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Genus of commuting graphs of some classes of finite rings

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ABSTRACT: In this paper, we compute the genus of commuting graphs of non-commutative rings of order p^4, p^5, p^2q and p^3q, where p and q are prime integers. We also characterize those finite rings such that their commuting graphs are planar or toroidal.

Key Words: Commuting graph, genus, planar graph, toroidal graph.

Contents

1. Introduction

The smallest non-negative integer n such that a graph $\mathcal{G}$ can be embedded on the surface obtained by attaching n handles to a sphere is called the genus of $\mathcal{G}$. We write $\gamma(\mathcal{G})$ to denote the genus of a graph $\mathcal{G}$. It is worth mentioning that

$$\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil \text{ if } n \geq 3 \text{ (see [13, Theorem 6-38])}, \tag{1.1}$$

where K_n is the complete graph on n vertices. Also, if $\mathcal{G} = \overset{m}{\underset{i=1}{\sqcup}} K_{n_i}$ then by [3, Corollary 2] we have

$$\gamma(\mathcal{G}) = \sum_{i=1}^{m} \gamma(K_{n_i}). \tag{1.2}$$

A graph $\mathcal{G}$ is called planar or toroidal if $\gamma(\mathcal{G}) = 0$ or 1, respectively. The following figures (Figures 1–2) show that K_4 is planar while K_5 is toroidal.

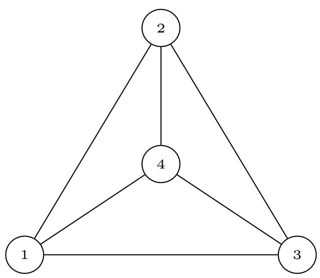

Figure 1: K_4

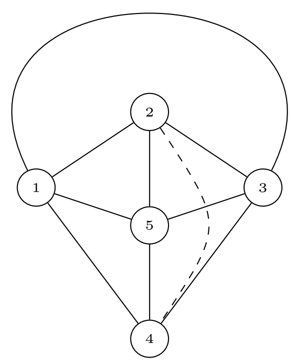

Figure 2: K_5

The commuting graph $\Gamma_c(R)$, of a finite non-commutative ring R with center $Z(R)$, is an undirected graph whose vertex set is $R \setminus Z(R)$ and two distinct vertices are adjacent if they commute. The concept of $\Gamma_c(R)$ was introduced by Akbari, Ghandehari, Hadian and Mohammadian [2] in 2004. However,

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they studied $\Gamma_c(R)$ for semisimple rings. In [1,8] Abdollahi and Mohammadian respectively, considered commuting graphs of some matrix rings and Omidi and Vatandoost, in [10], initiated the study of commuting graphs of any finite non-commutative rings. It is noteworthy that $\Gamma_c(R)$ is less explored compared to the study of commuting graphs of finite groups which was introduced by Brauer and Fowler [4] in 1955. In [5], Dutta, Fasfous and Nath have computed genus of commuting graphs of some classes of finite rings and characterized non-commutative rings of order p^2 and p^3 (for any prime) with unity such that their commuting graphs are planar or toroidal. In this paper, we consider non-commutative rings of order p^4, p^5, p^2q and p^3q, where p and q are two primes, and compute the genera of their commuting graphs. We also characterize those rings such that their commuting graphs are planar or toroidal. It is worth mentioning that the structures of $\Gamma_c(R)$ for the above mentioned classes of rings have been described in [11,12] and their spectral aspects have been explored in [6,7,9]. The following results are useful in our computations.

Theorem 1.1 [12, Theorem 2.5] *Let R have unity and $|R| = p^4$. Then $\Gamma_c(R) = (p^2+p+1)K_{(p^2-p)}$ or $l_1K_{(p^2-p)} \sqcup l_2K_{(p^3-p)}$ (where $l_1 + l_2(p+1) = p^2+p+1$) if $|Z(R)| = p$; and $\Gamma_c(R) = (p+1)K_{(p^3-p^2)}$ if $|Z(R)| = p^2$.*

We would like to remark that [12, Theorem 2.7(ii)] and [11, Theorem 2.9 (ii)] were printed incorrectly. The correct versions of [12, Theorem 2.7] and [11, Theorem 2.9] are given below.

Theorem 1.2 [12, Theorem 2.7] *Let R have unity, $|R| = p^5$ and $Z(R)$ is not a field. Then*

$$\Gamma_c(R) = (p^2+p+1)K_{(p^3-p^2)} \text{ or } l_1K_{(p^3-p^2)} \sqcup l_2K_{(p^4-p^2)}$$

(where $l_1 + l_2(p+1) = p^2+p+1$) if $|Z(R)| = p^2$; and $\Gamma_c(R) = (p+1)K_{(p^4-p^3)}$ if $|Z(R)| = p^3$.

Theorem 1.3 [11, Theorem 2.9] *Let $|R| = p^2q$ and $Z(R) = \{0\}$. Then $\Gamma_c(R) = \frac{p^2q-1}{t-1}K_{t-1}$ if $t \in \{p, q, p^2, pq\}$ and $(t-1) \mid (p^2q-1)$. Also,*

$$\Gamma_c(R) = l_1K_{p-1} \sqcup l_2K_{q-1} \sqcup l_3K_{p^2-1} \sqcup l_4K_{pq-1},$$

if $l_1(p-1) + l_2(q-1) + l_3(p^2-1) + l_4(pq-1) = p^2q-1$.

