



Interval-Valued Fuzzy Extremally Disconnected Spaces

S. Al Ghour, N. Rajesh* and B. Brundha

ABSTRACT: In this paper, we have study the characterizations of submaximal spaces and extremally disconnected spaces in interval-valued fuzzy topological spaces.

Key Words: Topological spaces, IVF open set, IVF regular open set.

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1. Introduction

The concept of a fuzzy subset was introduced and studied by L. A. Zadeh [9] in the year 1965. The subsequent research activities in this area and related areas have found applications in many branches of science and engineering. C. L. Chang [3] introduced and studied fuzzy topological spaces in 1968 as a generalization of topological spaces. Many researchers like this concept and many others have contributed to the development of fuzzy topological spaces. M. B. Gorzalczany [4] introduced the concept of interval-valued fuzzy set which is a generalization of fuzzy sets. T. K. Mondal and S. K. Samantha [8] introduced the topology of interval valued fuzzy sets. Y. B. Jun et. al. [5] introduced the concepts of IVF semiopen sets, IVF preopen sets and IVF α -open (=IVF strongly semiopen [6]) sets and studied some results about them. In this paper, we have study the characterizations of submaximal spaces and extremally disconnected spaces in interval-valued fuzzy topological spaces.

2. Preliminaries

Let $D[0, 1]$ be the set of all closed subintervals of the unit interval $[0, 1]$. The elements of $D[0, 1]$ are generally denoted by capital letters M, N, \dots , and note that $M = [M^L, M^U]$, where M^L and M^U are the lower and the upper end points respectively. Especially, we denote $\mathbf{0} = [0, 0]$, $\mathbf{1} = [1, 1]$, and $\mathbf{a} = [a, a]$ for every $a \in (0, 1)$. We also note that (i) $(\forall M, N \in D[0, 1])(M = N \Leftrightarrow M^L = N^L, M^U = N^U)$. (ii) $(\forall M, N \in D[0, 1])(M \subseteq N \Leftrightarrow M^L \subseteq N^L, M^U \subseteq N^U)$. For every $M \in D[0, 1]$, the complement of M , denoted by M^c , is defined by $M^c = 1 - M = [1 - M^U, 1 - M^L]$. Let X be a nonempty set. A mapping $A : X \rightarrow D[0, 1]$ is called an interval-valued fuzzy set (briefly, an IVF set) in X . For each $x \in X$, $A(x)$ is a closed interval whose lower and upper end points are denoted by $A(x)^L$ and $A(x)^U$ respectively. For any $[a, b] \in D[0, 1]$, the IVF set whose value is the interval $[a, b]$ for all $x \in X$ is denoted by $\widetilde{[a, b]}$. In particular, for any $a \in [0, 1]$, the IVF set whose value is $\mathbf{a} = [a, a]$ for all $x \in X$ is denoted by simply \tilde{a} . For a point $p \in X$ and for $[a, b] \in D[0, 1]$ with $b > 0$, the IVF set which takes the value $[a, b]$ at p and $\mathbf{0}$ elsewhere in X is called an interval-valued fuzzy point (briefly, an IVF point) and is denoted by $[a, b]_p$. In particular, if $b = a$, then it is also denoted by a_p . Denote by $IVF(X)$ the set of all IVF sets in X . For every $\mathcal{A}, \mathcal{B} \in IVF(X)$, we define

$$\mathcal{A} = \mathcal{B} \Leftrightarrow (\forall x \in X)([\mathcal{A}(x)]^L = [\mathcal{B}(x)]^L \text{ and } [\mathcal{A}(x)]^U = [\mathcal{B}(x)]^U),$$

$$\mathcal{A} \subset \mathcal{B} \Leftrightarrow (\forall x \in X)([\mathcal{A}(x)]^L \subseteq [\mathcal{B}(x)]^L \text{ and } [\mathcal{A}(x)]^U \subseteq [\mathcal{B}(x)]^U).$$

* Corresponding author.

2010 *Mathematics Subject Classification*: 54D10.

