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Second Order Discrete Boundary Value Problem With the $(p_1(k); p_2(k))$-Laplacian

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ABSTRACT: In this paper we investigate existence and non-existence of solutions for a Dirichlet boundary value problem involving the $(p_1(k), p_2(k))$-Laplacian operator when variational methods are applied to obtain the results.

Key Words: Anisotropic problem, Mountain pass Lemma, Variational methods.

Contents

1. Introduction

Let $T \geq 2$ be a positive integer and $[1,T]_{\mathbb{N}}$ be the discrete interval given by $[1,T]_{\mathbb{N}} := \{1,2,\ldots,T\}$. We consider the discrete anisotropic problem with the Dirichlet type boundary condition as follows:

$$\begin{cases} -\Delta\left(\sum_{i=1}^{2}(\mid \Delta u(k-1) \mid^{p_i(k-1)-2} \Delta u(k-1)\right) = f(k,u(k)) \ , \quad k \in [1,T]_{\mathbb{N}}, \\ \\ u(0) = u(T+1) = 0, \end{cases} \tag{1.1}$$

where Δ denotes the forward difference operator defined by $\Delta u(k) = u(k+1) - u(k)$.
$f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, i.e., for any fixed $k \in [0,T]_{\mathbb{N}}$, the function $f(k,.)$ is continuous and $p_1, p_2 : [0,T+1]_{\mathbb{N}} \to [2;+\infty)$ are given functions.

In the recent mathematical leterature a great deal of work has been devoted to the study of discrete boundary value problems because it was an interesting topic and it has been a very active area of research recently.

Problem (1.1) or similar may be seen as discretization of mathematical models arising in the study of elastic mechanics [24], electrorheological fluids [16], or image restoration [6]. Variational continuous anisotropic problems have been started by Fan and Zhang in [8] and later considered by many methods and authors (see [10]).

However, to the best of our knowledge, discrete problems like (1.1) involving anisotropic exponents have been discussed for the first time by Mihàilescu, Ràdulescu and Tersian [15] and for the second time by Kone and Ouaro [11], where known tools from the critical point theory are applied in order to get the existence of solutions. There are some related papers in the area of discrete problems. Paper [4] treats the discrete p-Laplacian problem and intervals for a nonlinear parameter are derived for which the existence and multiplicity are obtained. Let us also mention, far from being exhaustive, the following recent papers on discrete boundary value problems investigated via variational techniques and critical point theory [1], [5], [13], [18,24] and references therein.

In the present paper we are inspired by the results in [3] where authors are studying existence and muliplicity of a continuous problem by means of critical point theorems with Cerami condition and the theory of the variable exponent Sobolev spaces, by the way, we are trying to prove some of this results in discrete case, of course with necessary modifications.

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The rest of this article is arranged as folows, in section 2, we introduce some basic properties of the investigated space of solutions and provide several inequalities useful in our approach. After variational framework in section 3, we state and prove the main results.

Put

$$p_i^+ = \max_{k=1,..,T} p_i(k) \ , \quad p_i^- = \min_{k=1,..,T} p_i(k), \ \text{where } i = 1,2.$$
$$p_M^+ = \max_{i=1,2.} p_i^+ \ , \qquad p_m^- = \min_{i=1,2.} p_i^- .$$

Throughout the paper, we introduce the following assumptions:

(H_0) $f \in C\left([0,T]_{\mathbb{N}} \times \mathbb{R}; \mathbb{R}\right)$

(H_1) There exist $c > 0$ *and* $q(k) > p_M^+$ for all $k \in [0,T]_{\mathbb{N}}$ such that

$$\mid f(k,x) \mid \leq c(1+ \mid x \mid^{q(k)-1}) \text{ for all } (k,x) \in [0,T]_{\mathbb{N}} \times \mathbb{R}.$$

(H_2) $\lim\limits_{x\to 0} \dfrac{f(k,x)}{\mid x \mid^{p_M^+ -1}} = 0,$ uniformly for all $k \in [0,T]_{\mathbb{N}}$.

