Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–11.
ISSN-0037-8712
doi:10.5269/bspm.65602

# On Uniqueness Result for Meromorphic Functions Sharing Small Function Concerning Homogeneous Differential Polynomials

Veena* and Husna V.

ABSTRACT: The purpose of this paper is to study the uniqueness of meromorphic functions sharing a small function. In 2020 Molla Basie Ahmed and Santosh Linkha [14] proved the uniqueness of meromorphic functions sharing a small function of differential monomial. In this paper, we proved $f^d \equiv P[f]$ or $f^d - a \equiv c(P[f] - a)$ when homogeneous differential polynomials $f^d$ and $P[f]$ share 0 CM.

Key Words: Meromorphic Functions, Homogeneous differential polynomials, small function.

## Contents



## 1. Introduction

Throughout this paper, we use the standard notations of the Nevanlinna theory of meromorphic functions as explained in [9].
Let $f$ and $g$ be a two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for some $a \in \mathbb{C} \cup \{\infty\}$, $f$ and $g$ have the same set of $a$-points with the same multiplicities, then we say that $f$ and $g$ share the value $a$ counting multiplicities (in short, CM) and if we do not consider the multiplicities, then $f$ and $g$ are said to share the value $a$ ignoring multiplicities (in short, IM). If $a = \infty$, then the zeros of $f - a$ means the poles of $f$.
A meromorphic function $a = a(z)(\not\equiv 0, \infty)$ is called a small function with respect to $f$ provided that $T(r, a) = S(r, f)$ as $r \to \infty$, $r \notin \mathbb{E}$, where $\mathbb{E}$ is a set of positive real numbers with finite Lebesgue measure. If $a = a(z)$ is a small function, then we say that $f$ and $g$ share $a$ IM(resp. CM) according to $f - a$ and $g - a$ share 0 IM(resp. CM)
Now we give the following definitions which are used throughout the paper.

**Definition 1.1** *For a meromorphic function $f$ and for $a \in \mathbb{C} \cup \{\infty\}$, and for a positive integer $k$*
*(i) $N_{(k}(r, a; f)(\overline{N}_{(k}(r, a; f))$ denotes the counting function(resp. reduced counting function) of those a-points of $f$ whose multiplicities are not less than $k$;*
*(ii) $N_{k)}(r, a; f)(\overline{N}_{k)}(r, a; f))$ denotes the counting function(resp. reduced counting function) of those a-points of $f$ whose multiplicities are not greater than $k$;*
*(iii) $N_k(r, a; f)$ denotes the sum $\overline{N}(r, a; f) + \sum_{j=2}^{k} \overline{N}_{(j}(r, a; f)$.*
*It is clear that $N_k(r, a; f) \leq k\overline{N}(r, a; f)$.*

**Definition 1.2** *[4,5]. A function $a \equiv a(z)(\not\equiv 0, \infty)$ is called a small function of a meromorphic function $f$ if $T(r, a) = S(r, f)$.*

In 1926, Nevanlinna first showed that a non-constant meromorphic function on the complex plane $\mathbb{C}$ is uniqueness determined by the pre-images, ignoring multiplicities, of five distinct values (including infinity). The beauty of this result lies in the fact that there is no counterpart of it in the real function theory. A few years latter, he showed that when multiplicities are taken into consideration, four points

* Corresponding author.
2010 *Mathematics Subject Classification:* 30D35.
Submitted October 29, 2022. Published December 05, 2025

are enough and in that case either the two functions coincides or one is a bilinear transformation of the other one. Clearly these results initiated the study of the relation between two non-constant meromorphic functions $f$ and $g$. The study becomes more interesting if the function $g$ is related with the function $f$.

It was Brück [7], who first proved the following result by investigating the uniqueness problems of an entire function sharing a value counting multiplicities with its first derivative.

**Theorem 1.1** *[7]. Let $f$ be a nonconstant entire function. If $f$ and $f'$ share the value 1 CM, and $N(r, 0; f') = S(r, f)$, then $f - 1 = c(f' - 1)$, where $c$ is a non-zero constant.*

Later, Yang [13] proved the following result for finite ordered entire function by considering general $k$-th derivative instead of the first derivative.