Theorem 1.4 [11, Theorem 2.12] *Let $|R| = p^3q$ and R has unity. If $|Z(R)| = pq$ then $\Gamma_c(R) = (p+1)K_{p^2q-pq}$. If $|Z(R)| = p^2$ then*

$$\Gamma_c(R) = \begin{cases} \frac{pq-1}{p-1}K_{p^3-p^2}, & \text{when } (p-1) \mid (pq-1) \\ \frac{pq-1}{q-1}K_{p^2q-p^2}, & \text{when } (q-1) \mid (pq-1) \\ l_1K_{p^3-p^2} \sqcup l_2K_{p^2q-p^2}, & \text{when } l_1(p-1) + l_2(q-1) = pq-1. \end{cases}$$

2. Main results

We begin with the following result.

Theorem 2.1 *Let R be a non-commutative ring with unity and $|R| = p^4$.*

(a) *Let $|Z(R)| = p$.*

(i) If $p = 2$ then $\Gamma_c(R)$ is planar, toroidal or $\gamma(\Gamma_c(R)) = 2$.

(ii) If $p \geq 3$ then $\gamma(\Gamma_c(R)) = (p^2+p+1)\left\lceil \frac{(p^2-p-3)(p^2-p-4)}{12} \right\rceil$ or $l_1\left\lceil \frac{(p^2-p-3)(p^2-p-4)}{12} \right\rceil + l_2\left\lceil \frac{(p^3-p-3)(p^3-p-4)}{12} \right\rceil$ for some positive integers l_1, l_2 such that $l_1 + l_2(p+1) = p^2+p+1$; and hence $\Gamma_c(R)$ is neither planar nor toroidal.

(b) *Let $|Z(R)| = p^2$.*

(i) $\Gamma_c(R)$ *is planar if and only if* $p = 2$.

(ii) *If* $p \geq 3$ *then* $\gamma(\Gamma_c(R)) = (p+1)\left\lceil \frac{(p^3-p^2-3)(p^3-p^2-4)}{12} \right\rceil$*; and hence* $\Gamma_c(R)$ *is neither planar nor toroidal.*

Proof: (a) By Theorem 1.1, we have $\Gamma_c(R) = (p^2+p+1)K_{p^2-p}$ or $l_1K_{p^2-p} \sqcup l_2K_{p^3-p}$, where $l_1+l_2(p+1) = p^2+p+1$.

Case 1: $\Gamma_c(R) = (p^2+p+1)K_{p^2-p}$

By (1.2) we have $\gamma(\Gamma_c(R)) = (p^2+p+1)\gamma(K_{p^2-p})$. If $p = 2$ then $p^2 - p = 2$ and so $\gamma(\Gamma_c(R)) = 7\gamma(K_2) = 0$. Therefore, $\Gamma_c(R)$ is planar. If $p \geq 3$ then $p^2 - p \geq 6$. By (1.1) we have

$$\gamma(\Gamma_c(R)) = (p^2+p+1)\left\lceil \frac{(p^2-p-3)(p^2-p-4)}{12} \right\rceil.$$

Since $p \geq 3$ we have $\frac{(p^2-p-3)(p^2-p-4)}{12} \geq \frac{1}{2}$ and so $\gamma(\Gamma_c(R)) \geq 13$. Thus $\Gamma_c(R)$ is neither planar nor toroidal.

Case 2: $\Gamma_c(R) = l_1K_{p^2-p} \sqcup l_2K_{p^3-p}$

By (1.2) we have $\gamma(\Gamma_c(R)) = l_1\gamma(K_{p^2-p}) \sqcup l_2\gamma(K_{p^3-p})$. If $p = 2$ then $p^2 - p = 2$ and $p^3 - p = 6$. Also, $l_1+3l_2 = 7$ which gives $l_1 = 4$ and $l_2 = 1$ or $l_1 = 1$ and $l_2 = 2$. Therefore, $\gamma(\Gamma_c(R)) = 4\gamma(K_2)+\gamma(K_6) = 1$ or $\gamma(\Gamma_c(R)) = \gamma(K_2) + 2\gamma(K_6) = 2$. That is, $\Gamma_c(R)$ is toroidal or $\gamma(\Gamma_c(R)) = 2$. If $p \geq 3$ then $p^2 - p \geq 6$ and $p^3 - p \geq 24$. By (1.1) we have

$$\gamma(\Gamma_c(R)) = l_1\left\lceil \frac{(p^2-p-3)(p^2-p-4)}{12} \right\rceil + l_2\left\lceil \frac{(p^3-p-3)(p^3-p-4)}{12} \right\rceil.$$

Note that $\frac{(p^2-p-3)(p^2-p-4)}{12} \geq \frac{1}{2}$ and $\frac{(p^3-p-3)(p^3-p-4)}{12} \geq 35$. Therefore, $\gamma(\Gamma_c(R)) \geq l_1 + 35l_2 > 36$. That is, $\Gamma_c(R)$ is neither planar nor toroidal. This completes the proof of part (a).