(H_3) There exist constants $\mu > p_M^+$, C_1 and C_2 such that

$$F(k,x) \geq C_1 \mid x \mid^{\mu} -C_2, \text{ for all } (k,x) \in [0,T]_{\mathbb{N}} \times \mathbb{R}.$$

(H_4) $f(k,-t) = -f(k,t),$ for all $(k,t) \in [0,T]_{\mathbb{N}} \times \mathbb{R}$.

(H_5) $\lim\limits_{|x|\to\infty} \dfrac{f(k,x)x}{\mid x \mid^{p_M^+}} = +\infty,$ uniformly for all $k \in [0,T]_{\mathbb{N}}$.

2. Preliminaries

Solutions to (1.1) will be investigated in a space E with

$$E = \Big\{ u : [0,T+1]_{\mathbb{N}} \to \mathbb{R} \mid \ u(0) = u(T+1) = 0 \Big\},$$

which is a T-dimensional Hilbert space, with the inner product

$$< u,v >= \sum_{k=0}^{T} \Delta u(k-1)\Delta v(k-1),$$

the associated norm is defined by

$$\parallel u \parallel = \left(\sum_{k=0}^{T} \mid \Delta u(k-1) \mid^2 \right)^{\frac{1}{2}} .$$

Also, it is useful to introduce other norms on E,

$$\mid u \mid_m = \left(\sum_{k=1}^{T+1} \mid u(k) \mid^m \right)^{\frac{1}{m}}, \ \forall u \in E \text{ and } m \geq 2. \tag{2.1}$$

It can be verified that (see [5])

$$T^{\frac{2-m}{2m}} \mid u \mid_2 \leq \mid u \mid_m \leq T^{\frac{1}{m}} \mid u \mid_2, \quad \forall u \in E \text{ and } m \geq 2. \tag{2.2}$$

Morever, we introduce the Luxemburg norm on E, defined by

$$\| u \|_{p(.)} = \inf\Big\{\lambda > 0; \sum_{k=0}^{T+1} \mid \frac{\Delta u(k-1)}{\lambda} \mid^{p(k-1)} \leq 1\Big\}. \tag{2.3}$$

All norms on E are equivalent because it is a finit dimensional space.

Now we recall some auxiliary inequalities which we use later on (see [9]).

Proposition 2.1. *For every $u \in E$, we have:*

(A.1)- $\displaystyle\sum_{k=1}^{T+1} \mid \Delta u(k-1) \mid^{p(k-1)} \geq T^{\frac{p^+-2}{2}} \| u \|^{p^+}$, *with* $\| u \| < 1$.

(A.2)- $\displaystyle\sum_{k=1}^{T+1} \mid \Delta u(k-1) \mid^{p(k-1)} \leq 2^m \sum_{k=1}^{T} \mid u(k) \mid^m$, $\forall m \geq 2$.

(A.3)- $\displaystyle\max_{k\in[1,T]_{\mathbb{N}}} \mid u(k) \mid \leq (1+T)^{\frac{1}{q}} \left(\sum_{k=1}^{T+1} \mid \Delta u(k-1) \mid^p\right)^{\frac{1}{p}}$, $\forall\ p,q > 1$ *such that* $\frac{1}{P} + \frac{1}{q} = 1$.

(A.4)- $\displaystyle\sum_{k=1}^{T+1} \mid u(k) \mid^m \leq T(T+1)^{m-1} \sum_{k=1}^{T+1} \mid \Delta u(k-1) \mid^m$, $\forall m \geq 2$.

(A.5)- $\displaystyle\sum_{k=1}^{T+1} \mid \Delta u(k-1) \mid^m \leq (T+1) \| u \|^m$, $\forall m \geq 2$.

(A.6)- $\displaystyle(T+1)^{\frac{2-m}{2}} \| u \|^m \leq \sum_{k=1}^{T+1} \mid \Delta u(k-1) \mid^m$, $\forall m \geq 2$.