**Theorem 1.2** *[13]. Let $f$ be a non-constant entire function of finite order, and let $a(\neq 0, \infty)$ be a constant. If $f$ and $f^{(k)}$ share the value $a$ CM, then $f - a = c(f^{(k)} - a)$, where $c$ is a non-zero constant, and $k(\geq 1)$ is an integer.*

Regarding the non-integral hyper order, Brück [7] proposed the following famous conjecture which is known as Brück Conjecture, which inspired a number of people to work on the topic.

**Brück Conjecture.** Let $f$ be a non-constant entire function of finite non-integral hyper order. If $f$ and $f'$ share one finite value $a$ CM, then $f' - a = c(f - a)$ for some constant $c(\neq 0)$.

There are several results on the Brück Conjecture in the literature and many researchers are devoted to solve the conjecture and they put their valuable efforts to find different aspects of it by considering the general $k-$th derivative of an entire or meromorphic function $f$ , or some polynomials expressions in $f$ and its $k-$th derivative or sometimes a differential monomial or polynomials generated by $f$ but till now the original conjecture is open.
We recall here the definition of differential monomials and polynomials.

**Definition 1.3** *[15] Let $n_{0j}$, $n_{1j}$, $n_{2j}$,...,$n_{kj}$ are non-negative integers. The expression*

$$M_j\,[f] = (f)^{n_{0j}}(f^{(1)})^{n_{1j}}(f^{(2)})^{n_{2j}}\dots(f^{(k)})^{n_{kj}} \tag{1.1}$$

*is called a differential monomial generated by $f$ of degree $d(M_j) = \sum_{i=0}^{k} n_{ij}$ and weight $\Gamma_{M_j} = \sum_{i=0}^{k}(i+1)n_{ij}$. Let $a_j \in S(f)$ and $a_j \not\equiv 0 (j = 1, 2, \dots, t)$. The sum*

$$P\,[f] = \sum_{j=1}^{t} a_j M_j\,[f] \tag{1.2}$$

*is called a differential polynomial generated by $f$ of degree $\overline{d}(P) = max\{d(M_j) : 1 \leq j \leq t\}$ and weight $\Gamma_P = max\{\Gamma_{M_j} : 1 \leq j \leq t\}$. The numbers $\underline{d}(P) = min\{d(M_j) : 1 \leq j \leq t\}$ and $k$ (the highest order of the derivative of $f$ in $P\,[f]$) are called respectively the lower degree and the order of $P\,[f]$. $P\,[f]$ is said to be homogeneous differential polynomial of degree $d$ if $\overline{d}(P) = \underline{d}(P) = d$. $P\,[f]$ is called a linear differential Polynomial generated by $f$ if $\overline{d}(P) = 1$. Otherwise, $P\,[f]$ is called non-linear differential polynomial. Also, we denote by $Q$ the quantity $Q = max_{1\leq j\leq t}\sum_{i=0}^{k} i.n_{ij}$.*

Considering the sharing of small functions, Al-Khaladi [3] proved the following result.

**Theorem 1.3** *[3]. Let $f$ be a non-constant entire function satisfying $N(r, 0; f') = S(r, f)$ and $a(\not\equiv 0, \infty)$ be a small function of $f$. If $f - a$ and $f' - a$ share the value 0 CM, then $f - a = \left(1 + \frac{c}{a}\right)(f' - a)$, where $1 + \frac{c}{a} = e^{\beta}$, $c$ is a constant and $\beta$ is an entire function.*

For higher order derivative, Al-khaladi [3] proved the following result.

**Theorem 1.4** *[3]. Let $f$ be a non-constant entire function satisfying $\overline{N}(r,0;f^{(k)}) = S(r,f)$, $k(>1)$ be an integer and let $a(\not\equiv 0,\infty)$ be a meromorphic small function of $f$. If $f-a$ and $f^{(k)}-a$ share the value 0 CM, then $f-a = \left(1+\frac{P_{k-1}}{a}\right)(f^{(k)}-a)$, where $P_{k-1}(z)$, is a polynomial of degree at most $k-1$ and $1+\frac{P_{k-1}}{a} \not\equiv 0$.*

In 2011, Al-Khaladi [3] extended Theorem 1.1 to the class of meromorphic functions and obtained the following result.

