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New Best Proximity Point Results via Simulation Functions in Fuzzy Metric Spaces

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ABSTRACT: In this paper, we introduce the concepts of $\mathcal{FZ}$-proximal contraction of type-(I) and type-(II) and prove some $g-$best proximity point results involving $\mathcal{FZ}$-simulation functions under suitable conditions on the framework of fuzzy metric spaces. Our results unify, generalize, and improve a lot of solid papers in the same context and may lay the groundwork for new directions of best proximity and fuzzy fixed point theory. Also, an example is given to support the theoretical results.

Key Words: fixed point theory, best proximity point, fuzzy metric, fuzzy proximal contraction.

Contents

1. Introduction

The concept of fuzzy set was initiated by L.A. Zadeh [1] in 1965 as a new mathematical approach to deal with uncertainty and vagueness associated with the real-world context. It is based on the generalization of the classical concepts of crisp set and characteristic function. The theory of fuzzy sets is now well developed as an essential and practical modeling construct. One of the key issues in fuzzy topology is to obtain an appropriate and coherent concept of fuzzy metric space (FMS, for short). This problem has been considered by many authors in a number of different ways [3,4]. Kramosil and Michalek [9] defined fuzzy metric space by generalizing the notion of probabilistic metric space to the fuzzy setting. Furthermore, George and Veeramani [6] modified Kramosil and Michalek's definition of fuzzy metric space with the purpose to obtain a Hausdorff topology for this class of fuzzy metric space, which has significant applications in quantum mechanics, especially in connection with both string and $\epsilon^{(\infty)}$ theory [21,22]. Over the last years, there has been an intense interest in studying the fixed point (FP) theory in FMSs (see e.g, [7,12,13,16,18,19,20,28,29,31,32]).

On the other hand, the best proximity (BP) theory is a flourishing and influential aspect of fixed point theory which plays a crucial role in the study of conditions that ensure the existence of optimal approximate fixed point of non-self-mapping T when the functional equation $Tx = x$ has no solution. In fact, if $T : \mathcal{U} \longrightarrow \mathcal{V}$ is a non-self-mapping where $\mathcal{U}$ and $\mathcal{V}$ are two non-empty subsets of a metric space $(\mathcal{L}, d)$, it is crucial to furnish an optimal approximate solution $x \in \mathcal{U}$ which induces the minimum error $d(x, Tx)$. Taking into account the fact that $d(x, Tx)$ is at least $d(\mathcal{U}, \mathcal{V})$, a best proximity point of T is the optimal approximate solution x satisfying $d(x, Tx) = d(\mathcal{U}, \mathcal{V})$. In a natural way, the BP theory is a noteworthy generalization of FP theory. Precisely, a BP point turns out to be a FP if the mapping in question is a self-mapping. Further results of different type of contractions for the existence of a BP point in classical and FMSs can be found in ([2,19,24,25,26,27,30]).

In this manuscript, we introduce the ideas of $\mathcal{FZ}$-proximal contraction of type-(I) and (II) and prove some $g-$BP results involving $\mathcal{FZ}$-simulation functions under suitable conditions on the context of FMSs. Our results unify, generalize, and improve a lot of solid papers in the same direction. Also, an example is given to support the theoretical results.

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2. Preliminaries

This part has been specially prepared to provide some preliminary excerpts related to FMSs. Here, $\mathbb{N}$ and $\mathbb{R}$ refer to the set of all positive integer numbers and real numbers, respectively.

Definition 2.1 *[10] A binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is called a continuous t-norm if it satisfies the assertions below:*

1. *$*$ is continuous,*
2. *$*$ is commutative and associative,*
3. *$f_1 * 1 = f_1$ for all $f_1 \in [0,1]$,*
4. *$f_1 * f_2 \le f_3 * f_4$ whenever $f_1 \le f_3$ and $f_2 \le f_4$, for all $f_1, f_2, f_3, f_4 \in [0,1]$.*

Example 2.1 *The following instances are classical examples of continuous t-norm:*

i) $f_1 *_m f_2 = \min\{f_1, f_2\}$,

ii) $f_1 *_p f_2 = f_1.f_2$.

iii) $f_1 *_L f_2 = \max\{0, f_1 + f_2 - 1\}$,

Definition 2.2 *[6] The 3-tuple $(\mathcal{L}, \vartheta, *)$ is said to be an FMS if $\mathcal{L}$ is an arbitrary set, $*$ is a continuous t-norm and ϑ is fuzzy set on $\mathcal{L}^2 \times (0,\infty)$ satisfying the following axioms:*

(GV1) $\vartheta(x, y, \jmath) > 0$,

(GV2) $\vartheta(x, y, \jmath) = 1$ *if and only if* $x = y$,

(GV3) $\vartheta(x, y, \jmath) = \vartheta(y, x, \jmath)$,

(GV4) $\vartheta(x, y, \jmath) * \vartheta(y, z, \imath) \le \vartheta(x, z, \jmath + \imath)$,

(GV5) $\vartheta(x, y, .) : (0,\infty) \to [0,1]$ *is continuous .*

for all $x, y, z \in \mathcal{L}$ and $\jmath, \imath > 0$.

In the following instances, $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous increasing function, and d is a metric on $\mathcal{L}$.

Example 2.2 *[11] Define the mapping ϑ by*

$$\vartheta(x, y, \jmath) = \exp\Big(\frac{-d(x, y)}{\varphi(\jmath)}\Big) \text{ for all } x, y \in \mathcal{L}, \jmath > 0. \tag{2.1}$$

Then $(\vartheta, *_p)$ is a fuzzy metric on $\mathcal{L}$. In particular, if we use φ as the identity mapping, then (2.1) yields

$$\vartheta(x, y, \jmath) = \exp\Big(\frac{-d(x, y)}{\jmath}\Big) \text{ for all } x, y \in \mathcal{L}, \jmath > 0.$$

In this case, $(\vartheta, *_m)$ is a fuzzy metric on $\mathcal{L}$ (see Remark 2.8 in [6]).