(A.7)- $\displaystyle\sum_{k=1}^{T+1} \mid \Delta u(k-1) \mid^{p(k-1)} \geq T^{\frac{2-p^-}{2}} \| u \|^{p^-} - (T+1)$, *with* $\| u \| > 1$.

(A.8)- $\displaystyle\sum_{k=1}^{T+1} \mid \Delta u(k-1) \mid^{p(k-1)} \leq (T+1) \| u \|^{p^+} + (T+1)$.

Proposition 2.2. *(see [7]) Set* $\rho(u) = \displaystyle\sum_{k=1}^{T+1} \mid \Delta(u(k-1) \mid^{p(k-1)}$, *then for all* $u \in E$ *and* $(u_k) \subset E$, *we have:*

(1) $\| u \| < 1$ *(respectively* $= 1,\ > 1$*) if and only if* $\rho(u) < 1$ *(respectively* $= 1,\ > 1$*);*

(2) for $u \neq 0$, $\| u \| = \lambda$ *if and only if* $\rho(\frac{u}{\lambda}) = 1$*;*

(3) if $\| u \| > 1$, *then* $\| u \|^{p^-} \leq \rho(u) \leq \| u \|^{p^+}$*;*

(4) if $\| u \| < 1$, *then* $\| u \|^{p^+} \leq \rho(u) \leq \| u \|^{p^-}$*;*

(5) $\| u_k \| \to 0$ *(resp* $\to \infty$*) if and only if* $\rho(u_k) \to 0$ *(resp* $\to \infty$*) .*

Definition 2.3. *Let* $(X, \| \, . \, \|)$ *be a Banach space and* $J \in C^1(X, \mathbb{R})$, *we say that* J *satisfies the Palais-Smale condition (we denote* (PS) *condition), if any sequence* $(u_n) \subset X$ *such that* $\{J(u_n)\}$ *bounded and* $J'(u_n) \longrightarrow 0$, *the sequence* (u_n) *has a convergent subsequence.*

Proposition 2.4. *(Mountain Pass Lemma [2]). Let $(X, \|.\|)$ be a Banach space and $J \in C^1(X, \mathbb{R})$ satisfies (PS) condition with*

(1) $J(0) = 0$;

(2) *there exist $\rho, \alpha > 0$ such that $J(u) \geq \alpha$ for all $u \in E$ with $\|u\| = \rho$;*

(3) *there exists $u_1 \in E$ with $\|u_1\| > \rho$ such that $J(u_1) < \alpha$.*

Then J possesses a critical value $c \geq \alpha$ with

$$c = \inf_{g\in\Gamma} \max_{t\in[0,1]} J(g(t)),$$

where

$$\Gamma := \Big\{ g \in C\left([0,1], X\right) \mid \; g(0) = 0, \; g(1) = u_1 \Big\}.$$

We also introduce the Fountain Theorem which is a variant of [20], [25] .

Proposition 2.5. *Let X be a reflexive and separable Banach space.*
Then, from [22] there are $\{e_i\} \subset X$ and $\{e_i^*\} \subset X^*$ such that

$$X = \overline{< e_i \;,\; i \in \mathbb{N}^* >}, \;\; X^* = \overline{< e_i^* \;,\; i \in \mathbb{N}^* >}, \; < e_i, e_i^* >= \delta_{ij} \;,$$

where δ_{ij} denotes the Kroneker symbol. For $k \in \mathbb{N}^* >$, put

$$X_k = \mathbb{R}e_k, \;\; Y_k = \oplus_{i=1}^{k} X_i, \;\; Z_k = \overline{\oplus_{i=k}^{\infty} X_i}$$

Lemma 2.6. *([14]) Let $q > 1$. Define $\beta_k = \sup \Big\{ \mid u \mid_q \; \mid \; \| \; u \; \| = 1, \; u \in Z_k \Big\}$, then $\lim\limits_{k\to+\infty} \beta_k = 0$.*