(b) By Theorem 1.1, we have $\Gamma_c(R) = (p+1)K_{p^3-p^2}$. Therefore, using (1.2) we get $\gamma(\Gamma_c(R)) = (p+1)\gamma(K_{p^3-p^2})$. If $p = 2$ then $p^3 - p^2 = 4$. Therefore, $\gamma(\Gamma_c(R)) = 3\gamma(K_4) = 0$. That is, $\Gamma_c(R)$ is planar. If $p \geq 3$ then $p^3 - p^2 \geq 18$ and so by (1.1) we have

$$\gamma(\Gamma_c(R)) = (p+1)\left\lceil \frac{(p^3-p^2-3)(p^3-p^2-4)}{12} \right\rceil.$$

Note that $\frac{(p^3-p^2-3)(p^3-p^2-4)}{12} \geq \frac{35}{2}$ and so $\gamma(\Gamma_c(R)) \geq 72$. Thus $\Gamma_c(R)$ is neither planar nor toroidal. This completes the proof of part (b). $\square$

Theorem 2.2 *Let* R *be a non-commutative ring with unity where* $|R| = p^5$ *and* $Z(R)$ *not a field.*

(a) *Let* $|Z(R)| = p^2$.

(i) *If* $p = 2$ *then* $\Gamma_c(R)$ *is planar,* $\gamma(\Gamma_c(R)) = 6$ *or* $\gamma(\Gamma_c(R)) = 12$.

(ii) *If* $p \geq 3$ *then* $\gamma(\Gamma_c(R)) = (p^2+p+1)\left\lceil \frac{(p^3-p^2-3)(p^3-p^2-4)}{12} \right\rceil$ *or* $l_1\left\lceil \frac{(p^3-p^2-3)(p^3-p^2-4)}{12} \right\rceil + l_2\left\lceil \frac{(p^4-p^2-3)(p^4-p^2-4)}{12} \right\rceil$ *for some positive integers* l_1, l_2 *such that* $l_1 + l_2(p+1) = p^2+p+1$*; and hence* $\Gamma_c(R)$ *is neither planar nor toroidal.*

(b) *Let* $|Z(R)| = p^3$. *Then* $\gamma(\Gamma_c(R)) = (p+1)\left\lceil \frac{(p^4-p^3-3)(p^4-p^3-4)}{12} \right\rceil$*; and hence* $\Gamma_c(R)$ *is neither planar nor toroidal.*

Proof: (a) By Theorem 1.2, we have $\Gamma_c(R) = (p^2+p+1)K_{p^3-p^2}$ or $l_1K_{p^3-p^2} \sqcup l_2K_{p^4-p^2}$, where $l_1 + l_2(p+1) = p^2+p+1$.

Case 1: $\Gamma_c(R) = (p^2 + p + 1)K_{p^3-p^2}$

By (1.2) we have $\gamma(\Gamma_c(R)) = (p^2 + p + 1)\gamma(K_{p^3-p^2})$. If $p = 2$ then $p^3 - p^2 = 4$ and so $\gamma(\Gamma_c(R)) = 7\gamma(K_4) = 0$. Therefore, $\Gamma_c(R)$ is planar. If $p \geq 3$ then $p^3 - p^2 \geq 18$. By (1.1) we have

$$\gamma(\Gamma_c(R)) = (p^2 + p + 1)\left\lceil \frac{(p^3 - p^2 - 3)(p^3 - p^2 - 4)}{12} \right\rceil.$$

Since $p \geq 3$ we have $\frac{(p^3-p^2-3)(p^3-p^2-4)}{12} \geq \frac{35}{2}$ and so $\gamma(\Gamma_c(R)) \geq 234$. Thus $\Gamma_c(R)$ is neither planar nor toroidal.

Case 2: $\Gamma_c(R) = l_1K_{p^3-p^2} \sqcup l_2K_{p^4-p^2}$

By (1.2) we have $\gamma(\Gamma_c(R)) = l_1\gamma(K_{p^3-p^2}) \sqcup l_2\gamma(K_{p^4-p^2})$. If $p = 2$ then $p^3 - p^2 = 4$ and $p^4 - p^2 = 12$. Also, $l_1 + 3l_2 = 7$ which gives $l_1 = 4$ and $l_2 = 1$ or $l_1 = 1$ and $l_2 = 2$. Therefore, $\gamma(\Gamma_c(R)) = 4\gamma(K_4) + \gamma(K_{12}) = 6$ or $\gamma(\Gamma_c(R)) = \gamma(K_4) + 2\gamma(K_{12}) = 12$. If $p \geq 3$ then $p^3 - p^2 \geq 18$ and $p^4 - p^2 \geq 72$. By (1.1) we have

$$\gamma(\Gamma_c(R)) = l_1\left\lceil \frac{(p^3 - p^2 - 3)(p^3 - p^2 - 4)}{12} \right\rceil + l_2\left\lceil \frac{(p^4 - p^2 - 3)(p^4 - p^2 - 4)}{12} \right\rceil.$$

Note that $\frac{(p^3-p^2-3)(p^3-p^2-4)}{12} \geq \frac{35}{2}$ and $\frac{(p^4-p^2-3)(p^4-p^2-4)}{12} \geq 391$. Therefore, $\gamma(\Gamma_c(R)) \geq 18l_1 + 391l_2 > 409$. That is, $\Gamma_c(R)$ is neither planar nor toroidal. This completes the proof of part (a).