Example 2.3 *[11] Define the mapping ϑ by*

$$\vartheta(x, y, \jmath) = \frac{\varphi(\jmath)}{\varphi(\jmath) + md(x, y)} \text{ for all } x, y \in \mathcal{L}, \jmath > 0 \text{ with } m > 0. \tag{2.2}$$

Then $(\vartheta, *_p)$ is a fuzzy metric on $\mathcal{L}$.

Particularly, if we choose $\varphi(\jmath) = \jmath^n$, $n \in \mathbb{N}$ and $m = 1$, hence (2.2) becomes

$$\vartheta(x, y, \jmath) = \frac{\jmath^n}{\jmath^n + d(x, y)} \text{ for all } x, y \in \mathcal{L}, \jmath > 0.$$

and then $(\vartheta, *_m)$ is a fuzzy metric, as proven in [8]. Especially, for $n = 1$, the standard fuzzy metric is derived, as given in [6].

Lemma 2.1 *[5] $\vartheta(x, y, .)$ is nondecreasing for all x, y in $\mathcal{L}$.*

Definition 2.3 *[6] Let $(\mathcal{L}, \vartheta, *)$ be an FMS.*

1. *A sequence $\{y_n\} \subseteq \mathcal{L}$ is said to be convergent to $y \in \mathcal{L}$ if $\lim_{n\to\infty} \vartheta(y_n, y, \jmath) = 1$ for all $\jmath > 0$.*
2. *A sequence $\{y_n\} \subseteq \mathcal{L}$ is said to be a Cauchy sequence if for each $\varepsilon \in (0, 1)$ and $\jmath > 0$, there exists $n_0 \in \mathbb{N}$ such that $\vartheta(y_n, y_m, \jmath) > 1 - \varepsilon$ for all $n, m \geq n_0$.*
3. *A fuzzy metric space in which every Cauchy sequence is convergent is called a complete FMS.*

Definition 2.4 *[12] Let Ψ be the set of all mappings $\psi : (0, 1] \longrightarrow (0, 1]$ such that ψ is nondecreasing, continuous and $\psi(p) > p$, for all $p \in (0, 1)$.*

Definition 2.5 *[13] Let $\mathcal{H}$ be the family of all mappings $\eta : (0, 1] \longrightarrow [0, \infty)$ satisfying the following conditions :*

$\mathcal{H}_1$) *η transforms $(0, 1]$ onto $[0, \infty)$,*

$\mathcal{H}_2$) *η is strictly decreasing.*

The following class of control functions was initiated by Melliani and Moussaoui [16] (see also [17]) in order to present a new type of contraction known as $\mathcal{FZ}$-contractions.

Definition 2.6 *[16] The function $\zeta : (0, 1] \times (0, 1] \longrightarrow \mathbb{R}$ is said to be an $\mathcal{FZ}$-simulation function, if the following properties hold:*

(ζ1) *$\zeta(1, 1) = 0$,*

(ζ2) *$\zeta(t, s) < \frac{1}{s} - \frac{1}{t}$ for all $t, s \in (0, 1)$,*

(ζ3) *if $\{t_n\}, \{s_n\}$ are sequences in $(0, 1]$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n < 1$ then $\lim_{n\to\infty} \sup \zeta(t_n, s_n) < 0$.*

The collection of all $\mathcal{FZ}$-simulation functions is denoted by $\mathcal{FZ}$.

Definition 2.7 *[16] Let $(\mathcal{L}, \vartheta, *)$ be an FMS, $T : \mathcal{L} \longrightarrow \mathcal{L}$ a mapping and $\zeta \in \mathcal{FZ}$. Then T is called an $\mathcal{FZ}$-contraction with respect to ζ if the following condition is fulfilled:*

$$\zeta(\vartheta(Tx, Ty, \jmath), \vartheta(x, y, \jmath)) \geq 0 \text{ for all } x, y \in \mathcal{L}, \jmath > 0.$$

Example 2.4 *([16,17]) Let $\zeta_i : (0, 1] \times (0, 1] \longrightarrow \mathbb{R}$, $i = 1, 2, 3$ be such that*

1. *$\zeta_1(t, s) = k\left(\frac{1}{s} - 1\right) - \frac{1}{t} + 1$ for all $s, t \in (0, 1]$, where $k \in (0, 1)$,*
2. *$\zeta_2(t, s) = \frac{1}{\psi(s)} - \frac{1}{t}$ for all $s, t \in (0, 1]$ and $\psi \in \Psi$,*
3. *$\zeta_3(t, s) = \frac{1}{\eta^{-1}(k.\eta(s))} - \frac{1}{t}$ for all $t, s \in (0, 1]$, where $\eta \in \mathcal{H}$.*

Then the functions ζ_i for $i = 1, 2, 3$ are $\mathcal{FZ}$-simulation functions.

