setminus \psi(M)$. □

Corollary 4.4 *Let $M \in M_{max}I^*O(X)$ and U be a subset of X with $\psi(M) \subsetneqq U$. Then $Cl^*(U) = X$.*

Proof: Since $\psi(M) \subsetneqq U \subseteq X$, there exists a nonempty subset S of $X \setminus \psi(M)$ such that $U = S \cup \psi(M)$. Hence, we have $Cl^*(U) = Cl^*(S \cup \psi(M)) = Cl^*(S) \cup Cl^*(\psi(M)) \supseteq (X \setminus \psi(M)) \cup \psi(M) = X$ by Theorem 4.7. This implies $Cl^*(U) = X$. □

Theorem 4.8 *Let M be a proper maximal I^*-open set in a space XIT and assume that $|X \setminus \psi(M)| \geq 2$ where $|.|$ denotes the cardinality. Then $Cl^*(X \setminus \{a\}) = X$ for $a \in X \setminus \psi(M)$.*

Proof: Since $\psi(M) \subsetneqq X \setminus \{a\}$, then by our assumption, we get the required result by Corollary 4.4. □

Theorem 4.9 *Let M be a proper maximal I^*-open set in a space XIT and N be a proper subset of X with $\psi(M) \subseteq N$. Then $Int^*(N) = \psi(M)$.*

Proof: Since $\psi(M) \subseteq N$, then $Int^*(\psi(M)) \subseteq Int^*(N)$ and hence $\psi(M) \subseteq Int^*(N)$ as $\psi(M)$ is a $*$-open set. Also since $Int^*(N)$ is a $*$-open set and $\psi(M)$ is a proper maximal I^*-open set, then $\psi(M) = Int^*(N)$. □

Theorem 4.10 *Let M be a proper maximal I^*-open set in a space XIT and S be a nonempty subset of $X \setminus \psi(M)$. Then $X \setminus Cl^*(S) = Int^*(X \setminus S) = \psi(M)$.*

Proof: Since $\psi(M) \subseteq X \setminus S \subsetneqq X$ is our assumption, we get the required result by Theorems 4.7 and 4.9. □

A subset M of a topological space (X, τ) is called a pre-open set [13] if $M \subseteq Int(Cl(M))$. For a space XIT, the collection of all pre-open sets in $(X, \tau^*(I))$ is denoted as $PO^*(X, \tau^*(I))$.

Theorem 4.11 *Let M be a proper maximal I^*-open set in a space XIT and S be any subset of X with $\psi(M) \subseteq S$. Then $S \in PO^*(X, \tau^*(I))$.*

Proof: If $\psi(M) = S$, then S is a $*$-open set and hence $S = Int^*(S)$. Also $S \subseteq Cl^*(S)$. This implies $Int^*(S) \subseteq Int^*(Cl^*(S))$ and hence $S \subseteq Int^*(Cl^*(S))$. Thus $S \in PO^*(X, \tau^*(I))$.
Otherwise $\psi(M) \subsetneqq S$, then $Int^*(Cl^*(S)) = Int^*(X) \supseteq S$. Therefore $S \in PO^*(X, \tau^*(I))$. □

Corollary 4.5 *Let M be a proper maximal I^*-open set in a space XIT. Then $X \setminus \{a\} \in PO^*(X, \tau^*(I))$ for any element a of $X \setminus \psi(M)$.*

Proof: Since $\psi(M) \subseteq X \setminus \{a\}$, by our assumption, we get the required result by Theorem 4.11. □

5. Images of Maximal I-open sets

In this part, we have studied homeomorphic images of maximal I-open sets in the $Tspace\ XT$ with an ideal.

Lemma 5.1 ([9]) *Let $f : X \to Y$ be a function. If I is an ideal on X, then $f(I) = \{f(I_1) : I_1 \in I\}$ is also an ideal on Y.*

Lemma 5.2 ([9]) *Let $f : X \to Y$ be an injective function. If I is an ideal on Y, then $f^{-1}(I) = \{f^{-1}(I_1) : I_1 \in I\}$ is also an ideal on X.*

Proposition 5.1 ([10,17]) *Let (X, τ_1) and (Y, τ_2) be two T spaces and I be an ideal on X. If $f : X \to Y$ is a homeomorphism, then for any $M \in \wp(X)$, $(f(M))^{*f(I)} = f(M^{*I})$.*

Theorem 5.1 *Let (X, τ_1) and (Y, τ_2) be two T spaces and I be an ideal on X. If $f : X \to Y$ is a homeomorphism, then for any I-open set M in X, $f(M)$ is an $f(I)$-open set in Y.*