(b) By Theorem 1.2, we have $\Gamma_c(R) = (p+1)K_{p^4-p^3}$. By (1.2) we have $\gamma(\Gamma_c(R)) = (p+1)\gamma(K_{p^4-p^3})$. If $p \geq 2$ then $p^4 - p^3 \geq 8$ and so by (1.1) we have

$$\gamma(\Gamma_c(R)) = (p + 1)\left\lceil \frac{(p^4 - p^3 - 3)(p^4 - p^3 - 4)}{12} \right\rceil.$$

Note that $\frac{(p^4-p^3-3)(p^4-p^3-4)}{12} \geq \frac{5}{3}$ and so $\gamma(\Gamma_c(R)) \geq 6$. Thus $\Gamma_c(R)$ is neither planar nor toroidal. This completes the proof of part (b). $\square$

Theorem 2.3 *Let R be a non-commutative ring where $|R| = p^2q$ and $Z(R) = \{0\}$.*

(a) *Let $t \in \{p, q, p^2, pq\}$ and $(t - 1) \mid (p^2q - 1)$.*

- *(i) $\Gamma_c(R)$ is planar if $t = p = q = 2$; or $t = p^2 = 4$ and $q \geq 3$; or $t = p = 2, 3, 5$ and $q \geq 3$; or $t = q = 2, 3, 5$ and $p \geq 3$.*
- *(ii) If $t = p \geq 7$ and $q \geq 3$; or $t = q \geq 7$ and $p \geq 3$; or $p \geq 3$, $q \geq 3$ and $t = p^2$ or pq then $\gamma(\Gamma_c(R)) = \frac{p^2q-1}{t-1}\left\lceil \frac{(t-4)(t-5)}{12} \right\rceil$; and hence $\Gamma_c(R)$ is neither planar nor toroidal.*

(b) *Let $l_1(p-1) + l_2(q-1) + l_3(p^2-1) + l_4(pq-1) = p^2q - 1$ for some positive integers l_1, l_2, l_3 and l_4.*

- *(i) If $p = 2 = q$ then $\Gamma_c(R)$ is planar.*
- *(ii) If $p = 2$ and $q = 3$ then $\gamma(\Gamma_c(R)) = l_4$; and hence $\Gamma_c(R)$ is not planar but toroidal if and only if $l_4 = 1$.*
- *(iii) If $p = 2$ and $q \geq 5$ then $\gamma(\Gamma_c(R)) = l_2\left\lceil \frac{(q-4)(q-5)}{12} \right\rceil + l_4\left\lceil \frac{(2q-4)(2q-5)}{12} \right\rceil$.*
- *(iv) If $q = 2$ and $p = 3$ then $\gamma(\Gamma_c(R)) = 2l_3 + l_4$.*
- *(v) If $q = 2$ and $p \geq 5$ then $\gamma(\Gamma_c(R)) = l_1\left\lceil \frac{(p-4)(p-5)}{12} \right\rceil + l_3\left\lceil \frac{(p^2-4)(p^2-5)}{12} \right\rceil + l_4\left\lceil \frac{(2p-4)(2p-5)}{12} \right\rceil$.*
- *(vi) If $p = 3 = q$ then $\gamma(\Gamma_c(R)) = 2(l_3 + l_4)$.*
- *(vii) If $p = 3$ and $q \geq 5$ then $\gamma(\Gamma_c(R)) = l_2\left\lceil \frac{(q-4)(q-5)}{12} \right\rceil + 2l_3 + l_4\left\lceil \frac{(3q-4)(3q-5)}{12} \right\rceil$.*
- *(viii) If $p \geq 5$ and $q = 3$ then $\gamma(\Gamma_c(R)) = l_1\left\lceil \frac{(p-4)(p-5)}{12} \right\rceil + l_3\left\lceil \frac{(p^2-4)(p^2-5)}{12} \right\rceil + l_4\left\lceil \frac{(3p-4)(3p-5)}{12} \right\rceil$.*

(ix) If $p \geq 5$ and $q \geq 5$ then $\gamma(\Gamma_c(R)) = l_1 \left\lceil \frac{(p-4)(p-5)}{12} \right\rceil + l_2 \left\lceil \frac{(q-4)(q-5)}{12} \right\rceil + l_3 \left\lceil \frac{(p^2-4)(p^2-5)}{12} \right\rceil + l_4 \left\lceil \frac{(pq-4)(pq-5)}{12} \right\rceil$.

It follows that $\Gamma_c(R)$ is neither planar nor toroidal in all the cases $(iii)-(ix)$.

Proof: (a) By Theorem 1.3, we have $\Gamma_c(R) = \frac{p^2q-1}{t-1}K_{t-1}$. By (1.2) we have $\gamma(\Gamma_c(R)) = \frac{p^2q-1}{t-1}\gamma(K_{t-1})$.

Case 1: $p = 2 = q$

In this case $t = 2$, since $(t-1)|(p^2q-1) = 7$. Therefore, $\gamma(\Gamma_c(R)) = 7\gamma(K_1) = 0$. That is, $\Gamma_c(R)$ is planar.

Case 2: $p = 2$ and $q \geq 3$

We have $(q-1) \nmid (4q-1)$ and $(2q-1) \nmid (4q-1)$. Therefore, $t \neq q$ and $t \neq pq = 2q$. If $t = p = 2$ or $t = p^2 = 4$ and $q \geq 3$ then $t-1 = 1$ or 3 and so $\gamma(\Gamma_c(R)) = 0$. That is, $\Gamma_c(R)$ is planar.

Case 3: $p \geq 3$ and $q = 2$

We have $(p-1) \nmid (2p^2-1)$, $(p^2-1) \nmid (2p^2-1)$ and $(2p-1) \nmid (2p^2-1)$. Therefore, $t \neq p$, p^2 and $2p = pq$. If $t = q = 2$ then $t-1 = 1$ and so $\gamma(\Gamma_c(R)) = 0$. That is, $\Gamma_c(R)$ is planar.