Proposition 2.7. *([12]) Let $(X, \|.\|)$ be a reflexive and separable Banach space and J is an even functional and satisfies (PS) condition. For each k=1, 2, ... there exist $\rho_k > r_k > 0$ such that:*

$$(F_1) \;\; b_k = \inf \Big\{ J(u) \;,\; u \in Z_k, \;\; \| \; u \; \| = r_k \Big\} \to +\infty \;\; as \;\; k \to +\infty,$$

$$(F_2) \;\; a_k = \max \Big\{ J(u) \;,\; u \in Y_k, \;\; \| \; u \; \| = \rho_k \Big\} \leq \;\; 0 \;\; as \;\; k \to +\infty \;.$$

Then J has a sequence of critical values which tends to $+\infty$.

Variational framework

We have the following lemma.

Proposition 2.8. *Let E a finite dimensional Banach space, let $J \in C^1(E, \mathbb{R})$ an anti-coercive functional. Then J satisfies (PS) condition.*

Proof. Suppose to the contrary, i.e., suppose that J does not satisfy (PS) condition. Then there exists an unbounded sequence (u_n) in E such that $J'(u_n) \to 0$ as $n \to \infty$ and the sequence $J(u_n)$ is bounded. There exists a subsequence (u_{n_k}) such that $u_{n_k} \to +\infty$ as $k \to \infty$ (because (u_n) is unbounded) and by anti-coercivity of J we get $J(u_{n_k}) \to -\infty$, we obtain the contradiction.

The functional associated to problem (1.1) is defined by $J : E \to \mathbb{R}$

$$J(u) = \Phi(u) - \sum_{k=1}^{T} F(k, u(k)), \tag{2.4}$$

with

$$\Phi(u) = \sum_{i=1}^{2} \sum_{k=1}^{T+1} \frac{1}{p_i(k-1)} \mid \Delta u(k-1) \mid^{p_i(k-1)},$$

and

$$F(k,x)=\int_0^x f(k,s)ds, \quad \text{for all} \quad k\in[0,T]_{\mathbb{N}}.$$

Under the assumption (H_0) the functional is well defined, of class C^1 and its Gâteaux derivative is given by:

$$(J'(u),v)=\sum_{i=1}^{2}\sum_{k=1}^{T+1}|\Delta u(k-1)|^{p_i(k-1)-2}\Delta u(k-1)\Delta v(k-1) \;-\sum_{k=1}^{T} f(k,u(k))v(k), \tag{2.5}$$

for all $u,v\in E$.

Lemma 2.9. *Assume that (H_0) holds, then $u\in E$ is a critical point of J if and only if u is a solution of problem (1.1).*

Proof.([17]) Let us fix u, $h\in E$. We consider a function $\Psi:\mathbb{R}\to\mathbb{R}$ defined by

$$\begin{aligned}\Psi(\epsilon)&=J(u+\epsilon h)\\ &=\sum_{i=1}^{2}\sum_{k=1}^{T+1}\frac{1}{p_i(k-1)}\mid \Delta(u(k-1)+\epsilon h(k-1)\mid^{p_i(k-1)}-\sum_{k=1}^{T}F(k;u(k)+\epsilon h(k))\end{aligned}$$

Recalling that $h(0)=h(T+1)=0$ we deduce by summation by parts that :