Let $\mathcal{U}$ and $\mathcal{V}$ be two non-empty subsets of an FMS $(\mathcal{L}, \vartheta, *)$ and $T : \mathcal{U} \longrightarrow \mathcal{V}$ be a non-self-mapping. The following notations will be used in the sequel.

$$\mathcal{U}_0(\jmath) = \{u \in \mathcal{U} : \vartheta(u, v, \jmath) = \vartheta(\mathcal{U}, \mathcal{V}, \jmath) \text{ for some } v \in \mathcal{V}\},$$

$$\mathcal{V}_0(\jmath) = \{v \in \mathcal{V} : \vartheta(u, v, t) = \vartheta(\mathcal{U}, \mathcal{V}, \jmath) \text{ for some } u \in \mathcal{U}\},$$

where

$$\vartheta(\mathcal{U}, \mathcal{V}, \jmath) = \sup\{\vartheta(u, v, \jmath) : u \in \mathcal{U}, v \in \mathcal{V}\},$$

and

$$\vartheta(u,\mathcal{V},\jmath)=\sup_{v\in\mathcal{V}}\vartheta(u,v,\jmath),\ \text{for } \jmath>0.$$

The set of all BP points of a non-self mapping $T:\mathcal{U}\to\mathcal{V}$ will be denoted by

$$B_{est}(T)=\{u\in\mathcal{U},\vartheta(u,Tu,\jmath)=\vartheta(\mathcal{U},\mathcal{V},\jmath)\}.$$

Furthermore, if $g:\mathcal{U}\longrightarrow\mathcal{U}$ then we have

$$B^{g}_{est}(T)=\{u\in\mathcal{U},\vartheta(gu,Tu,\jmath)=\vartheta(\mathcal{U},\mathcal{V},\jmath)\}.$$

3. Main results

In this section, we employ $\mathcal{FZ}$-simulation functions to introduce a very general types of fuzzy proximal contractions on FMSs, and we establish related BP points theorems. We begin with the definitions below:

Definition 3.1 *Let $\mathcal{U}$ and $\mathcal{V}$ be two non-empty subsets of a FMS $(\mathcal{L},\vartheta,*)$, $T:\mathcal{U}\to\mathcal{V}$ be a non-self mapping and $g:\mathcal{U}\longrightarrow\mathcal{U}$ be a self-mapping. We define $\Lambda_{g,\vartheta}$ and $\Omega_{\mathcal{U},\vartheta}$ as follows:*

$$\Lambda_{g,\vartheta}=\{T:\mathcal{U}\longrightarrow\mathcal{V}:\ \vartheta(Tgx,Tgy,\jmath)\le\vartheta(Tx,Ty,\jmath),\forall x,y\in\mathcal{U}\},$$
$$\Omega_{\mathcal{U},\vartheta}=\{g:\mathcal{U}\longrightarrow\mathcal{U}\ \textit{is continuous}:\ \vartheta(gx,gy,\jmath)\le\vartheta(x,y,\jmath),\forall x,y\in\mathcal{U}\}.$$

Definition 3.2 *Let $\mathcal{U}$ and $\mathcal{V}$ be two non-empty subsets of FMS $(\mathcal{L},\vartheta,*)$. We say that a non-self mapping $T:\mathcal{U}\to\mathcal{V}$ is an $\mathcal{FZ}$-proximal contraction of type (I) with respect to $\xi\in\mathcal{FZ}$ if*

$$\begin{cases}\vartheta(u,Tx,\jmath)=\vartheta(\mathcal{U},\mathcal{V},\jmath)\\ \vartheta(v,Ty,\jmath)=\vartheta(\mathcal{U},\mathcal{V},\jmath)\end{cases}\Rightarrow\xi(\vartheta(u,v,\jmath),\vartheta(x,y,\jmath))\ge0, \tag{3.1}$$

for all $u,v,x,y\in\mathcal{U}$ and $\jmath>0$.

Definition 3.3 *Let $\mathcal{U}$ and $\mathcal{V}$ be two non-empty subsets of FMS $(\mathcal{L},\vartheta,*)$. We say that a non-self mapping $T:\mathcal{U}\to\mathcal{V}$ is an $\mathcal{FZ}$-proximal contraction of type (II) with respect to $\xi\in\mathcal{FZ}$ if*

$$\begin{cases}\vartheta(u,Tx,\jmath)=\vartheta(\mathcal{U},\mathcal{V},\jmath)\\ \vartheta(v,Ty,\jmath)=\vartheta(\mathcal{U},\mathcal{V},\jmath)\end{cases}\Rightarrow\xi(\vartheta(Tu,Tv,\jmath),\vartheta(x,y,\jmath))\ge0, \tag{3.2}$$

for all $u,v,x,y\in\mathcal{U}$ and $\jmath>0$.

Our first main result is as follows:

Theorem 3.1 *Let $\mathcal{U}$ and $\mathcal{V}$ be non-empty subsets of a complete FMS $(\mathcal{L},\vartheta,*)$ such that $\mathcal{U}_0$ is non-empty and closed. Suppose also that the mappings $T:\mathcal{U}\longrightarrow\mathcal{V}$ and $g:\mathcal{U}\longrightarrow\mathcal{U}$ satisfy the followings*

(i) *T is an $\mathcal{FZ}$-proximal contraction of type (I),*

(ii) *$g\in\Omega_{\mathcal{U},\vartheta}$,*

(iii) *$T(\mathcal{U}_0)\subseteq\mathcal{V}_0$,*

(iv) *$\mathcal{U}_0\subseteq g(\mathcal{U}_0)$.*

Then there exists a unique $x\in\mathcal{U}$ such that $\vartheta(gx,Tx,\jmath)=\vartheta(\mathcal{U},\mathcal{V},\jmath)$ for all $\jmath>0$. Moreover, for each $x_0\in\mathcal{U}_0$ there exists a sequence $\{x_n\}\subseteq\mathcal{U}$ such that $\vartheta(gx_{n+1},Tx_n,\jmath)=\vartheta(\mathcal{U},\mathcal{V},\jmath)$ for every $n\in\mathbb{N}$ and $x_n\to x$.