Proof: Since M is an I-open set in X, then $M \subseteq Int(M^{*I})$. This implies,

$$\begin{aligned} f(M) &\subseteq f(Int(M^{*I})) \\ &= Int(f((M^{*I}))) \ (since\ f\ is\ a\ homeomorphism) \\ &= Int(f(M)^{*f(I)}) \ (by\ Proposition\ 5.1) \end{aligned}$$

Hence $f(M)$ is an $f(I)$-open set in Y. □

Theorem 5.2 *Let (X, τ_1) and (Y, τ_2) be two T spaces and I be an ideal on Y. If $f : X \to Y$ is a homeomorphism, then for any I-open set M in Y, $f^{-1}(M)$ is an $f^{-1}(I)$-open set in X.*

Proof: Follows from Theorem 5.1. □

Theorem 5.3 *Let (X, τ_1) and (Y, τ_2) be two T spaces and I be an ideal on X. If $f : X \to Y$ is a homeomorphism, then for any maximal I-open set M in X, $f(M)$ is a maximal $f(I)$-open set in Y.*

Proof: If possible, suppose $f(M)$ is not a maximal $f(I)$-open set in Y, then there exists a nonempty proper $f(I)$-open set V in Y such that $f(M) \subsetneqq V$. Since f is a homeomorphism, then $M \subsetneqq f^{-1}(V) \subsetneqq X$. Also since f is a homeomorphism and V is a nonempty $f(I)$-open set in Y, then $f^{-1}(V)$ is a nonempty $f^{-1}(f(I))(= I)$-open set in X by Theorem 5.2, which contradicts that M is a maximal I-open set. Thus our assumption is wrong. Hence the result. □

Theorem 5.4 *Let (X, τ_1) and (Y, τ_2) be two T spaces and I be an ideal on Y. If $f : X \to Y$ is a homeomorphism, then for any maximal I-open set M in Y, $f^{-1}(M)$ is a maximal $f^{-1}(I)$-open set in X.*

Proof: Follows from Theorem 5.3. □

Note 5 *Let (X, τ_1) and (Y, τ_2) be two T spaces and I be an ideal on X. If $f : X \to Y$ is a function, then for any maximal I-open set M in X, $f(M)$ is not always a maximal $f(I)$-open set in Y.*

We are now giving an example in support of the Note 5:

Example 5.1 *Consider (X, τ_1, I) be a T spaces with an ideal where $X = \{e^1, e^2, e^3, e^4\}$, $\tau_1 = \{\emptyset, X, \{e^1, e^2\}, \{e^3\}, \{e^1, e^2, e^3\}\}$ and $I = \{\emptyset, \{e^2\}\}$. Then $M = \{e^1, e^2, e^3\}$ is a maximal I-open set in X. Again consider (Y, τ_2) be another T spaces where $Y = X = \{e^1, e^2, e^3, e^4\}$ and $\tau_2 = \tau_1$.*
Let us define a function $f : X \to Y$ by $f(e^1) = e^1$, $f(e^2) = e^2$, $f(e^3) = e^2$ and $f(e^4) = e^3$. Then $f(I) = \{f(I_1) : I_1 \in I\} = \{\emptyset, \{e^2\}\}$ is an ideal on Y. Now $f(M) = \{f(x) : x \in M\} = \{e^1, e^2\}$ is not a maximal $f(I)$-open set in Y.

Note 6 **([11,26])** *Let $f : X \to Y$ be a function. If I is an ideal on Y, then $f^{\leftarrow}(I) = \{A : A \subset f^{-1}(I_1) \in f^{-1}(I)\}$ is also an ideal on X.*

Proposition 5.2 *Let (X, τ_1) and (Y, τ_2) be two T spaces and I be an ideal on Y. If $f : X \to Y$ is an injective function, then $f^{-1}(I) = f^{\leftarrow}(I)$*

Proof: Proof is obvious and hence omitted. □

Note 7 *Let (X, τ_1) and (Y, τ_2) be two T spaces and I be an ideal on Y. If $f : X \to Y$ is a function, then for any maximal I-open set M in Y, $f^{-1}(M)$ is not always a maximal $f^{\leftarrow}(I)$-open set in X.*

We are now giving an example in support of the Note 7:

Example 5.2 *Consider (Y, τ_2, I) be a T spaces with an ideal where $Y = \{e^1, e^2, e^3, e^4\}$, $\tau_2 = \{\emptyset, X, \{e^1, e^2\}, \{e^3\}, \{e^1, e^2, e^3\}\}$ and $I = \{\emptyset, \{e^2\}\}$. Therefore $M = \{e^1, e^2, e^3\}$ is a maximal I-open set in Y. Again consider (X, τ_1) be another T space where $X = \{e^1, e^2, e^3, e^4\}$ and $\tau_1 = \tau_2 = \{\emptyset, X, \{e^1, e^2\}, \{e^3\}, \{e^1, e^2, e^3\}\}$.*
Let us define a function $f : X \to Y$ by $f(e^1) = e^1$, $f(e^2) = e^2$, $f(e^3) = e^4$ and $f(e^4) = e^3$. Then $f^{-1}(I) = \{f^{-1}(I_1) : I_1 \in I\} = \{\emptyset, \{e^2\}\}$ and hence $f^{\leftarrow}(I) = \{\emptyset, \{e^2\}\}$ is an ideal on X by Note 9. Now $f^{-1}(M) = \{e^1, e^2, e^4\}$. Thus $(f^{-1}(M))^{*f^{\leftarrow}(I)} = \{e^1, e^2, e^4\}$ and hence $Int((f^{-1}(M))^{*f^{\leftarrow}(I)}) = \{e^1, e^2\}$. This implies $f^{-1}(M) \subsetneqq Int((f^{-1}(M))^{*f^{\leftarrow}(I)})$ and hence $f^{-1}(M)$ is not a maximal $(f^{\leftarrow}(I))$-open set in X.

6. Images of Maximal I^*-open sets

In this part, we have studied homeomorphic images of proper maximal I^*-open sets in the $T space$ XT with an ideal.

Proposition 6.1 **([17])** *Let (X, τ_1) and (Y, τ_2) be two T spaces and I be an ideal on X. If $f : X \to Y$ is a homeomorphism, then for any $M \in \wp(X)$, $\psi^{f(I)}_{\tau_2}(f(M)) = f(\psi^I_{\tau_1}(M))$.*

Proposition 6.2 **([10,17])** *Let (X, τ_1) and (Y, τ_2) be two $T spaces$ and I be an ideal on Y. If $f : X \to Y$ is a homeomorphism, then for any $M \in \wp(Y)$, $f^{-1}(M^{*I}) = (f^{-1}(M))^{*f^{-1}(I)}$.*

Proposition 6.3 **([10,17])** *Let (X, τ_1) and (Y, τ_2) be two T spaces and I be an ideal on Y. If $f : X \to Y$ is a homeomorphism, then for any $M \in \wp(Y)$, $\psi^{f^{-1}(I)}_{\tau_1}(f^{-1}(M)) = f^{-1}(\psi^I_{\tau_2}(M))$.*

Theorem 6.1 *Let (X, τ_1) and (Y, τ_2) be two T spaces and I be an ideal on X. If $f : X \to Y$ is a homeomorphism, then for any proper maximal I^*-open set M in X, $f(M)$ is a proper maximal $(f(I))^*$-open set in Y.*

Proof: If possible, suppose $f(M)$ is not a maximal $(f(I))^*$-open set in Y, then there exists $\emptyset \neq V \in \tau_2^*(f(I))$ and $V \neq Y$ such that $\psi^{f(I)}_{\tau_2}(f(M)) \subsetneqq V$. This implies $f(\psi^I_{\tau_1}(M)) \subsetneqq V$. Since f is a homeomorphism, $\psi^I_{\tau_1}(M) \subsetneqq f^{-1}(V) \subseteq X$. Also since f is a homeomorphism, $V \in \tau_2^*(f(I))$ and hence $\psi^I_{\tau_1}(M) \subsetneqq f^{-1}(V) \in \tau_1^*(I)$ leads a contradiction as M is a proper maximal I^*-open set. Hence the result. □

Theorem 6.2 *Let (X, τ_1) and (Y, τ_2) be two T spaces and I be an ideal on Y. If $f : X \to Y$ is a homeomorphism, then for any proper maximal I^*-open set M in Y, $f^{-1}(M)$ is a proper maximal $(f^{-1}(I))^*$-open set in X.*

Proof: Follows from Theorem 6.1. □

Note 8 *Let (X, τ_1) and (Y, τ_2) be two T spaces and I be an ideal on X. If $f : X \to Y$ is a function, then for any proper maximal I^*-open set M in X, $f(M)$ is not always a maximal $f(I)^*$-open set in Y.*

We are now giving an example in support of the Note 8:

Example 6.1 *Consider* (X, τ_1, I) *be a* T *spaces with an ideal where* $X = \{e^1, e^2, e^3\}$*,* $\tau_1 = \{\emptyset, X, \{e^1\}, \{e^2\}, \{e^1, e^2\}\}$ *and* $I = \{\emptyset, \{e^1\}\}$*. Then* $\tau^*(I) = \{\emptyset, X, \{e^1\}, \{e^2\}, \{e^1, e^2\}, \{e^2, e^3\}\}$*. Clearly* $M = \{e^1, e^2\}$ *is a proper maximal* I^**-open set in* X*. Again consider* (Y, τ_2) *be another* T *spaces where* $Y = \{a, b\}$ *and* $\tau_2 = \{\emptyset, \{a\}, Y\}$*.*
Let us define a function $f : X \to Y$ *by* $f(e^1) = b$*,* $f(e^2) = b$ *and* $f(e^3) = a$*. Then* $f(I) = \{f(I_1) : I_1 \in I\} = \{\emptyset, \{b\}\}$ *is an ideal on* Y*. Thus* $\tau_2^*(f(I)) = \{\emptyset, \{a\}, Y\}$*. Now* $f(M) = \{f(x) : x \in M\} = \{b\}$*. Therefore* $\psi_{\tau_2}^{f(I)}(f(M)) = Y \setminus (Y \setminus \{b\})^{*f(I)} = Y \setminus \{a\}^{*f(I)} = Y \setminus \{a, b\} = \emptyset$*. This implies* $f(M)$ *is not a maximal* $f(I)^*$*-open set in* Y*.*

Note 9 *Let* (X, τ_1) *and* (Y, τ_2) *be two* T *spaces and* I *be an ideal on* Y*. If* $f : X \to Y$ *is a function, then for any proper maximal* I^**-open set* M *in* Y*,* $f^{-1}(M)$ *is not always a maximal* $(f^{\leftarrow}(I))^*$*-open set in* X*.*

We are now giving an example in support of the Note 9:

Example 6.2 *Consider* (Y, τ_2, I) *be a* T *spaces with an ideal where* $Y = \{e^1, e^2, e^3\}$*,* $\tau_2 = \{\emptyset, Y, \{e^1\}, \{e^2\}, \{e^1, e^2\}\}$ *and* $I = \{\emptyset, \{e^2\}\}$*. Therefore* $\tau_2^*(I) = \{\emptyset, Y, \{e^1\}, \{e^2\}, \{e^1, e^2\}, \{e^1, e^3\}\}$*. Clearly* $M = \{e^1, e^2\}$ *is a proper maximal* I^**-open set in* Y*. Again consider* (X, τ_1) *be another* T *space where* $X = \{a, b\}$ *and* $\tau_1 = \{\emptyset, \{b\}, X\}$*.*
Let us define a function $f : X \to Y$ *by* $f(a) = e^2$ *and* $f(b) = e^3$ *. Then* $f^{-1}(I) = \{f(I_1) : I_1 \in I\} = \{\emptyset, \{a\}\}$ *and hence* $f^{\leftarrow}(I) = \{\emptyset, \{a\}\}$ *is an ideal on* X *by Note 9. Thus* $\tau_1^*(f^{\leftarrow}(I)) = \{\emptyset, \{b\}, X\}$*. Now* $f^{-1}(M) = \{a\}$*. Therefore* $\psi_{\tau_1}^{f^{\leftarrow}(I)}(f^{-1}(M)) = X \setminus (X \setminus \{a\})^{*f^{\leftarrow}(I)} = X \setminus \{b\}^{*f^{\leftarrow}(I)} = X \setminus \{a, b\} = \emptyset$*. This implies* $f^{-1}(M)$ *is not a maximal* $(f^{\leftarrow}(I))^*$*-open set in* X*.*

7. Conclusion

In this write up, we have added some new kinds of open sets called maximal I-open sets and maximal I^*-open sets in ideal topological spaces and discussed their various properties. Using this idea, we have discussed the relationships between pre-open sets, maximal open sets, maximal I-open sets and maximal I^*-open sets. Furthermore, images of maximal I-open sets and maximal I^*-open sets under homeomorphisms have been discussed here. The other properties of these types of sets can be found and one can introduce some other relations to these types of sets to develop the skills of learning mathematics. These ideas may be defined with help of grill [6,14,15], filter [28,14,15] and other related mathematical structures. Furthermore it may be considered for the nature of $M_{max}IO(X)$ and $M_{max}I^*O(X)$ to constitute a filter, ultrafilter, ideal, universal ideal etc. These may be the way of the future research work.

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9. Conflicts of Interest

The authors declare that there is no conflict of interest.

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