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Bounds for the Minimum Degree Eigenvalues of Graphs

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ABSTRACT: In this article, we obtain several upper bounds for the minimum degree eigenvalues of graph G.

Key Words: Minimum degree matrix, Minimum degree eigenvalues of graph.

Contents

1. Introduction

Let G be a simple graph and let its vertex set be $V(G) = \{v_1, v_2, \ldots, v_n\}$. The square matrix $A(G)$ of order n whose $(i, j)-$ entry equal to unity if the vertices v_i and v_j are adjacent and is equal to zero otherwise is called adjacency matrix of graph G. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $A(G)$, assumed in non increasing order are the eigenvalues of the graph G.
In 1978 Ivan Gutman [3] introduced Energy of graph G as

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

In [1,8], author introduced the minimum degree matrix $m(G)$ associated with a graph G and studied its spectrum. Let G be a simple graph with n vertices $v_1, v_2, \ldots, v_n$ and let d_i be the degree of v_i, $i = 1, 2, 3, \ldots, n$. Define

$$d_{ij} = \begin{cases} min\{d_i, d_j\}, & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{otherwise.} \end{cases}$$

Then the $n \times n$ matrix $m(G) = (d_{ij})$ is called the minimum degree matrix of G. The characteristic polynomial of the minimum degree matrix $m(G)$ is defined by

$$\begin{aligned} \phi(G; \mu) &= det(\mu I - m(G)) \\ &= \mu^n + c_1\mu^{n-1} + c_2\mu^{n-2} + \cdots + c_{n-1}\mu + c_n, \end{aligned}$$

where I is the unit matrix of order n. The minimum degree eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$ of the graph G are the eigenvalues of its minimum degree matrix $m(G)$. The minimum degree energy of a graph G is defined as

$$E_m(G) = \sum_{i=1}^{n} |\mu_i|.$$

Since $m(G)$ is real symmetric matrix with zero trace, these minimum degree eigenvalues are all real with sum equal to zero.
The largest eigenvalue λ_1 of the graph G is often called the Spectral radius of G. In literature there are several upper bounds for the spectral radius λ_1 (see [2,4,5,6,7,9])
In this paper we give upper bounds for minimum degree eigenvalues of G.

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2. Bounds for Minimum degree eigenvalues

We now give the explicit expression for the co-efficient c_i of $\mu^{n-i}(i = 0, 1, 2)$ in the characteristic polynomial of the minimum degree matrix $m(G)$. It is clear that $c_0 = 1$ and $c_1 =$ trace of $m(G) = 0$. We have,

$$c_2 = \sum_{i \le j \le k \le n} \begin{vmatrix} 0 & d_{kj} \\ d_{jk} & 0 \end{vmatrix}.$$

But

$$\begin{vmatrix} 0 & d_{kj} \\ d_{jk} & 0 \end{vmatrix} = \begin{cases} -(min\{d_j, d_k\})^2, & \text{if } v_j \text{ and } v_k \text{ are adjacent,} \\ 0, & \text{otherwise.} \end{cases}$$

Thus,

$$c_2 = -\sum_{i=1}^{n}(a_i + b_i)d_i^2$$

where, $a_i =$ the number of vertices in the neighborhood of v_i, whose degrees are greater than d_i and $b_i =$ the number of vertices $v_j(j > i)$ in the neighborhood of v_i, whose degrees are equal to d_i. Note that c_2 and c_2' are negative and so $-c_2 = |c_2|$, $-c_2' = |c_2'|$.

Theorem 2.1 *If $\mu_1, \mu_2, \ldots, \mu_n$ are the minimum degree eigenvalues of G, then*

$$\sum_{i=1}^{n} \mu_i^2 = 2|c_2|.$$

Proof: We have

$$\begin{aligned} \sum_{i=1}^{n} \mu_i^2 = \text{trace of } m(G)^2 &= \sum_{i=1}^{n}\left(\sum_{k=1}^{n} d_{ik}d_{ki}\right) \\ &= 2\sum_{i=1}^{n}(a_i + b_i)d_i^2 \\ &= -2c_2 \\ &= 2|c_2|. \end{aligned}$$