Proof: Let $x_0 \in \mathcal{U}_0$. Since $T(\mathcal{U}_0) \subseteq \mathcal{V}_0$ and $\mathcal{U}_0 \subseteq g(\mathcal{U}_0)$, there exists $x_1 \in \mathcal{U}_0$ such that

$$\vartheta(gx_1, Tx_0, \jmath) = \vartheta(\mathcal{U}, \mathcal{V}, \jmath),$$

and for $x_1 \in \mathcal{U}_0$, there exists $x_2 \in \mathcal{U}_0$ such that

$$\vartheta(gx_2, Tx_1, t) = \vartheta(\mathcal{U}, \mathcal{V}, \jmath).$$

Recursively, a sequence $\{x_n\} \subseteq \mathcal{U}_0$ can be defined as follows

$$\vartheta(gx_{n+1}, Tx_n, \jmath) = \vartheta(\mathcal{U}, \mathcal{V}, \jmath) \text{ for all } n \in \mathbb{N}. \tag{3.3}$$

In the construction procedure of $\{x_n\}$, if for some $p > n$, we have $Tx_p = Tx_n$, then we take $x_{p+1} = x_{n+1}$. Moreover, if there exist $p \in \mathbb{N}$ such that $\vartheta(gx_{p+1}, gx_p, \jmath) = 1$, that is, $x_{p+1} = x_p$ and so $Tx_{p+1} = Tx_p$ and $x_{p+2} = x_{p+1}$. Hence, $x_n = x_p$ for all $n \geq p$, which means that the sequence $\{x_n\}$ converges to $x_p \in \mathcal{U}$, we have $\vartheta(gx_p, Tx_p, \jmath) = \vartheta(\mathcal{U}, \mathcal{V}, \jmath)$ as well. Therefore, to continue our proof, we assume that $\vartheta(gx_{n+1}, gx_n, \jmath) \leq \vartheta(x_{n+1}, x_n, \jmath) < 1$ for all $n \in \mathbb{N}$, $\jmath > 0$.

Regarding that T is a $\mathcal{FZ}$-proximal contraction of type (I) with respect to $\xi \in \mathcal{FZ}$ and ($\xi 2$), we have

$$\begin{aligned} 0 &\leq \xi(\vartheta(gx_{n+1}, gx_n, \jmath), \vartheta(x_n, x_{n-1}, \jmath)) \\ &< \frac{1}{\vartheta(x_n, x_{n-1}, \jmath)} - \frac{1}{\vartheta(gx_{n+1}, gx_n, \jmath)}. \end{aligned} \tag{3.4}$$

As $g \in \Omega_{\mathcal{U},\vartheta}$, we derive that

$$\vartheta(x_{n-1}, x_n, \jmath) < \vartheta(gx_{n+1}, gx_n, \jmath) \leq \vartheta(x_{n+1}, x_n, \jmath). \tag{3.5}$$

This means that $\{\vartheta(x_n, x_{n-1}, \jmath)\}$ is a nondecreasing sequence of positive real numbers in $(0, 1]$. Then, there exists $l(\jmath) \leq 1$ such that $\lim_{n\to\infty} \vartheta(x_n, x_{n-1}, \jmath) = l(\jmath)$ for all $\jmath > 0$. We shall prove that $l(\jmath) = 1$. Reasoning by contradiction, suppose that $l(\jmath_0) < 1$ for some $\jmath_0 > 0$. By (3.5), we get also that

$$\lim_{n\to\infty} \vartheta(gx_{n+1}, gx_n, \jmath) = l(\jmath_0).$$

Now, if we take the sequences $\{\vartheta(gx_{n+1}, gx_n, \jmath_0)$ and $\{\vartheta(x_n, x_{n-1}, \jmath_0)\}$ and considering ($\xi 3$), we get

$$0 \leq \lim_{n\to\infty} \sup \xi(\vartheta(gx_{n+1}, gx_n, \jmath_0), \vartheta(x_n, x_{n-1}, \jmath_0)) < 0.$$

This is a contraction, which implies that

$$\lim_{n\to\infty} \vartheta(x_n, x_{n-1}, \jmath) = 1 \text{ for all } \jmath > 0. \tag{3.6}$$

Next, we show that the sequence $\{x_n\}$ is Cauchy. Reasoning by contradiction, suppose that $\{x_n\}$ is not a Cauchy sequence. Then, there exist $\epsilon \in (0, 1)$, $\jmath_0 > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ with $n_k > m_k \geq k$ for all $k \in \mathbb{N}$ such that

$$\vartheta(x_{m_k}, x_{n_k}, \jmath_0) \leq 1 - \epsilon. \tag{3.7}$$

Taking in account Lemma 2.1 we derive that

$$\vartheta(x_{m_k}, x_{n_k}, \frac{\jmath_0}{2}) \leq 1 - \epsilon. \tag{3.8}$$

Taking m_k as the lowest value satisfying (3.8), we have

$$\vartheta(x_{m_k}, x_{n_k - 1}, \frac{\jmath_0}{2}) > 1 - \epsilon. \tag{3.9}$$

On account of (3.7), (3.9) and the triangular inequality, we have

$$\begin{aligned}1-\epsilon &\geq \vartheta(x_{m_k}, x_{n_k}, \jmath_0)\\ &\geq \vartheta(x_{m_k}, x_{n_k-1}, \frac{\jmath_0}{2}) * \vartheta(x_{n_k-1}, x_{n_k}, \frac{\jmath_0}{2})\\ &> (1-\epsilon) * \vartheta(x_{n_k-1}, x_{n_k}, \frac{\jmath_0}{2}).\end{aligned}$$

Taking the limit as $k \to \infty$ in both sides of the above inequality and using (3.6), we derive that

$$\lim_{n\to\infty} \vartheta(x_{m_k}, x_{n_k}, \jmath_0) = 1-\epsilon. \tag{3.10}$$