$$\begin{aligned}\Psi'(0)&=\sum_{i=1}^{2}\sum_{k=1}^{T+1}\mid\Delta(u(k-1)\mid^{p_i(k-1)-2}\Delta u(k-1).\Delta h(k-1)-\sum_{k=1}^{T}f(k,u(k))h(k)\\ &=\sum_{i=1}^{2}\left(\mid\Delta u(T)\mid^{p_i(T)-2}\Delta u(T)\Delta h(T)+\sum_{k=1}^{T}\mid\Delta(u(k-1)\mid^{p_i(k-1)-2}\Delta u(k-1).\Delta h(k-1)\right)-\sum_{k=1}^{T}f(k,u(k))h(k)\\ &=\sum_{i=1}^{2}\Big\{\mid\Delta u(T)\mid^{p_i(T)-2}\Delta u(T)\Delta h(T)+\Big[\mid\Delta u(k-1)\mid^{p_i(k-1)-2}\Delta u(k-1).h(k-1)\Big]_{k=1}^{k=T+1}\\ &\quad-\sum_{k=1}^{T}\Delta\Big(\mid\Delta(u(k-1)\mid^{p_i(k-1)-2}.\Delta u(k-1)\Big)h(k)\Big\}-\sum_{k=1}^{T}f(k,u(k))h(k)\\ &=\sum_{k=1}^{T}\left(-\Delta\left(\sum_{i=1}^{2}\mid\Delta(u(k-1)\mid^{p_i(k-1)-2}\Delta u(k-1)\right)-f(k,u(k))\right).h(k)\end{aligned}$$

Since h was arbitrarily fixed, we arrive to the assertion.

Throughout the sequel, the letters $c,\ \tilde{c},\ c_i,\ i=1,2,...$ denote positive constants which may vary from line to line.

3. Main results and their proofs

We state our main result as follows.

Lemma 3.1. *Assume that (H_0) and (H_3) hold. Then the fonctionnal J satisfies (PS) condition.*

In fact, by (H_3), (A.8), (A.2) and (A.6) we obtain for any $u \in E$,

$$\begin{aligned} J(u) &= \sum_{i=1}^{2}\sum_{k=1}^{T+1}\frac{1}{p_i(k-1)} \mid \Delta(u(k-1)) \mid^{p_i(k-1)} - \sum_{k=1}^{T} F(k,u(k)) \\ &\leq \sum_{i=1}^{2}\frac{1}{p_i^-}\sum_{k=1}^{T+1} \mid \Delta(u(k-1)) \mid^{p_i(k-1)} - \sum_{k=1}^{T}(C_1 \mid u(k) \mid^{\mu} - C_2) \\ &\leq \sum_{i=1}^{2}\frac{1}{p_i^-}(T+1) \parallel u \parallel^{p_i^+} + \frac{1}{p_i^-}(T+1) - C_1\sum_{k=1}^{T} \mid u(k) \mid^{\mu} + C_2 T \\ &\leq \frac{2}{p_m^-}(T+1) \parallel u \parallel^{p_M^+} - \frac{c_1}{2^{\mu}}(T+1)^{\frac{2-\mu}{\mu}} \parallel u \parallel^{\mu} + \frac{1}{p_i^-}(T+1) + C_2 T. \end{aligned}$$

Since $\mu > p_M^+$, then $J(u) \to -\infty$ as $\|u\| \longrightarrow +\infty$.
By lemma (2.8), it follows that J satisfaies (PS) condition.

Theorem 3.2. *Suppose that condition $H(0)-H(3)$ are hold, then the problem has at least one non-trivial solution.*

Proof. We shall show that the functionnal J as defined above satisfies the assumptions of a Mountain Pass Lemma wich is proved by A. Ambrosetti and H. Rabinowitz (see [2]).

From Lemma (3.1) we are proving that J holds the (PS) condition.
By (H_2), For any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\mid x \mid \leq \delta$ we have

$$\mid f(k,x) \mid \leq \varepsilon \mid x \mid^{p_M^+ - 1} \quad \forall k \in [1,T]_{\mathbb{N}}$$

for $0 < x \leq \delta$ we obtain

$$\mid F(k,x) \mid = \mid \int_0^x f(k,s)ds \mid \leq \int_0^x \mid f(k,s) \mid ds$$

$$\int_0^x \varepsilon \mid s \mid^{p_M^+ - 1} ds = \varepsilon\int_0^x s^{p_M^+ - 1} ds = [\varepsilon\frac{s^{p_M^+}}{p_M^+}]_0^x = \varepsilon\frac{x^{p_M^+}}{p_M^+} = \varepsilon\frac{\mid x \mid^{p_M^+}}{p_M^+}.$$