Case 4: $p \geq 3$ and $q \geq 3$

If $t = p = 3$ or 5 then $t-1 = 2$ or 4 and so $\gamma(\Gamma_c(R)) = 0$. That is, $\Gamma_c(R)$ is planar. If $t = p \geq 7$ then $t-1 \geq 6$. Therefore, by (1.1) we have

$$\gamma(\Gamma_c(R)) = \frac{p^2q-1}{t-1}\left\lceil \frac{(t-4)(t-5)}{12} \right\rceil.$$

Since $\frac{(t-4)(t-5)}{12} \geq \frac{1}{2}$ and $\frac{p^2q-1}{t-1} > 2$ we have $\gamma(\Gamma_c(R)) > 2$. That is, $\Gamma_c(R)$ is neither planar nor toroidal. If $t = p^2$ then $t-1 \geq 8$. Therefore, by (1.1) we have

$$\gamma(\Gamma_c(R)) = \frac{p^2q-1}{t-1}\left\lceil \frac{(t-4)(t-5)}{12} \right\rceil.$$

Since $\frac{(t-4)(t-5)}{12} \geq \frac{5}{3}$ and $\frac{p^2q-1}{t-1} > 2$ we have $\gamma(\Gamma_c(R)) > 4$. That is, $\Gamma_c(R)$ is neither planar nor toroidal. If $t = q = 3$ or 5 then $t-1 = 2$ or 4 and so $\gamma(\Gamma_c(R)) = 0$. That is, $\Gamma_c(R)$ is planar. If $t = q \geq 7$ then $t-1 \geq 6$. Therefore, by (1.1) we have

$$\gamma(\Gamma_c(R)) = \frac{p^2q-1}{t-1}\left\lceil \frac{(t-4)(t-5)}{12} \right\rceil.$$

Since $\frac{(t-4)(t-5)}{12} \geq \frac{1}{6}$ and $\frac{p^2q-1}{t-1} > 2$ we have $\gamma(\Gamma_c(R)) > 2$. That is, $\Gamma_c(R)$ is neither planar nor toroidal. If $t = pq$ then $t-1 \geq 8$. Therefore, by (1.1) we have

$$\gamma(\Gamma_c(R)) = \frac{p^2q-1}{t-1}\left\lceil \frac{(t-4)(t-5)}{12} \right\rceil.$$

Since $\frac{(t-4)(t-5)}{12} \geq \frac{5}{3}$ and $\frac{p^2q-1}{t-1} > 2$ we have $\gamma(\Gamma_c(R)) > 4$. That is, $\Gamma_c(R)$ is neither planar nor toroidal.

(b) By Theorem 1.3, we have $\Gamma_c(R) = l_1K_{p-1} \sqcup l_2K_{q-1} \sqcup l_3K_{p^2-1} \sqcup l_4K_{pq-1}$. By (1.2) we have

$$\gamma(\Gamma_c(R)) = l_1\gamma(K_{p-1}) + l_2\gamma(K_{q-1}) + l_3\gamma(K_{p^2-1}) + l_4\gamma(K_{pq-1}).$$

Case 1: $p = 2 = q$

In this case $\gamma(\Gamma_c(R)) = l_1\gamma(K_1) + l_2\gamma(K_1) + l_3\gamma(K_3) + l_4\gamma(K_3) = 0$. Therefore, $\Gamma_c(R)$ is planar.

Case 2: $p = 2$ and $q \geq 3$

In this case $\gamma(\Gamma_c(R)) = l_2\gamma(K_{q-1}) + l_4\gamma(K_{2q-1})$. If $q = 3$ then $\gamma(\Gamma_c(R)) = l_2\gamma(K_2) + l_4\gamma(K_5) = l_4$. Therefore, $\Gamma_c(R)$ is not planar since $l_4 \neq 0$ and $\Gamma_c(R)$ is toroidal if $l_4 = 1$. If $q \geq 5$ then $q-1 \geq 4$ and $2q-1 \geq 9$. Therefore, by (1.1) we have

$$\gamma(\Gamma_c(R)) = l_2\left\lceil\frac{(q-4)(q-5)}{12}\right\rceil + l_4\left\lceil\frac{(2q-4)(2q-5)}{12}\right\rceil.$$

Since $\frac{(q-4)(q-5)}{12} \geq 0$ and $\frac{(2q-4)(2q-5)}{12} \geq \frac{5}{2}$ we have $\gamma(\Gamma_c(R)) \geq 3l_4 \geq 3$. That is, $\Gamma_c(R)$ is neither planar nor toroidal.