On other hand, we have

$$\vartheta(x_{m_k+1}, x_{n_k+1}, \jmath_0) \geq \vartheta(x_{m_k+1}, x_{m_k}, \frac{\jmath_0}{3}) * \vartheta(x_{m_k}, x_{n_k}, \frac{\jmath_0}{3}) * \vartheta(x_{n_k}, x_{n_k+1}, \frac{\jmath_0}{3}),$$

and

$$\vartheta(x_{m_k}, x_{n_k}, \jmath_0) \geq \vartheta(x_{m_k}, x_{m_k+1}, \frac{\jmath_0}{3}) * \vartheta(x_{m_k+1}, x_{n_k+1}, \frac{\jmath_0}{3}) * \vartheta(x_{n_k+1}, x_{n_k}, \frac{\jmath_0}{3}),$$

which imply that

$$\lim_{n\to\infty} \vartheta(x_{m_k+1}, x_{n_k+1}, \jmath_0) = 1-\epsilon. \tag{3.11}$$

Regarding the fact that T is an $\mathcal{FZ}$-proximal contraction of type (I) with respect to $\xi \in \mathcal{FZ}$ and $\vartheta(gx_{m_k+1}, Tx_{m_k}, \jmath_0) = \vartheta(gx_{n_k+1}, Tx_{n_k}, \jmath_0) = \vartheta(\mathcal{U}, \mathcal{V}, \jmath_0)$, we have

$$\begin{aligned}0 &\leq \xi(\vartheta(gx_{m_k+1}, gx_{m_k+1}, \jmath_0), \vartheta(x_{n_k}, x_{mk}, \jmath_0))\\ &< \frac{1}{\vartheta(x_{n_k}, x_{m_k}, \jmath_0)} - \frac{1}{\vartheta(gx_{n_k+1}, gx_{m_k+1}, \jmath_0)},\end{aligned} \tag{3.12}$$

for all $k \in \mathbb{N}$. Using (3.12) and taking into account that $g \in \Omega_{\mathcal{U},\vartheta}$, we get

$$\vartheta(x_{n_k}, x_{m_k}, \jmath_0) < \vartheta(gx_{n_k+1}, gx_{m_k+1}, \jmath_0) \leq \vartheta(x_{n_k+1}, x_{m_k+1}, \jmath_0).$$

Hence,

$$\lim_{n\to\infty} \vartheta(gx_{n_k+1}, gx_{m_k+1}, \jmath_0) = 1-\epsilon. \tag{3.13}$$

From (3.10) and (3.13), we see that the sequences $\{\mu_k = \vartheta(gx_{n_k+1}, gx_{m_k+1}, \jmath_0)\}$ and $\{\nu_k = \vartheta(x_{m_k}, x_{n_k}, \jmath_0)\}$ have the same limit $1-\epsilon < 1$, by the property ($\xi 3$), we get

$$0 \leq \lim_{k\to\infty} \sup \xi(\mu_k, \nu_k) < 0.$$

This is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence. Since $\mathcal{U}_0$ is closed subset of a complete FMS $(\mathcal{L}, \vartheta, *)$, there exists $x \in \mathcal{U}_0$ such that $x_n \to x$. As g is continuous, $gx_n \in \mathcal{U}_0$ for all $n \in \mathbb{N}$ and $\mathcal{U}_0$ is closed, it follows that $gx_n \to gx$ and then $gx \in \mathcal{U}_0$. Now, since $x \in \mathcal{U}_0$ and $T(\mathcal{U}_0) \subseteq \mathcal{V}_0$, there exists $z \in \mathcal{U}_0$ such that $\vartheta(z, Tx, \jmath) = \vartheta(\mathcal{U}, \mathcal{V}, \jmath)$. Next, if $z = gx_n$ for infinite $n \in \mathbb{N}$, it follows that $z = gx$. We suppose that $z \neq gx_n$ for all $n \in \mathbb{N}$. Also, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \neq x$ for all $k \in \mathbb{N}$. Next, since T is $\mathcal{FZ}$-proximal contraction of type (I) with respect to $\xi \in \mathcal{FZ}$, we have

$$\begin{aligned}0 &\leq \xi(\vartheta(z, gx_{n_k+1}, \jmath), \vartheta(x, x_{n_k}, \jmath))\\ &< \frac{1}{\vartheta(x, x_{n_k}, \jmath)} - \frac{1}{\vartheta(z, gx_{n_k+1}, \jmath)}.\end{aligned}$$

Thus, we have

$$\vartheta(x, x_{n_k}, \jmath) < \vartheta(z, gx_{n_k+1}, \jmath) \text{ for all } k \in \mathbb{N}.$$

Passing to the limit as $k \to \infty$, it follows that

$$\lim_{n\to\infty} \vartheta(z, gx_{n_k+1}, \jmath) = 1 \text{ and then } z = gx,$$

which implies that

$$\vartheta(gx, Tx, \jmath) = \vartheta(\mathcal{U}, \mathcal{V}, \jmath).$$

In the next step we prove the uniqueness, we argue by contradiction, suppose that x^* be another point in $\mathcal{U}_0$ distinct of x such that

$$\vartheta(gx^*, Tx^*, \jmath) = \vartheta(\mathcal{U}, \mathcal{V}, \jmath).$$

Regarding $g \in \Omega_{\mathcal{U},\vartheta}$ and T is an $\mathcal{FZ}$-proximal contraction with respect to $\xi \in \mathcal{FZ}$

$$\begin{aligned}
0 &\le \xi(\vartheta(gx, gx^*, \jmath), \vartheta(x, x^*, \jmath)) \\
&< \frac{1}{\vartheta(x, x^*, \jmath)} - \frac{1}{\vartheta(gx, gx^*, \jmath)} \\
&\le \frac{1}{\vartheta(x, x^*, \jmath)} - \frac{1}{\vartheta(x, x^*, \jmath)} = 0
\end{aligned}$$

and hence $x = x^*$, which is a contradiction. □

Corollary 3.1 *Let $\mathcal{U}$ and $\mathcal{V}$ be non-empty subsets of a complete FMS $(\mathcal{L}, \vartheta, *)$ such that $\mathcal{U}_0$ is non-empty and closed. Assume that the mappings $T : \mathcal{U} \longrightarrow \mathcal{V}$ satisfies the following conditions:*