Submitted October 28, 2022. Published December 05, 2025

The complement \mathcal{A}^c of \mathcal{A} is defined by

$$[\mathcal{A}^c(x)]^L = 1 - [\mathcal{A}(x)]^U \text{ and } [\mathcal{A}^c(x)]^U = 1 - [\mathcal{A}(x)]^L$$

for all $x \in X$. For a family of IVF sets $\{\mathcal{A}_i : i \in \Lambda\}$, where Λ is an index set, the union $G = \bigcup_{i \in \Lambda} \mathcal{A}_i$ and the intersection $F = \bigcap_{i \in \Lambda} \mathcal{A}_i$ are defined by

$$(\forall x \in X)([G(x)]^L = \sup_{i \in \Lambda} [\mathcal{A}_i(x)]^L, [G(x)]^U = \sup_{i \in \Lambda} [\mathcal{A}_i(x)]^U),$$

$$(\forall x \in X)([F(x)]^L = \inf_{i \in \Lambda} [\mathcal{A}_i(x)]^L, [F(x)]^U = \inf_{i \in \Lambda} [\mathcal{A}_i(x)]^U),$$

respectively.

Definition 2.1 [8] *A family τ of IVF sets in X is called an interval-valued fuzzy topology (briefly, IVF topology) for X if it satisfies:*

1. $\mathbf{0}, \mathbf{1} \in \tau$,
2. $\mathcal{A}, \mathcal{B} \in \tau \Rightarrow \mathcal{A} \cap \mathcal{B} \in \tau$,
3. $\mathcal{A}_i \in \tau, i \in \Lambda \Rightarrow \bigcup_{i \in \Lambda} \mathcal{A}_i \in \tau$.

Every member of τ is called an IVF open set. An IVF set \mathcal{A} in X is called an IVF closed set if the complement of \mathcal{A} is an IVF open set, that is, $\mathcal{A}^c \in \tau$. Moreover, (X, τ) is called an interval-valued fuzzy topological space (briefly, IVF topological space).

Definition 2.2 [8] *For an IVF set \mathcal{A} in an IVF topological space (X, τ) , the IVF closure and the IVF interior of \mathcal{A} , denoted by $\text{Cl}(\mathcal{A})$, $\text{Int}(\mathcal{A})$ respectively, are defined as $\text{Cl}(\mathcal{A}) = \cap\{\mathcal{B} \in I^X : \mathcal{B} \text{ is IVF closed and } \mathcal{A} \subset \mathcal{B}\}$, $\text{Int}(\mathcal{A}) = \cup\{\mathcal{B} \in I^X : \mathcal{B} \text{ is IVF open and } \mathcal{B} \subset \mathcal{A}\}$, respectively. Note that $\text{Int}(\mathcal{A})$ is the largest IVF open set which is contained in \mathcal{A} , and that \mathcal{A} is IVF open if and only if $\mathcal{A} = \text{Int}(\mathcal{A})$.*

Definition 2.3 [2,4,5,7] *An IVF set \mathcal{A} of an IVF topological space (X, τ) is said to be*

1. *strongly semiopen if $\mathcal{A} \subseteq \text{Int}(\text{Cl}(\text{Int}(\mathcal{A})))$,*
2. *semiopen if $\mathcal{A} \subseteq \text{Cl}(\text{Int}(\mathcal{A}))$,*
3. *preopen if $\mathcal{A} \subseteq \text{Int}(\text{Cl}(\mathcal{A}))$,*
4. *semi-preopen if $\mathcal{A} \subseteq \text{Cl}(\text{Int}(\text{Cl}(\mathcal{A})))$.*
5. *t -set if $\text{Int}(\mathcal{A}) = \text{Int}(\text{Cl}(\mathcal{A}))$.*
6. *semiregular set if \mathcal{A} is an IVF t -set and IVF semiopen.*
7. *AB -set if $\mathcal{A} = \mathcal{B} \cap \mathcal{C}$, where $\mathcal{B} \in \tau$ and \mathcal{C} is γ -semiregular.*
8. *dense if $\text{Cl}(\mathcal{A}) = X$.*

The complement of an IVF strongly semiopen (resp. semiopen, preopen, semi-preopen) set is called an IVF strongly semiclosed (resp. semiclosed, preclosed, semi-preclosed) set. The family of all IVF regular open (resp. IVF preopen, IVF semiopen, IVF β -open) sets of (X, τ) is denoted by $RO(X)$ (resp. $PO(X)$, $SO(X)$, $\beta O(X)$).