**Theorem 1.5** *[3]. Let $f$ be a non-constant meromorphic function satisfying the condition $N(r,0;f') = S(r,f)$. If $f$ and $f'$ share the value 1 CM, then $f-1 = c(f'-1)$ for some non-zero constant.*

Thus in order to replace value 1 by a small function some extra conditions are required. We refer our reader for the more details to see [1,5,6]

Chen - Wu [8] extended the result of Al-Khaladi up to a linear differential polynomial, and obtained the following result.

**Theorem 1.6** *[8]. Let $f$ be a non-constant entire function satisfying $N(r,0;f') = S(r,f)$, $a(\not\equiv 0,\infty)$ be a small function of $f$ and $L \equiv L(f) = \sum_{j=1}^{k} a_i f^{(j)}$, where $k \in \mathbb{N}$, and $a_1,\ a_2,\ a_3,\dots a_k(\not\equiv 0)$ are small entire functions of $f$. If $f-a$ and $L-a$ share 0 CM, then $f-a = \left(1+\frac{c}{a}\right)(L-a)$, where $1+\frac{c}{a} = e^{\beta}$, $c$ is a constant and $\beta$ is an entire function.*
*We now define*

$$L\left(f^{(k)}\right) = a_0 f^{(k)} + a_1 f^{(k+1)} + \ldots + a_p f^{(k+p)},$$

*where $a_0,\ a_1,\dots,a_p(\neq 0)$ are constants, and $k(\geq 1)$ and $p(\geq 0)$ are integers such that $p=0$ if $k=1$ and $0 \leq p \leq k-2$ if $k \geq 2$.*

Recently Lahiri - Pal [11] considered the problem of sharing small function of a meromorphic function and its linear differential polynomial, and obtained the following result.

**Theorem 1.7** *[11]. Let $f$ be a transcendental meromorphic function be such that $f-a$ and $L(f^{(k)})-a$ share the value 0 CM, where $a(\not\equiv 0,\infty)$ is a small function of $f$. If $N(r,0;f^{(k)}) = S(r,f)$, then*

$$f-a = \left(1+\frac{P_{k-1}}{a}\right)\left(L(f^{(k)})-a\right),$$

*where $P_{k-1}$ is a polynomial of degree $k-1$ and $1+\frac{P_{k-1}}{a} \not\equiv 0$,*

In 2020 M.B. Ahamed and Santosh Linkha consider the problem of sharing a small function $a(z)$ by certain power $f^{d_M}$ of a meromorphic function and its differential monomial $M[f]$ in conformity with Brück Conjecture and got the following result.

**Theorem 1.8** *[14]. Let $f$ be meromorphic function with $\overline{N}(r,0;f') + \overline{N}(r,\infty;f) = S(r,f)$. Suppose that $M[f]$ , as defined by (1.1), is a non-constant and $k(\geq 2)$ is a positive integer. Let $a \equiv a(z)(\neq 0,\infty)$ be a small function of $f$ such that $\overline{N}(r,\infty;a) \leq \lambda T(r,a) + S(r,a)$, where $0 < \lambda < 1 - \frac{1}{\Gamma_M}$. If $f^{d_M} - a$ and $M[f] - a$ share 0 CM, then $f^{d_M} \equiv M[f]$. Furthermore $f(z)$ assumes the form $f(z) = ce^{\mu z}$, where $c(\neq 0)$ a constant and $\mu^{Q_M} = 1$.*

They have the following corollary of Theorem 1.8.

**Corollary 1.1** *Let $f$ be meromorphic function with $\overline{N}(r,0;f') + \overline{N}(r,\infty;f) = S(r,f)$. Suppose that $f^{(k)}$ is non-constant and $k(\geq 2)$ is a positive integer. Let $a \equiv a(z)(\neq 0,\infty)$ be a small function of $f$ such that $\overline{N}(r,\infty;a) \leq \lambda T(r,a) + S(r,a)$, where $0 < \lambda < 1 - \frac{1}{k}$. If $f-a$ and $f^{(k)}-a$ share 0 CM, then $f \equiv f^{(k)}$. Furthermore $f(z)$ assumes the form $f(z) = ce^{\mu z}$, where $c$ a constant and $\mu^k = 1$.*