- **(i)** *T is a $\mathcal{FZ}$-proximal contraction of type (I),*
- **(ii)** *$T(\mathcal{U}_0) \subseteq \mathcal{V}_0$.*

Then there exists $x \in \mathcal{U}$ such that $\vartheta(x, Tx, \jmath) = \vartheta(\mathcal{U}, \mathcal{V}, \jmath)$ for all $\jmath > 0$. Moreover, for each $x_0 \in \mathcal{U}_0$ there exists a sequence $\{x_n\} \subseteq \mathcal{U}$ such that $\vartheta(x_{n+1}, Tx_n, \jmath) = \vartheta(\mathcal{U}, \mathcal{V}, \jmath)$ for every $n \in \mathbb{N}$ and $x_n \to x$.

Proof: It follows from Theorem 3.1 by taking g as the identity mapping on $\mathcal{L}$. □

Theorem 3.2 *Let $\mathcal{U}$ and $\mathcal{V}$ be non-empty subsets of a complete FMS $(\mathcal{L}, \vartheta, *)$ such that $\mathcal{F}\mathcal{U}_0$ is non-empty and closed. Suppose also that the mappings $T : \mathcal{U} \longrightarrow \mathcal{V}$ and $g : \mathcal{U} \longrightarrow \mathcal{U}$ satisfy the followings:*

- **(i)** *T is an $\mathcal{FZ}$-proximal contraction of type (II),*
- **(ii)** *T is injective on $\mathcal{U}_0$,*
- **(iii)** *$T \in \Lambda_{g,\vartheta}$,*
- **(iv)** *$T(\mathcal{U}_0) \subseteq \mathcal{V}_0$,*
- **(v)** *$\mathcal{U}_0 \subseteq g(\mathcal{U}_0)$.*

Then there exists $x \in \mathcal{U}$ such that $\vartheta(gx, Tx, \jmath) = \vartheta(\mathcal{U}, \mathcal{V}, \jmath)$ for all $\jmath > 0$. Moreover, for each $x_0 \in \mathcal{U}_0$ there exists a sequence $\{x_n\} \subseteq \mathcal{U}$ such that $\vartheta(gx_{n+1}, Tx_n, \jmath) = \vartheta(\mathcal{U}, \mathcal{V}, \jmath)$ for every $n \in \mathbb{N}$ and $x_n \to x$.

Proof: Following the lines of the proof of Theorem 3.1, one can construct a sequence $\{x_n\} \subset \mathcal{U}_0$ such that $\vartheta(gx_{n+1}, Tx_n, \jmath) = \vartheta(\mathcal{U}, \mathcal{V}, \jmath)$ for all $n \in \mathbb{N}$. if for some $m > n$ we have $Tx_m = Tx_n$, then we choose $x_{m+1} = x_{n+1}$. Since T is an $\mathcal{FZ}$-proximal contraction type (II) with respect to $\xi \in \mathcal{FZ}$, we have

$$0 \leq \xi(\vartheta(Tgx_{n+1}, Tgx_n, \jmath), \vartheta(Tx_n, Tx_{n-1}, \jmath)).$$

Due to the injectivity of T on $\mathcal{U}_0$, $T \in \Lambda_{g,\vartheta}$ and considering (ξ_3), we derive that

$$\begin{aligned} 0 &\leq \xi(\vartheta(Tgx_{n+1}, Tgx_n, \jmath), \vartheta(Tx_n, Tx_{n-1}, \jmath)) \\ &< \frac{1}{\vartheta(Tx_n, Tx_{n-1}, \jmath)} - \frac{1}{\vartheta(Tgx_{n+1}, Tgx_n, \jmath)} \\ &\leq \frac{1}{\vartheta(Tx_n, Tx_{n-1}, \jmath)} - \frac{1}{\vartheta(Tx_{n+1}, Tx_n, \jmath)}. \end{aligned} \tag{3.14}$$

Consequently,

$$\vartheta(Tx_n, Tx_{n-1}, \jmath) < \vartheta(Tx_{n+1}, Tx_n, \jmath). \tag{3.15}$$

Hence $\{\vartheta(Tx_n, Tx_{n-1}, \jmath)\}$ is a nondecreasing sequence of positive real numbers in $(0, 1]$. Then, there exists $h(\jmath) \leq 1$ such that $\lim_{n\to\infty} \vartheta(Tx_n, Tx_{n-1}, \jmath) = h(\jmath) \geq 1$ for all $\jmath > 0$. We shall prove that $h(\jmath) = 1$. Suppose that $h(\jmath) < 1$ for some $\jmath > 0$. From (14), we derive

$$\vartheta(Tx_n, Tx_{n-1}, \jmath) < \vartheta(Tgx_{n+1}, Tgx_n, \jmath). \tag{3.16}$$

Using the fact that $T \in \Lambda_{g,\vartheta}$, we obtain

$$\vartheta(Tx_n, Tx_{n-1}, \jmath) < \vartheta(Tgx_{n+1}, Tgx_n, \jmath) \leq \vartheta(Tx_{n+1}, Tx_n, \jmath), \tag{3.17}$$

for all $n \in \mathbb{N}$. Letting $k \to \infty$, it follows that

$$\lim_{n\to\infty} \vartheta(gx_{n+1}, gx_n, \jmath) = h(\jmath).$$

Taking the sequences $\{\vartheta(Tgx_{n+1}, Tgx_n, \jmath)$ and $\{\vartheta(x_n, x_{n-1}, \jmath)\}$ and applying (ξ_3), we deduce that

$$0 \leq \lim_{n\to\infty} \sup \xi(\vartheta(Tgx_{n+1}, Tgx_n, \jmath), \vartheta(x_n, x_{n-1}, \jmath)) < 0,$$

which is a contradiction. Thus

$$\lim_{n\to\infty} \vartheta(Tx_n, Tx_{n-1}, \jmath) = 1 \text{ for all } \jmath > 0. \tag{3.18}$$