3. On interval-valued fuzzy submaximal spaces

Definition 3.1 An IVF topological space (X, τ) is said to be IVF submaximal if every IVF dense subset of X is IVF open.

Lemma 3.1 If (X, τ) is IVF submaximal, then $\tau = PO(X)$.

Proof: Clearly $\tau \subset PO(X)$. Now, $\mathcal{A} \in PO(X)$, then $\mathcal{A} = \mathcal{B} \cap \mathcal{C}$ for some $\mathcal{B} \in \tau$ and dense set \mathcal{C} of X . Therefore, if (X, τ) is IVF submaximal, $\mathcal{C} \in \tau$, then $\mathcal{A} \in \tau$. \square

Theorem 3.1 For an IVF topological space (X, τ) , the following are equivalent:

1. X is IVF submaximal.
2. Every IVF preopen set is IVF open.
3. Every IVF preopen set is IVF semiopen and every IVF strongly semiopen set is IVF open.

Proof: (1) \Rightarrow (2): It follows from Lemma 3.1.

(2) \Rightarrow (3): Suppose that every IVF preopen set is IVF open. Then every IVF preopen set is IVF semiopen.

(3) \Rightarrow (1): Let \mathcal{A} be a dense subset of X . Since $\text{Cl}(\mathcal{A}) = X$, then \mathcal{A} is IVF preopen. By (3), \mathcal{A} is IVF semiopen. Since a set is IVF strongly semiopen if and only if it is IVF preopen and IVF semiopen, then \mathcal{A} is IVF strongly semiopen. Thus, by (3), \mathcal{A} is IVF open; hence X is IVF submaximal. \square

Theorem 3.2 For an IVF topological space (X, τ) , the following are equivalent:

1. X is IVF submaximal.
2. Every IVF preopen set is an IVF \mathcal{AB} -set.
3. Every IVF dense set is an IVF \mathcal{AB} -set.

Proof: (1) \Rightarrow (2): Let \mathcal{A} be an IVF preopen set. Since X is IVF submaximal, by Theorem 3.1, \mathcal{A} is IVF open. Since X is both IVF open and IVF semiregular, every IVF open set is IVF \mathcal{AB} -set. It follows that \mathcal{A} is an IVF \mathcal{AB} -set.

(2) \Rightarrow (3): Let \mathcal{A} be an IVF dense set. Since every IVF dense set is IVF preopen, by (2) \mathcal{A} is an IVF \mathcal{AB} -set.

(3) \Rightarrow (1): Let \mathcal{A} be an IVF dense set. By (3), \mathcal{A} is an IVF \mathcal{AB} -set. Since every IVF dense set is IVF preopen, \mathcal{A} is IVF preopen. Then $\mathcal{A} \subset \text{Int}(\text{Cl}(\mathcal{A}))$. Furthermore, because \mathcal{A} is an IVF \mathcal{AB} -set, we have $\mathcal{A} = \mathcal{B} \cap \mathcal{C}$, where $\mathcal{B} \in \tau$ and \mathcal{C} is IVF semiregular. Then $\mathcal{A} \subset \text{Int}(\text{Cl}(\mathcal{B} \cap \mathcal{C})) = \text{Int}(\text{Cl}(\mathcal{B}) \cap \text{Cl}(\mathcal{C})) = \text{Int}(\text{Cl}(\mathcal{B})) \cap \text{Int}(\text{Cl}(\mathcal{C}))$. Since \mathcal{C} is IVF semiregular, \mathcal{C} is also IVF t -set. Then we have $\mathcal{A} \subset \text{Int}(\text{Cl}(\mathcal{B})) \cap \text{Int}(\mathcal{C})$. Also $\mathcal{A} = \mathcal{B} \cap \mathcal{A} \subset \mathcal{B} \cap (\text{Int}(\text{Cl}(\mathcal{B})) \cap \text{Int}(\mathcal{C})) = \mathcal{B} \cap (\text{Int}(\text{Cl}(\mathcal{B}))) \cap \text{Int}(\mathcal{C}) = \mathcal{B} \cap \text{Int}(\mathcal{C}) = \text{Int}(\mathcal{B} \cap \mathcal{C}) = \text{Int}(\mathcal{A})$. Hence, $\mathcal{A} \in \tau$. Therefore, X is IVF submaximal. \square