Case 3: $q = 2$ and $p \geq 3$

In this case $\gamma(\Gamma_c(R)) = l_1\gamma(K_{p-1}) + l_3\gamma(K_{p^2-1}) + l_4\gamma(K_{2p-1})$. If $p = 3$ then $\gamma(\Gamma_c(R)) = l_1\gamma(K_2) + l_3\gamma(K_8) + l_4\gamma(K_5) = 2l_3 + l_4 \geq 3$. Therefore, $\Gamma_c(R)$ is neither planar nor toroidal. If $p \geq 5$ then $p-1 \geq 4$, $p^2-1 \geq 24$ and $2p-1 \geq 9$. Therefore, by (1.1) we have

$$\gamma(\Gamma_c(R)) = l_1\left\lceil\frac{(p-4)(p-5)}{12}\right\rceil + l_3\left\lceil\frac{(p^2-4)(p^2-5)}{12}\right\rceil + l_4\left\lceil\frac{(2p-4)(2p-5)}{12}\right\rceil.$$

Since $\frac{(p-4)(p-5)}{12} \geq 0$, $\frac{(p^2-4)(p^2-5)}{12} \geq 35$ and $\frac{(2p-4)(2p-5)}{12} \geq \frac{5}{2}$ we have $\gamma(\Gamma_c(R)) \geq 35l_3 + 3l_4 \geq 38$. That is, $\Gamma_c(R)$ is neither planar nor toroidal.

Case 4: $p \geq 3$ and $q \geq 3$

If $p = 3 = q$ then $\gamma(\Gamma_c(R)) = l_3\gamma(K_8) + l_4\gamma(K_8) = 2(l_3 + l_4) \geq 4$. Therefore, $\Gamma_c(R)$ is neither planar nor toroidal. If $p = 3$ and $q \geq 5$ then

$$\begin{aligned}\gamma(\Gamma_c(R)) &= l_1\gamma(K_2) + l_2\gamma(K_{q-1}) + l_3\gamma(K_8) + l_4\gamma(K_{3q-1})\\ &= l_2\left\lceil\frac{(q-4)(q-5)}{12}\right\rceil + 2l_3 + l_4\left\lceil\frac{(3q-4)(3q-5)}{12}\right\rceil.\end{aligned}$$

Since $\frac{(q-4)(q-5)}{12} \geq 0$ and $\frac{(3q-4)(3q-5)}{12} \geq \frac{55}{6}$ we have $\gamma(\Gamma_c(R)) \geq 2l_3 + 10l_4 \geq 12$. That is, $\Gamma_c(R)$ is neither planar nor toroidal. If $q = 3$ and $p \geq 5$ then

$$\begin{aligned}\gamma(\Gamma_c(R)) &= l_1\gamma(K_{p-1}) + l_2\gamma(K_2) + l_3\gamma(K_{p^2-1}) + l_4\gamma(K_{3p-1})\\ &= l_1\left\lceil\frac{(p-4)(p-5)}{12}\right\rceil + l_3\left\lceil\frac{(p^2-4)(p^2-5)}{12}\right\rceil + l_4\left\lceil\frac{(3p-4)(3p-5)}{12}\right\rceil.\end{aligned}$$

Since $\frac{(p-4)(p-5)}{12} \geq 0$, $\frac{(p^2-4)(p^2-5)}{12} \geq 35$ and $\frac{(3q-4)(3q-5)}{12} \geq \frac{55}{6}$ we have $\gamma(\Gamma_c(R)) \geq 35l_3 + 10l_4 \geq 45$. That is, $\Gamma_c(R)$ is neither planar nor toroidal. If $p \geq 5$ and $q \geq 5$ then

$$\gamma(\Gamma_c(R)) = l_1\left\lceil\frac{(p-4)(p-5)}{12}\right\rceil + l_2\left\lceil\frac{(q-4)(q-5)}{12}\right\rceil + l_3\left\lceil\frac{(p^2-4)(p^2-5)}{12}\right\rceil + l_4\left\lceil\frac{(pq-4)(pq-5)}{12}\right\rceil.$$

Since $\frac{(p-4)(p-5)}{12} \geq 0$, $\frac{(q-4)(q-5)}{12} \geq 0$, $\frac{(p^2-4)(p^2-5)}{12} \geq 35$ and $\frac{(pq-4)(pq-5)}{12} \geq 35$ we have $\gamma(\Gamma_c(R)) \geq 35l_3 + 35l_4 \geq 70$. That is, $\Gamma_c(R)$ is neither planar nor toroidal. $\square$

Theorem 2.4 *Let R be a non-commutative ring with unity where $|R| = p^3q$ and $|Z(R)| = pq$.*

(a) *If $p = 2 = q$ then $\Gamma_c(R)$ is planar.*

(b) *If $p = 2$ and $q \geq 3$ then $\gamma(\Gamma_c(R)) = 3\left\lceil\frac{(2q-3)(2q-4)}{12}\right\rceil$.*

(c) *If $q = 2$ and $p \geq 3$ then $\gamma(\Gamma_c(R)) = (p+1)\left\lceil\frac{(2p^2-2p-3)(2p^2-2p-4)}{12}\right\rceil$.*

(d) *If* $p \geq 3$ *and* $q \geq 3$ *then* $\gamma(\Gamma_c(R)) = (p+1)\left\lceil \frac{(p^2q-pq-3)(p^2q-pq-4)}{12} \right\rceil$.

It follows that $\Gamma_c(R)$ *is neither planar nor toroidal in all the cases* $(b)-(d)$.

Proof: By Theorem 1.4, we have $\Gamma_c(R) = (p+1)K_{p^2q-pq}$. By (1.2) we have $\gamma(\Gamma_c(R)) = (p+1)\gamma(K_{p^2q-pq})$.

Case 1: $p = 2 = q$

In this case $\gamma(\Gamma_c(R)) = 3\gamma(K_4) = 0$. Therefore, $\Gamma_c(R)$ is planar.