And for $-\delta < x \leq 0$ we observe that

$$\mid F(k,x) \mid = \mid \int_0^x f(k,s)ds \mid \leq \mid \int_x^0 -f(k,s)ds \mid$$

$$\int_x^0 \varepsilon \mid s \mid^{p_M^+ - 1} ds = \varepsilon\int_x^0 (-s)^{p_M^+ - 1} ds = [-\varepsilon\frac{(-s)^{p_M^+}}{p_M^+}]_x^0 = \varepsilon\frac{\mid x \mid^{p_M^+}}{p_M^+}.$$

We choose ε such that $0 < \varepsilon < \frac{(T+1)^{\frac{2-p_M^+}{2}}}{T(T+1)^{p_M^+}}$.

So, there exists $\delta > 0$ such that for all $\mid x \mid \leq \delta$ we have

$$\mid F(k,x) \mid \leq \varepsilon\frac{\mid x \mid^{p_M^+}}{p_M^+}, \ \forall k \in [1,T]_{\mathbb{N}}. \tag{3.1}$$

Let $u \in E$ with $\|u\| \leq 1$, then $\mid \Delta u(k-1) \mid \leq 1, \ \forall k \in [1,T]_{\mathbb{N}}$.
By (A.6) we obtain

$$\sum_{k=1}^{T+1} \mid \Delta u(k-1) \mid^{p_i(k-1)} \geq \sum_{k=1}^{T+1} \mid \Delta u(k-1) \mid^{p_M^+} \geq (T+1)^{\frac{2-p_M^+}{2}} \|u\|^{p_M^+}. \tag{3.2}$$

So,

$$\sum_{i=1}^{2}\sum_{k=1}^{T+1} \mid \Delta u(k-1) \mid^{p_i(k-1)} \geq 2(T+1)^{\frac{2-p_M^+}{2}} \parallel u \parallel^{p_M^+}$$

Put $\eta = \min\left(2\delta(T+1)^{\frac{1}{2}},\ 1\right)$.
For $u \in E$ with $\parallel u \parallel \leq \eta$, by (3.1), (3.2), (A.4) and (A.5) it follows that :

$$\begin{aligned} J(u) &= \sum_{i=1}^{2}\sum_{k=1}^{T+1} \frac{1}{p_i(k-1)} \mid \Delta(u(k-1) \mid^{p_i(k-1)} - \sum_{k=1}^{T} F(k,u(k)) \\ &\geq \frac{1}{p_M^+}\sum_{i=1}^{2}\sum_{k=1}^{T+1} \mid \Delta(u(k-1) \mid^{p_i(k-1)} - \varepsilon\frac{1}{p_M^+}\sum_{k=1}^{T} \mid u(k) \mid^{p_M^+} \\ &\geq \frac{1}{p_M^+} 2(T+1)^{\frac{2-p_M^+}{2}} \parallel u \parallel^{p_M^+} - \varepsilon\frac{1}{p_M^+} T(T+1)^{p_M^+} \parallel u \parallel^{p_M^+} \\ &= \frac{\parallel u \parallel^{p_M^+}}{p_M^+}\left(2(T+1)^{\frac{2-p_M^+}{2}} - \varepsilon T(T+1)^{p_M^+}\right). \end{aligned}$$

So, there exist positive numbers $0 < \rho < \eta$ and $\alpha = \dfrac{\rho^{p_M^+}}{p_M^+}\left(2(T+1)^{\frac{2-p_M^+}{2}} - \varepsilon T(T+1)^{p_M^+}\right)$ we obtain $J(u) \geq \alpha$ for all $u \in E$ with $\parallel u \parallel = \rho$. It is obvious that $J(0) = 0$.