Definition 3.2 An IVF topological space (X, τ) is said to be IVF extremally disconnected if $\text{Cl}(\mathcal{A}) \in \tau$ for every $\mathcal{A} \in \tau$.

Theorem 3.3 For an IVF topological space (X, τ) , the following are equivalent:

1. X is IVF extremally disconnected.
2. $\text{Int}(\mathcal{A})$ is IVF closed for every IVF closed subset \mathcal{A} of X .
3. $\text{Cl}(\text{Int}(\mathcal{A})) \subset \text{Int}(\text{Cl}(\mathcal{A}))$ for every $\mathcal{A} \in I^X$.
4. Every IVF semiopen set is IVF preopen.

5. The IVF closure of every IVF β -open subset of X is IVF open.
6. Every IVF β -open set is IVF preopen.
7. For every $\mathcal{A} \in I^X$, \mathcal{A} is IVF strongly semiopen if and only if it is IVF semiopen.

Proof: (1) \Rightarrow (2): Let \mathcal{A} be an IVF closed set. Then $\mathbf{1} - \mathcal{A}$ is IVF open. By (1), $\text{Cl}(\mathbf{1} - \mathcal{A}) = \mathbf{1} - \text{Int}(\mathcal{A})$ is IVF open. Thus, $\text{Int}(\mathcal{A})$ is IVF closed.
 (2) \Rightarrow (3): Let $\mathcal{A} \in I^X$. Then $\mathbf{1} - \text{Int}(\mathcal{A})$ is IVF closed in X and by (2) $\text{Int}(\mathbf{1} - \text{Int}(\mathcal{A}))$ is IVF closed in X . Therefore, $\text{Cl}(\text{Int}(\mathcal{A}))$ is IVF open in X and hence, $\text{Cl}(\text{Int}(\mathcal{A})) \subset \text{Int}(\text{Cl}(\mathcal{A}))$.
 (3) \Rightarrow (4): Let \mathcal{A} be IVF semiopen. By (3), we have $\mathcal{A} \subset \text{Cl}(\mathcal{A}) \subset \text{Int}(\text{Cl}(\mathcal{A}))$. Thus, \mathcal{A} is IVF preopen.
 (4) \Rightarrow (5): Let \mathcal{A} be an IVF β -open set. Then $\text{Cl}(\mathcal{A})$ is IVF semiopen. By (4), $\text{Cl}(\mathcal{A})$ is IVF preopen. Thus, $\text{Cl}(\mathcal{A}) \subset \text{Int}(\text{Cl}(\mathcal{A}))$ and hence $\text{Cl}(\mathcal{A})$ is IVF open.
 (5) \Rightarrow (6): Let \mathcal{A} be IVF β -open. By (5), $\text{Cl}(\mathcal{A}) = \text{Int}(\text{Cl}(\mathcal{A}))$ and hence \mathcal{A} is IVF preopen.
 (6) \Rightarrow (7): Let \mathcal{A} be IVF semiopen set. Since an IVF semiopen set is IVF β -open, then by (6), it is IVF preopen. Since \mathcal{A} is IVF semiopen and IVF preopen, \mathcal{A} is IVF strongly semiopen.
 (7) \Rightarrow (1): Let \mathcal{A} be an IVF open set of X . Then $\text{Cl}(\mathcal{A})$ is IVF semiopen and by (7) $\text{Cl}(\mathcal{A})$ is IVF strongly semiopen. Therefore, $\text{Cl}(\mathcal{A}) \subset \text{Int}(\text{Cl}(\text{Int}(\text{Cl}(\mathcal{A})))) = \text{Int}(\text{Cl}(\mathcal{A}))$ and hence, $\text{Cl}(\mathcal{A}) = \text{Int}(\text{Cl}(\mathcal{A}))$. Hence $\text{Cl}(\mathcal{A})$ is IVF open and X is IVF extremally disconnected. \square