Next, we show that the sequence $\{Tx_n\}$ is Cauchy. Reasoning by contradiction, suppose that $\{Tx_n\}$ is not a Cauchy sequence. Then, there exists $\epsilon \in (0, 1)$, $\jmath_0 > 0$ and two subsequences $\{Tx_{n_k}\}$ and $\{Tx_{m_k}\}$ of $\{Tx_n\}$ with $n_k > m_k \geq k$ for all $k \in \mathbb{N}$ such that

$$\vartheta(Tx_{m_k}, Tx_{n_k}, \jmath_0) \leq 1 - \epsilon. \tag{3.19}$$

By Lemma 2.1, we have

$$\vartheta(Tx_{m_k}, Tx_{n_k}, \frac{\jmath_0}{2}) \leq 1 - \epsilon. \tag{3.20}$$

Choosing m_k as the smallest value fulfilling (3.20) and using a similar reasoning to that used in the demonstration of Theorem 3.1, we get

$$\lim_{k\to\infty} \vartheta(Tx_{m_k}, Tx_{n_k}, \jmath_0) = 1 - \epsilon = \lim_{n\to\infty} \vartheta(Tx_{m_k+1}, Tx_{n_k+1}, \jmath_0). \tag{3.21}$$

Regarding the fact that T is $\mathcal{FZ}$-proximal contraction type (II) with respect to $\xi \in \mathcal{FZ}$ and $\vartheta(gx_{m_k+1}, Tx_{m_k}, \jmath_0) = \vartheta(gx_{n_k+1}, Tx_{n_k}, \jmath_0) = \vartheta(\mathcal{U}, \mathcal{V}, \jmath_0)$, we have

$$\begin{aligned} 0 &\leq \xi(\vartheta(Tgx_{n_k+1}, Tgx_{m_k+1}, \jmath_0), \vartheta(Tx_{n_k}, Tx_{mk}, \jmath_0)), \\ &< \frac{1}{\vartheta(Tx_{n_k}, Tx_{m_k}, \jmath_0)} - \frac{1}{\vartheta(Tgx_{n_k+1}, Tgx_{m_k+1}, \jmath_0)}, \end{aligned} \tag{3.22}$$

for all $k \in \mathbb{N}$. Using (3.22) and $T \in \Lambda_{g,\vartheta}$, we obtain that

$$\vartheta(Tx_{n_k}, Tx_{m_k}, \jmath_0) < \vartheta(Tgx_{n_k+1}, Tgx_{m_k+1}, \jmath_0) \leq \vartheta(Tx_{n_k+1}, Tx_{m_k+1}, \jmath_0).$$

Hence,

$$\lim_{n\to\infty} \vartheta(Tgx_{n_k+1}, Tgx_{m_k+1}, \jmath_0) = 1 - \epsilon. \tag{3.23}$$

Regarding that the sequences $\{\vartheta(Tgx_{n_k+1}, Tgx_{m_k+1}, \jmath_0)\}$ and $\{\vartheta(Tx_{m_k}, Tx_{n_k}, \jmath_0)\}$ have the same limit $1 - \epsilon < 1$, the property ξ_3 yields that

$$0 \leq \lim_{k\to\infty} \sup \xi(\vartheta(Tgx_{n_k+1}, Tgx_{m_k+1}, \jmath_0), \vartheta(Tx_{m_k}, Tx_{n_k}, \jmath_0)) < 0.$$

Which is a contradiction. Therefore, $\{\mathcal{T}x_n\}$ is a Cauchy sequence. Since $T\mathcal{U}_0$ is closed subset of a complete FMS $(\mathcal{L}, \vartheta, *)$, there exists $u \in \mathcal{U}_0$ such that $Tx_n \to Tu \in T(\mathcal{U}_0) \subseteq \mathcal{V}_0$. Moreover, there exists $z \in \mathcal{U}_0$ such that

$$\vartheta(z, Tu, \jmath) = \vartheta(\mathcal{U}, \mathcal{V}, \jmath).$$

It follows from $\mathcal{U}_0 \subseteq g(\mathcal{U}_0)$ that $z = gx$ for some $x \in \mathcal{U}_0$, and then

$$\vartheta(gx, Tu, \jmath) = \vartheta(\mathcal{U}, \mathcal{V}, \jmath).$$

Now, if $x = x_n$ for infinite $n \in \mathbb{N}$, it follows that $Tu = Tx$. Hence, we suppose that $x \neq x_n$ for all $n \in \mathbb{N}$. Also, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $Tu \neq Tx_{n_k}$ for all $k \in \mathbb{N}$. Since T is $\mathcal{FZ}$-proximal contraction of type (II) with respect to $\xi \in \mathcal{FZ}$,

$$\begin{aligned} 0 &\leq \xi(\vartheta(Tgx, Tgx_{n_k+1}, \jmath), \vartheta(Tu, Tx_{n_k}, \jmath)) \\ &< \frac{1}{\vartheta(Tu, Tx_{n_k}, \jmath)} - \frac{1}{\vartheta(Tgx, Tgx_{n_k+1}, \jmath)}. \end{aligned}$$