Since J is anti-coercive, there exists u_1 which satisfied condition three from the proposition2.4, therefor the fuctionnal J has a critical value $c > 0$ i.e., there exists $\tilde{u} \in E$ such that $J(\tilde{u}) = c$ and $J'(\tilde{u}) = 0$. It is clear that $\tilde{u} \neq 0$, because $J(0) = 0$.
The critical value c can be caracterized by

$$c = J(\tilde{u}) = \inf_{g\in\Gamma}\max_{t\in[0,1]} J(g(t)). \tag{3.3}$$

Where

$$\Gamma = \{g \in C([0,1],E) \mid g(0) = 0,\ g(1) = u_1\}.$$

then we have shown the existence of at least one solution to problem (1.1). □

Theorem 3.3. *Assume that assumptions* (H_0), (H_1), $(H_3) - (H_5)$ *are hold, then the problem has a sequence of solutions.*

Proof. In this proof, we will use the Fountain Theorem. According to Lemma (3.1) and (H_4), J is an even functional satisfies (PS) condition.
We will prove that if k is large enough, then there exist $\rho_k > r_k > 0$ such that:

$$(F_1)\ b_k = \inf\left\{J(u)\ \mid\ u \in Z_k,\ \|u\| = r_k\right\} \longrightarrow +\infty\ as\ k \to +\infty$$
$$(F_2)\ a_k = \max\left\{J(u)\ \mid\ u \in Y_k,\ \|u\| = \rho_k\right\} \leq 0\ as\ k \to +\infty$$

For (F_1): For any $u \in Z_k$ such that $\|u\| = r_k$ is big enough to ensure that $\|u\|_{p_1(.)} \geq 1$ and $\|u\|_{p_2(.)} \geq 1$ (r_k will specified bellow). By condition (H_1) we have

$$J(u) = \sum_{i=1}^{2}\sum_{k=1}^{T+1} \frac{1}{p_i(k-1)} \mid \Delta(u(k-1) \mid^{p_i(k-1)} - \sum_{k=1}^{T} F(k,u(k)) \tag{3.4}$$

$$\geq \frac{1}{p_M^+}(\parallel u \parallel_{p_1(.)}^{p_1^-} + \parallel u \parallel_{p_2(.)}^{p_2^-}) - c_7\sum_{k=1}^{T} \mid u \mid^{q(k)} - c_8$$

$$\begin{cases} \geq \frac{\tilde{c}}{p_M^+} \parallel u \parallel^{p_m^-} - c_7 - c_9 \text{ if } \mid u \mid_q \leq 1 , \\ \geq \frac{\tilde{c}}{p_M^+} \parallel u \parallel^{p_m^-} - c_7(\beta_k \parallel u \parallel)^{q^+} - c_9 \text{ if } \mid u \mid_q \geq 1 , \end{cases} \tag{3.5}$$

$$\begin{aligned} &\geq \frac{\tilde{c}}{p_M^+} \parallel u \parallel^{p_m^-} - c_7(\beta_k \parallel u \parallel)^{q^+} - c_{10} \\ &= \tilde{c}\left(\frac{1}{p_M^+} \parallel u \parallel^{p_m^-} - c_{11}\beta_k^{q^+} \parallel u \parallel^{q^+}\right) - c_{10}. \end{aligned}$$

We choose r_k as follows

$$r_k = \left(c_{11}\beta_k^{q^+} \parallel u \parallel^{q^+}\right)^{\frac{1}{p_m^- - q^+}}.$$

Then

$$\begin{aligned} J(u) &\geq \tilde{c}\left(\frac{p_m^-}{p_M^+}\left(c_{11}\beta_k^{q^+} \parallel u \parallel^{q^+}\right)^{\frac{1}{p_m^- - q^+}} - c_{11}\beta_k^{q^+} \parallel u \parallel^{q^+}\right) - c_{10} \\ &\geq \tilde{c} r_k^{p_m^-}\left(\frac{1}{q^+}\right) - c_{10}. \end{aligned}$$

From the lemma (2.6) we know that $\beta_k \to 0$, then since $1 < p_m^- < p_M^+ < q^+$, it follows that $r_k \to 0$ *as* $k \to +\infty$, then $J(u) \to +\infty$ *as* $\parallel u \parallel \to +\infty$ with $u \in Z_k$. The assertion (F_1) is valid.