Lemma 3.2 A subset \mathcal{A} of an IVF topological space (X, τ) is IVF semiopen if and only if $\text{Cl}(\mathcal{A}) = \text{Cl}(\text{Int}(\mathcal{A}))$.

Proof: Let \mathcal{A} be an IVF semiopen set. We have $\mathcal{A} \subset \text{Cl}(\text{Int}(\mathcal{A}))$ and hence $\text{Cl}(\mathcal{A}) \subset \text{Cl}(\text{Int}(\mathcal{A}))$. Since $\text{Cl}(\text{Int}(\mathcal{A})) \subset \text{Cl}(\mathcal{A})$, $\text{Cl}(\mathcal{A}) = \text{Cl}(\text{Int}(\mathcal{A}))$. Conversely, since $\text{Cl}(\mathcal{A}) = \text{Cl}(\text{Int}(\mathcal{A}))$, $\mathcal{A} \subset \text{Cl}(\mathcal{A}) = \text{Cl}(\text{Int}(\mathcal{A}))$. Thus, \mathcal{A} is IVF semiopen. \square

Theorem 3.4 For an IVF topological space (X, τ) , the following properties are equivalent:

1. X is IVF submaximal and IVF extremally disconnected.
2. Any subset of X is IVF β -open if and only if IVF open.

Proof: (1) \Rightarrow (2): Let X be IVF submaximal and IVF extremally disconnected. By Theorem 3.3, every IVF β -open set is IVF preopen. By Theorem 3.1, every IVF preopen set is IVF open. Thus, every IVF β -open set is IVF open. The converse follows from the fact that every IVF open set is IVF β -open.
 (2) \Rightarrow (1): Suppose that any subset of X is IVF β -open if and only if it is IVF open. Since every IVF β -open set is IVF open and so IVF preopen, by Theorem 3.3, X is IVF extremally disconnected. Since every IVF preopen set is IVF open, by Theorem 3.1 X is IVF submaximal. \square

Corollary 3.1 For a IVF submaximal and IVF extremally disconnected space (X, τ) , the following properties are equivalent:

1. \mathcal{A} is IVF β -open,
2. \mathcal{A} is IVF semiopen,
3. \mathcal{A} is IVF preopen,
4. \mathcal{A} is IVF strongly semiopen,
5. \mathcal{A} is IVF open.

Proof: The proof follows from Theorem 3.4. \square

Theorem 3.5 *An IVF topological space (X, τ) is IVF extremally disconnected if and only if every IVF regular open sets coincide with IVF regular closed sets.*