Case 2: $p = 2$ and $q \geq 3$

In this case we have $\gamma(\Gamma_c(R)) = 3\gamma(K_{2q})$ and $2q \geq 6$. Therefore, by (1.1) we have

$$\gamma(\Gamma_c(R)) = 3\left\lceil \frac{(2q-3)(2q-4)}{12} \right\rceil.$$

Since $\frac{(2q-3)(2q-4)}{12} \geq \frac{1}{2}$ we have $\gamma(\Gamma_c(R)) \geq 3$. That is, $\Gamma_c(R)$ is neither planar nor toroidal.

Case 3: $q = 2$ and $p \geq 3$

In this case we have $\gamma(\Gamma_c(R)) = (p+1)\gamma(K_{2p^2-2p})$ and $2p^2 - 2p \geq 12$. Therefore, by (1.1) we have

$$\gamma(\Gamma_c(R)) = (p+1)\left\lceil \frac{(2p^2-2p-3)(2p^2-2p-4)}{12} \right\rceil.$$

Since $\frac{(2p^2-2p-3)(2p^2-2p-4)}{12} \geq 6$ and $p+1 \geq 4$ we have $\gamma(\Gamma_c(R)) \geq 24$. That is, $\Gamma_c(R)$ is neither planar nor toroidal.

Case 4: $p \geq 3$ and $q \geq 3$

We have $p^2q - pq \geq 18$. Therefore, by (1.1) we have

$$\gamma(\Gamma_c(R)) = (p+1)\left\lceil \frac{(p^2q-pq-3)(p^2q-pq-4)}{12} \right\rceil.$$

Since $\frac{(p^2q-pq-3)(p^2q-pq-4)}{12} \geq \frac{35}{2}$ and $p+1 \geq 4$ we have $\gamma(\Gamma_c(R)) \geq 72$. That is, $\Gamma_c(R)$ is neither planar nor toroidal. □

Theorem 2.5 *Let* R *be a non-commutative ring where* $|R| = p^3q$ *and* $|Z(R)| = p^2$.

(a) *Let* $(q-1) \mid (pq-1)$.

  (i) If $p = 2 = q$ *then* $\Gamma_c(R)$ *is planar.*

  (ii) If $q = 2$ *and* $p \geq 3$ *then* $\gamma(\Gamma_c(R)) = 2p - 1\left\lceil \frac{(p^2-3)(p^2-4)}{12} \right\rceil$.

  (iii) If $p \geq 3$ *and* $q \geq 3$ *then* $\gamma(\Gamma_c(R)) = \frac{pq-1}{q-1}\left\lceil \frac{(p^2q-p^2-3)(p^2q-p^2-4)}{12} \right\rceil$.

  It follows that $\Gamma_c(R)$ *is neither planar nor toroidal in the cases (ii) and (iii).*

(b) *Let* $(p-1) \mid (pq-1)$.

  (i) If $p = 2$ *and* $q \geq 2$ *then* $\Gamma_c(R)$ *is planar.*

  (ii) If $p \geq 3$ *and* $q \geq 3$ *then* $\gamma(\Gamma_c(R)) = \frac{pq-1}{p-1}\left\lceil \frac{(p^3-p^2-3)(p^3-p^2-4)}{12} \right\rceil$*; and hence* $\Gamma_c(R)$ *is neither planar nor toroidal.*

(c) *Let* $l_1(p-1) + l_2(q-1) = pq-1$*, where* l_1 *and* l_2 *are positive integers.*

  (i) If $p = 2 = q$ *then* $\Gamma_c(R)$ *is planar.*

  (ii) If $p = 2$ *and* $q \geq 3$ *then* $\gamma(\Gamma_c(R)) = l_2\left\lceil \frac{(4q-7)(4q-8)}{12} \right\rceil$.

(iii) *If $q = 2$ and $p \geq 3$ then* $\gamma(\Gamma_c(R)) = l_1 \left\lceil \frac{(p^3-p^2-3)(p^3-p^2-4)}{12} \right\rceil + l_2 \left\lceil \frac{(p^2-3)(p^2-4)}{12} \right\rceil$.

(iv) *If $p \geq 3$ and $q \geq 3$ then*
$\gamma(\Gamma_c(R)) = l_1 \left\lceil \frac{(p^3-p^2-3)(p^3-p^2-4)}{12} \right\rceil + l_2 \left\lceil \frac{(p^2q-p^2-3)(p^2q-p^2-4)}{12} \right\rceil$.

It follows that $\Gamma_c(R)$ is neither planar nor toroidal in all the cases $(ii) - (iv)$.

Proof: (a) By Theorem 1.4, we have $\Gamma_c(R) = \frac{pq-1}{q-1} K_{p^2q-p^2}$. By (1.2) we have $\gamma(\Gamma_c(R)) = \frac{pq-1}{q-1}\gamma(K_{p^2q-p^2})$. We shall complete the proof by considering the following cases. Note that the case $p = 2$ and $q \geq 3$ does not arise since $(q-1) \nmid (2q-1)$.

Case 1: $p = 2 = q$

In this case we have $\Gamma_c(R) = 3K_4$ and so $\gamma(\Gamma_c(R)) = 0$. That is, $\Gamma_c(R)$ is planar.