□

Theorem 2.2 *Let G and H be two graphs with n vertices. If $\mu_1, \mu_2, \ldots, \mu_n$ are the minimum degree eigenvalues of G and $\mu_1', \mu_2', \ldots, \mu_n'$ are the minimum degree eigenvalues of H, then*

$$\sum_{i=1}^{n} \mu_i \mu_i' \le 2\sqrt{|c_2|\,|c_2'|}.$$

Proof: By Cauchy-Schwarz inequality, we have

$$\left(\sum_{i=1}^{n} \mu_i \mu_i'\right)^2 \le \left(\sum_{i=1}^{n} \mu_i^2\right)\left(\sum_{i=1}^{n} \mu_i'^2\right).$$

On using Theorem 2.1 in the above inequality, we obtain

$$\left(\sum_{i=1}^{n} \mu_i \mu_i'\right)^2 \le 4|c_2|\,|c_2'|.$$

Hence,

$$\sum_{i=1}^{n} \mu_i \mu_i' \leq 2\sqrt{|c_2|\,|c_2'|}.$$

□

Theorem 2.3 *If G is a graph with n vertices and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ are the minimum degree eigenvalues of G, then*

$$\mu_1 \leq \frac{1}{p-1}\left\{\sqrt{2\,|c_2|\,p(p-1)} + \sum_{i=1}^{n} \mu_{n-p+i}\right\}, \qquad 2 \leq p \leq n.$$

Proof: Let $\mu_1, \mu_2, \ldots, \mu_{n-p+1}, \mu_{n-p+2}, \ldots, \mu_n, \quad 2 \leq p \leq n$ be the minimum degree eigenvalues of G. Let $H = K_p \bigcup \overline{K_{n-p}}$. The minimum degree eigenvalues of H are $(p-1)^2$, $0(n-p$ times), and $-(p-1)(p-1$ times).

Now on employing Theorem 2.2, we obtain

$$\mu_1(p-1)^2 + \mu_2(0) + \cdots + \mu_{n-p+1}(0) - \mu_{n-p+2}(p-1) - \cdots - \mu_n(p-1) \leq 2\sqrt{|c_2|\,\frac{p(p-1)^3}{2}}$$

and so

$$\mu_1(p-1)^2 - (p-1)\sum_{i=2}^{p} \mu_{n-p+i} \leq \sqrt{2\,|c_2|\,p(p-1)^3}.$$

Thus,

$$\mu_1 \leq \frac{1}{p-1}\left\{\sqrt{2\,|c_2|\,p(p-1)} + \sum_{i=2}^{p} \mu_{n-p+i}\right\}. \tag{2.1}$$

This completes the proof of the theorem. □

Remark 2.1 If we put $p = n$ in (2.1), we get

$$\mu_1 \leq \frac{1}{n-1}\left\{\sqrt{2\,|c_2|\,n(n-1)} + \sum_{i=2}^{n} \mu_i\right\}.$$

Since

$$\sum_{i=1}^{n} \mu_i = 0,$$

we have

$$\mu_1 \leq \frac{1}{n-1}\left\{\sqrt{2\,|c_2|\,n(n-1)} - \mu_1\right\}$$

and hence,

$$\mu_1 \leq \frac{1}{n}\sqrt{2\,|c_2|\,n(n-1)}.$$

Remark 2.2 Now putting $p = 2$ in (2.1), we get

$$\mu_1 - \mu_n \leq \sqrt{4\,|c_2|}. \tag{2.2}$$

Corollary 2.1 *If G is r-regular with n vertices, then*

$$\mu_n \geq r^2 - 2\sqrt{nr^3}.$$

Proof: Let G be an r-regular graph with n vertices. It is known that $|c_2| = \dfrac{nr^3}{2}$ and $\mu_1 = r^2$. Therefore from (2.2), we get

$$\mu_n \geq r^2 - \sqrt{2nr^3}.$$

□

Theorem 2.4 *Let G be a graph with n vertices and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ be the minimum degree eigenvalues, then*

$$\sum_{i=1}^{k} \mu_i \leq \sqrt{\frac{2\,|c_2|\,k(n-k)}{n}}, \qquad 1 \leq k \leq n.$$