Proof: Suppose \mathcal{A} is a IVF regular open subset of X . Since IVF regular open sets are IVF open, by (1), $A = \text{Cl}(\mathcal{A}) = \text{Cl}(\text{Int}(\mathcal{A}))$ and so \mathcal{A} is IVF regular closed. If \mathcal{A} is IVF regular closed, then $A = \text{Cl}(\text{Int}(\mathcal{A})) = \text{Int}(\text{Cl}(\text{Int}(\mathcal{A}))) = \text{Int}(\mathcal{A})$ so \mathcal{A} is IVF open. Also, $A = \text{Cl}(\text{Int}(\mathcal{A})) = \text{Int}(\text{Cl}(\text{Int}(\mathcal{A}))) = \text{Int}(\text{Cl}(\mathcal{A}))$. Hence \mathcal{A} is IVF regular open. Conversely, let \mathcal{A} be a IVF open subset of X . Then $\text{Int}(\text{Cl}(\mathcal{A}))$ is IVF regular open and so it is IVF regular closed. Hence $\text{Int}(\text{Cl}(\text{Int}(\text{Cl}(\mathcal{A})))) = \text{Int}(\text{Cl}(\mathcal{A}))$ which implies that $\text{Cl}(\text{Int}(\text{Cl}(\mathcal{A}))) = \text{Int}(\text{Cl}(\mathcal{A}))$. Therefore, $\text{Cl}(\mathcal{A}) = \text{Cl}(\text{Int}(\mathcal{A})) \subset \text{Cl}(\text{Int}(\text{Cl}(\mathcal{A}))) = \text{Int}(\text{Cl}(\mathcal{A}))$ and so $\text{Cl}(\mathcal{A}) = \text{Cl}(\text{Int}(\mathcal{A}))$. Hence $\text{Cl}(\mathcal{A})$ is IVF open. This shows that X is IVF extremally disconnected. \square

Theorem 3.6 *An IVF topological space (X, τ) is IVF extremally disconnected if and only if every IVF regular closed sets is IVF preopen.*

Proof: The proof follows from the fact that every IVF regular open set is IVF preopen set. Conversely, if \mathcal{A} is IVF open, then $\text{Cl}(\text{Int}(\mathcal{A}))$ is IVF regular closed and so it is IVF preopen. Therefore, $\text{Cl}(\mathcal{A}) = \text{Cl}(\text{Int}(\mathcal{A})) \subset \text{Int}(\text{Cl}(\text{Cl}(\text{Int}(\mathcal{A})))) = \text{Int}(\text{Cl}(\text{Int}(\mathcal{A}))) = \text{Int}(\text{Cl}(\mathcal{A}))$. Thus, $\text{Cl}(\mathcal{A}) = \text{Int}(\text{Cl}(\mathcal{A}))$ which implies that $\text{Cl}(\mathcal{A})$ is IVF open. Hence X is IVF extremally disconnected. \square

References

1. M. Ali, A. Kılıçman and A.Z. Khameneh, *Separation Axioms of Interval-Valued Fuzzy Soft Topology via Quasi-Neighborhood Structure*, Mathematics, 8 (2020), 178.
2. S. Al Ghour, J. Princivishvamalar and N. Rajesh, *Interval-valued fuzzy b-open sets* (under preparation).
3. C.L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl., 24 (1968), 182-190.
4. M. B. Gorzalczy, *A method of inference in approximate reasoning based on interval-valued fuzzy sets*, J. Fuzzy Math., 21 (1987), 1-17.
5. Y. B. Jun, G. C. Kang and M. A. Ozturk, *Interval-valued fuzzy semiopen, preopen and α -open mappings*, Honam Math. J., 28 (2) (2006), 241-259.
6. Y. B. Jun, J. H. Bae, S. H. Cho and C. S. Kim, *Interval-valued fuzzy strong semiopen sets* East Asian Math. J., 23 (1) (2007), 45-58.
7. Y. B. Jun, S. S. Kim and C. S. Kim, *Interval-valued fuzzy semi-preopen sets and interval-valued semi-precontinuous mappings*, Honam Math. J., 29 (2) (2007), 223-244.
8. T. K. Mondal and S. K. Samanta, *Topology of interval-valued fuzzy sets*, Indian J. Pure Appl. Math., 30(1) (1999), 23-38.
9. L. A. Zadeh, *Fuzzy sets*, Information and Control, 8 (1965), 338-353.

S. Al Ghour,
Department of Mathematics and Statistics,
Jordan University of Science and Technology,
Irbid 22110, Jordan.
E-mail address: algore@just.edu.jo

and

N. Rajesh,
Department of Mathematics
Rajah Serfoji Government College,
Thanjavur-613005
Tamilnadu, India.
E-mail address: nrajesh_topology@yahoo.co.in

and

B. Brundha,
Government Arts College for Women,
Orathanadu-614625
Tamilnadu, India.
E-mail address: `brindamithunraj@gmail.com`