For (F_2): Let $u \in Y_k$ such that $\parallel u \parallel$ is big enough to ensure that $\parallel u \parallel_{p_1(.)} \geq 1$ and $\parallel u \parallel_{p_2(.)} \geq 1$, we have

$$\begin{aligned} \Phi(u) &= \sum_{i=1}^{2}\sum_{k=1}^{T+1} \frac{1}{p_i(k-1)} \mid \Delta(u(k-1) \mid^{p_i(k-1)} \\ &\leq \frac{1}{p_m^-}(\parallel u \parallel_{p_1(.)}^{p_1^+} + \parallel u \parallel_{p_2(.)}^{p_2^+}) \\ &\leq \frac{c_1}{p_m^-} \parallel u \parallel^{p_1^+} + \frac{c_2}{p_m^-} \parallel u \parallel^{p_2^+}) \\ &\leq \frac{\max(c_1,\ c_2)}{p_m^-} \parallel u \parallel^{p_M^+} \leq d_k \mid u \mid_{p_M^+}^{p_M^+} . \end{aligned} \tag{3.6}$$

All the norms are equivalent, so there exists a constant d_k such that

$$\parallel u \parallel \leq c_3 \mid u \mid_{p_M^+} .$$

Then

$$\Phi(u) \leq d_k \mid u \mid_{p_M^+}^{p_M^+} \ \ with\ d_k = c_3\frac{\max\left(c_1\ ,\ c_2\right)}{p_m^-}.$$

From (H_5), there exists $R_k > 0$ such that for all $\mid s \mid \geq R_k$, we have

$$F(k,s) \geq 2d_k \mid s \mid^{p_M^+} .$$

From (H_1), there exists a positive constant M_k such that

$$F(k,s) \leq M_k \ for\ all\ (k,s) \in [1,T]_{\mathbb{N}} \times [-R_k, R_k].$$

Then for all $(k,s) \in [1,T]_{\mathbb{N}} \times [-R_k, R_k]$ we have

$$F(k,s) \geq 2d_k \mid s \mid^{p_M^+} - M_k. \tag{3.7}$$

Combining (3.6) and (3.7), for all $u \in Y_k$ such that $\| u \|= \rho_k > r_k$ we have

$$J(u) = \Phi(u) - \sum_{k=1}^{T} F(k, u(k))$$

$$\leq -d_k \mid u \mid_{p_M^+}^{p_M^+} + M_k T.$$

Therefore, for ρ_k large enough ($\rho_k > r_k$) we get from the above that (F_2) is satisfied i.e.,

$$a_k = \max \left\{ \; J(u) \;\; | \;\; u \in Y_k, \; \|u\| = \rho_k \right\} \leq 0 \; as \; k \to +\infty$$

Finally we apply the Fountain Theorem to acheive the proof of Theorem 3.3. □

Theorem 3.4. *Suppose that condition (H_0) holds, if*

$$xf(k, x) < 0 \;\; \text{for all} \;\; (k, x) \in [0, T]_{\mathbb{N}} \times \mathbb{R}^*. \tag{3.8}$$

Then the problem has no nontrivial solution.

Proof. Assume that the problem (1.1) has a nonzero solution. Then J has a non trivial critical point $\tilde{u}$, by (2.5) and lemma (2.9) we have

$$0 = (J'(\tilde{u}), \tilde{u}) = \sum_{i=1}^{2} \sum_{k=1}^{T+1} |\Delta \tilde{u}(k-1)|^{p_i(k-1)} - \sum_{k=1}^{T} f\left(k, \tilde{u}(k)\right) \tilde{u}(k),$$

since the assumptions bellow we have

$$0 > \sum_{k=1}^{T} f\left(k, \tilde{u}(k)\right) \tilde{u}(k) = \sum_{i=1}^{2} \sum_{k=1}^{T+1} |\Delta \tilde{u}(k-1)|^{p_i(k-1)} \geq 0.$$

It is impossible, so the problem (1.1) has no nonzero solution. □

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