Case 2: $q = 2$ and $p \geq 3$

In this case $\gamma(\Gamma_c(R)) = (2p-1)\gamma(K_{p^2})$ where $p^2 \geq 9$. Therefore, by (1.1) we have

$$\gamma(\Gamma_c(R)) = (2p-1) \left\lceil \frac{(p^2-3)(p^2-4)}{12} \right\rceil.$$

Since $\frac{(p^2-3)(p^2-4)}{12} \geq \frac{5}{2}$ and $2p-1 \geq 5$ we have $\gamma(\Gamma_c(R)) > 15$. That is, $\Gamma_c(R)$ is neither planar nor toroidal.

Case 3: $p \geq 3$ and $q \geq 3$

We have $p^2q - p^2 \geq 18$. Therefore, by (1.1) we have

$$\gamma(\Gamma_c(R)) = \frac{pq-1}{q-1} \left\lceil \frac{(p^2q-p^2-3)(p^2q-p^2-4)}{12} \right\rceil.$$

Since $\frac{(p^2q-p^2-3)(p^2q-p^2-4)}{12} \geq \frac{35}{2}$ and $\frac{pq-1}{q-1} > 2$ we have $\gamma(\Gamma_c(R)) > 36$. That is, $\Gamma_c(R)$ is neither planar nor toroidal.

(b) By Theorem 1.4, we have $\Gamma_c(R) = \frac{pq-1}{p-1} K_{p^3-p^2}$. By (1.2) we have $\gamma(\Gamma_c(R)) = \frac{pq-1}{p-1}\gamma(K_{p^3-p^2})$. We shall complete the proof by considering the following cases. Note that the case $q = 2$ and $p \geq 3$ does not arise since $(p-1) \nmid (2p-1)$.

Case 1: $p = 2$ and $q \geq 2$

In this case we have $\gamma(\Gamma_c(R)) = (2q-1)\gamma(K_4) = 0$. That is, $\Gamma_c(R)$ is planar.

Case 2: $p \geq 3$ and $q \geq 3$

We have $p^3 - p^2 \geq 18$. Therefore, by (1.1) we have

$$\gamma(\Gamma_c(R)) = \frac{pq-1}{p-1} \left\lceil \frac{(p^3-p^2-3)(p^3-p^2-4)}{12} \right\rceil.$$

Since $\frac{(p^3-p^2-3)(p^3-p^2-4)}{12} \geq \frac{35}{2}$ and $\frac{pq-1}{p-1} > 2$ we have $\gamma(\Gamma_c(R)) > 36$. That is, $\Gamma_c(R)$ is neither planar nor toroidal.

(c) By Theorem 1.4, we have $\Gamma_c(R) = l_1 K_{p^3-p^2} \sqcup l_2 K_{p^2q-p^2}$. By (1.2) we have $\gamma(\Gamma_c(R)) = l_1\gamma(K_{p^3-p^2}) + l_2\gamma(K_{p^2q-p^2})$.

Case 1: $p = 2 = q$

In this case $\gamma(\Gamma_c(R)) = l_1\gamma(K_4) + l_2\gamma(K_4) = 0$. Therefore, $\Gamma_c(R)$ is planar.

Case 2: $p = 2$ and $q \geq 3$

In this case $\gamma(\Gamma_c(R)) = l_1\gamma(K_4) + l_2\gamma(K_{4q-4}) = l_2\gamma(K_{4q-4})$. We have $4q - 4 \geq 8$. Therefore, by (1.1) we have

$$\gamma(\Gamma_c(R)) = l_2 \left\lceil \frac{(4q-7)(4q-8)}{12} \right\rceil.$$

Since $\frac{(4q-7)(4q-8)}{12} \geq \frac{5}{3}$ we have $\gamma(\Gamma_c(R)) \geq 2l_2 \geq 2$. That is, $\Gamma_c(R)$ is neither planar nor toroidal.

Case 3: $q = 2$ and $p \geq 3$

In this case $\gamma(\Gamma_c(R)) = l_1\gamma(K_{p^3-p^2}) + l_2\gamma(K_{p^2})$. Since $p^3 - p^2 \geq 18$ and $p^2 \geq 9$, by (1.1) we have

$$\gamma(\Gamma_c(R)) = l_1 \left\lceil \frac{(p^3-p^2-3)(p^3-p^2-4)}{12} \right\rceil + l_2 \left\lceil \frac{(p^2-3)(p^2-4)}{12} \right\rceil.$$

Since $\frac{(p^3-p^2-3)(p^3-p^2-4)}{12} \geq \frac{35}{2}$ and $\frac{(p^2-3)(p^2-4)}{12} \geq \frac{5}{2}$ we have $\gamma(\Gamma_c(R)) \geq 18l_1 + 3l_2 \geq 21$. That is, $\Gamma_c(R)$ is neither planar nor toroidal.

Case 4: $p \geq 3$ and $q \geq 3$

We have $p^3 - p^2 \geq 18$ and $p^2q - p^2 \geq 18$. Therefore, by (1.1) we have

$$\gamma(\Gamma_c(R)) = l_1 \left\lceil \frac{(p^3-p^2-3)(p^3-p^2-4)}{12} \right\rceil + l_2 \left\lceil \frac{(p^2q-p^2-3)(p^2q-p^2-4)}{12} \right\rceil.$$

Since $\frac{(p^3-p^2-3)(p^3-p^2-4)}{12} \geq \frac{35}{2}$ and $\frac{(p^2q-p^2-3)(p^2q-p^2-4)}{12} \geq \frac{35}{2}$ we have $\gamma(\Gamma_c(R)) \geq 18l_1 + 18l_2 \geq 36$. That is, $\Gamma_c(R)$ is neither planar nor toroidal. □

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