Proof: Let G be a graph with minimum degree eigenvalues $\mu_1, \mu_2, \ldots, \mu_k, \mu_{k+1}, \ldots, \mu_n$. Let H be a graph with n vertices and k components each is complete graph K_p i.e.,$H = \bigcup_k K_p$. The minimum degree eigenvalues of H are $(p-1)^2$ (k times) and $-(p-1)[(p-1)k$ times] and the number of vertices and edges of H are $n = pk$ and $\dfrac{kp(p-1)}{2}$ respectively. Therefore from theorem 2.2, we obtain

$$(p-1)\left\{(p-1)\mu_1 + (p-1)\mu_2 + \cdots + (p-1)\mu_k - [\mu_{k+1} + \cdots + \mu_n]\right\} \leq 2\sqrt{|c_2|\,\frac{kp(p-1)^3}{2}}.$$

i.e.,

$$p\sum_{i=1}^{k} \mu_i - \sum_{i=1}^{n} \mu_i \leq \sqrt{2\,|c_2|\,kp(p-1)}.$$

Since

$$\sum_{i=1}^{n} \mu_i = 0 \text{ and } n = pk,$$

we deduce that,

$$\sum_{i=1}^{k} \mu_i \leq \sqrt{\frac{2\,|c_2|\,k(n-k)}{n}}. \tag{2.3}$$

□

Corollary 2.2 *If G is r-regular graph with n vertices,then*

$$\mu_2 \leq r\sqrt{2r(n-2)} - r^2.$$

Proof: Putting $k = 2$ in equation (2.3), we see that

$$\mu_1 + \mu_2 \leq 2\sqrt{\frac{|c_2|\,(n-2)}{n}}.$$

Since G is r-regular, we have $|c_2| = \dfrac{nr^3}{2}$ and $\mu_1 = r^2$.
Thus,

$$\mu_2 \leq r\sqrt{2r(n-2)} - r^2.$$

□

Theorem 2.5 *Let G be a graph with n vertices and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ be the minimum degree eigenvalues, then*

$$\sum_{i=1}^{k} [\mu_i - \mu_{n-k+i}] \leq 2\sqrt{|c_2|\,k}, \qquad 1 \leq k \leq \left[\frac{n}{2}\right].$$

Proof: Let G be a graph with minimum degree eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k \geq \mu_{k+1} \geq \cdots \geq \mu_{n-k} \geq \mu_{n-k+1} \geq \cdots \geq \mu_n$. Let H be a graph with n vertices and k components each is complete bipartite graph $K_{p,q}$ i.e., $H = \bigcup_k K_{p,q}$.

The minimum degree eigenvalues of H are $p\sqrt{pq}$ [k times], $0[(n-2k)$ times] and $-p\sqrt{pq}$ [k times] and the number of vertices and edges of H are $n = k(p+q)$ and kpq respectively.
On employing Theorem 2.2, we get

$$p\sqrt{pq}\sum_{i=1}^{k}\mu_i - p\sqrt{pq}\sum_{i=1}^{k}\mu_{n-k+i} \leq 2\sqrt{|c_2|\,(kp^3q)}.$$

Thus,

$$\sum_{i=1}^{k}[\mu_i - \mu_{n-k+i}] \leq 2\sqrt{|c_2|\,k}. \tag{2.4}$$

□

Corollary 2.3 *If G is r-regular bipartite graph with $n \geq 6$ vertices, then*

$$\mu_2 \leq r\sqrt{nr} - r^2.$$

Proof: Putting $k = 2$ in (2.4), we get

$$\mu_1 + \mu_2 - \mu_{n-1} - \mu_n \leq 2\sqrt{2\,|c_2|}.$$

Since G is bipartite, we have

$$\mu_1 = -\mu_n, \quad \mu_2 = -\mu_{n-1}$$

and

$$\mu_1 + \mu_2 \leq \sqrt{2\,|c_2|}.$$

Since,

$$|c_2| = \frac{nr^3}{2} \text{ and } \mu_1 = r^2,$$

we have,

$$\mu_2 \leq r\sqrt{nr} - r^2.$$

□

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