Since $\mathcal{T} \in \Lambda_{g,\vartheta}$, the obtained inequality yields that

$$\vartheta(Tu, Tx_{n_k}, \jmath) < \vartheta(Tgx, Tgx_{n_k+1}, \jmath) \leq \vartheta(Tx, Tx_{n_k+1}, \jmath) \text{ for all } k \in \mathbb{N}.$$

Passing to the limit as $k \to \infty$, we get $\vartheta(Tx, Tx_{n_k+1}, \jmath) \to 1$ and then $Tx = Tu$. Consequently,

$$\vartheta(gx, Tx, \jmath) = \vartheta(\mathcal{U}, \mathcal{V}, \jmath).$$

Next, we prove the uniqueness, suppose that $z, z^* \in X$ are two distinct points in $\mathcal{U}_0$ such that

$$\vartheta(gz^*, Tz^*, \jmath) = \vartheta(\mathcal{U}, \mathcal{V}, \jmath).$$

Regarding $T \in \Lambda_{g,\vartheta}$ is injective on $\mathcal{U}_0$ and T is an $\mathcal{FZ}$-proximal contraction of type (II) with respect to $\xi \in \mathcal{FZ}$

$$\begin{aligned} 0 &\leq \xi(\vartheta(Tgz, Tgz^*, \jmath), \vartheta(Tz, Tz^*, \jmath)) \\ &< \frac{1}{\vartheta(Tz, Tz^*, \jmath)} - \frac{1}{\vartheta(Tgz, Tgz^*, \jmath)} \\ &\leq \frac{1}{\vartheta(Tz, Tz^*, \jmath)} - \frac{1}{\vartheta(Tz, Tz^*, \jmath)} = 0. \end{aligned}$$

This is a contradiction, and hence $Tz = Tz^*$ which means that $z = z^*$. □

Corollary 3.2 *Let $\mathcal{U}$ and $\mathcal{V}$ be non-empty subsets of a complete FMS $(\mathcal{L}, \vartheta, *)$ such that $T\mathcal{U}_0$ is non-empty and closed. Suppose also that a mapping $T : \mathcal{U} \longrightarrow \mathcal{V}$ satisfies the followings:*

(i) *T is an $\mathcal{FZ}$-proximal contraction of type (II),*

(ii) *T is injective on $\mathcal{U}_0$,*

(iii) *$T(\mathcal{U}_0) \subseteq \mathcal{V}_0$.*

Then there exists a unique point $x \in \mathcal{U}$ such that $\vartheta(x, Tx, \jmath) = \vartheta(\mathcal{U}, \mathcal{V}, \jmath)$ for all $\jmath > 0$. Moreover, for each $x_0 \in \mathcal{U}_0$ there exists a sequence $\{x_n\} \subseteq \mathcal{U}$ such that $\vartheta(x_{n+1}, Tx_n, \jmath) = \vartheta(\mathcal{U}, \mathcal{V}, \jmath)$ for every $n \in \mathbb{N}$ and $x_n \to x$.

Proof: By taking g as the identity mapping on $\mathcal{U}$ in Theorem 3.2 we get the desired result. □

Corollary 3.3 *Let $\mathcal{U}$ and $\mathcal{V}$ be non-empty subsets of a complete FMS $(\mathcal{L}, \vartheta, *)$ such that $T(\mathcal{U}_0)$ is non-empty and closed. Suppose also that a mapping $T : \mathcal{U} \longrightarrow \mathcal{V}$ satisfies the followings:*

$$\begin{cases} \vartheta(u, Tx, \jmath) = \vartheta(\mathcal{U}, \mathcal{V}, \jmath) \\ \vartheta(v, Ty, \jmath) = \vartheta(\mathcal{U}, \mathcal{V}, \jmath) \end{cases} \Rightarrow \vartheta(Tu, Tv, \jmath) \geq \psi(\vartheta(Tx, Ty, \jmath)). \tag{3.24}$$

Then there exists a unique point $x \in \mathcal{U}$ such that $\vartheta(x, Tx, \jmath) = \vartheta(\mathcal{U}, \mathcal{V}, \jmath)$ for all $t > 0$ provided that T is injective on $\mathcal{U}_0$.

Example 3.1 *Let $\mathcal{L} = [0, 1] \times \mathbb{R}$, $\mathcal{U} = \{(0, x) : 0 \leq x \leq 1, x \in \mathbb{R}\}$ and $\mathcal{V} = \{(1, y) : 0 \leq y \leq 1, y \in \mathbb{R}\}$. Note that,*

$$\vartheta_d(\mathcal{U}, \mathcal{V}, \jmath) = \frac{\jmath}{\jmath + 1}, \ \mathcal{U}_0(\jmath) = \mathcal{U} \text{ and } \mathcal{V}_0(\jmath) = \mathcal{V}.$$

Define $T : \mathcal{U} \longrightarrow \mathcal{V}$ by $T(0, \eta) = (1, \frac{\eta}{4})$. Clearly, $T(\mathcal{U}_0) \subseteq \mathcal{V}_0$. Now, if $u = (0, \eta)$, $v = (0, \theta)$, $x = (0, \mu)$ and $y = (0, \nu) \in U$ satisfy

$$\vartheta(u, Tx, \jmath) = \vartheta(\mathcal{U}, \mathcal{V}, \jmath), \text{ and } \vartheta(v, Ty, \jmath) = \vartheta(\mathcal{U}, \mathcal{V}, \jmath),$$

hence, $\eta = \frac{\mu}{4}$ and $\theta = \frac{\nu}{4}$. It is straightforward to check that

$$\vartheta(Tu, Tv, \jmath) \geq \psi(\vartheta(Tx, Ty, \jmath))$$

holds with $\psi(l) = \sqrt{l}$. Then T satisfies all the conditions of Corollary 3.3. Moreover $(0, 0)$ is a BP point of T.

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