**Theorem 1.9** *[14]. Let $f$ be meromorphic function with $\overline{N}(r,0;f^{(2)})+\overline{N}_{(2}(r,\infty;f)=S(r,f)$. Suppose that $M[f]$ , as defined by (1.1), is a non-constant, where $n_1=0$ and $k(\geq 2)$ is a positive integer. Let $a\equiv a(z)(\not\equiv 0,\infty)$ be a small function of $f$ such that $\overline{N}(r,\infty;a)\leq \lambda T(r,a)+S(r,a)$, where $0<\lambda<1-\frac{1}{\Gamma_M}$. If $f^{d_M}-a$ and $M[f]-a$ share 0 CM, then $f^{d_M}-a\equiv c(M[f]-a)$, where $c$ a non-zero constant.*

M.B.Ahamed and Santosh Linkha posed the following open questions for the further study in this direction.
**Question 1.** Is it possible to prove the main results of this paper up to a general differential polynomial $P[f]$?
**Question 2.**What should we set with the function $f$, so that when that setting shares a small function with its differential polynomial $P[f]$ we get a certain solution of the identical relation?

In connection to the above questions, the following Theorems are the main results of this paper which extends Theorem 1.8 and Theorem 1.9 for homogeneous differential polynomials.

**Theorem 1.10** *Let $f$ be meromorphic function with $\overline{N}(r,0;f')+\overline{N}(r,\infty;f)=S(r,f)$. Suppose that $P[f]$ , as defined by (1.2), is a non-constant and $k(\geq 2)$ is a positive integer. Let $a\equiv a(z)(\not\equiv 0,\infty)$ be a small function of $f$ such that $\overline{N}(r,\infty;a)\leq \lambda T(r,a)+S(r,a)$, where $0<\lambda<1-\frac{1}{\Gamma_P}$. If $f^d-a$ and $P[f]-a$ share 0 CM, then $f^d\equiv P[f]$. Furthermore $f(z)$ assumes the form $f(z)=ce^{\mu z}$, where $c(\neq 0)$ a constant and $\mu^Q=1$.*

We have the following corollary of Theorem 1.10.

**Corollary 1.2** *Let $f$ be meromorphic function with $\overline{N}(r,0;f')+\overline{N}(r,\infty;f)=S(r,f)$. Suppose that $M[f]$, as defined by (1.1), is a non-constant and $k(\geq 2)$ is a positive integer. Let $a\equiv a(z)(\not\equiv 0,\infty)$ be a small function of $f$ such that $\overline{N}(r,\infty;a)\leq \lambda T(r,a)+S(r,a)$, where $0<\lambda<1-\frac{1}{\Gamma_M}$. If $f^{d_M}-a$ and $M[f]-a$ share 0 CM, then $f^{d_M}\equiv M[f]$. Furthermore $f(z)$ assumes the form $f(z)=ce^{\mu z}$, where $c(\neq 0)$ a constant and $\mu^{Q_M}=1$.*

**Theorem 1.11** *Let $f$ be meromorphic function with $\overline{N}(r,0;f^{(2)})+\overline{N}_{(2}(r,\infty;f)=S(r,f)$. Suppose that $P[f]$, as defined by (1.2), is a non-constant, where $n_1=0$ and $k(\geq 2)$ is a positive integer. Let $a\equiv a(z)(\not\equiv 0,\infty)$ be a small function of $f$ such that $\overline{N}(r,\infty;a)\leq \lambda T(r,a)+S(r,a)$, where $0<\lambda<1-\frac{1}{\Gamma_P}$. If $f^d-a$ and $P[f]-a$ share 0 CM, then $f^d-a\equiv c(P[f]-a)$, where $c$ a non-zero constant.*

## 2. Lemmas

In this section, we present some lemmas which will be needed in sequel.

**Lemma 2.1** *[3]. Let $k(\geq 2)$ be a positive integer, and $f$ be a non-constant meromorphic function. If $\overline{N}(r,0;f^{(k)})+\overline{N}_{(2}(r,\infty;f)=S(r,f)$, then either $N_{1)}(r,\infty;f)=S(r,f)$ or $f(z)=\frac{-(k+1)^{k+1}}{k[c(z+d(k+1)]}+P_{k-1}(z)$, where $c(\neq 0)$, $d$ are constants and $P_{k-1}(z)$ is a polynomial of degree at most $k-1$.*

**Lemma 2.2** *Let $f$ be a non-constant meromorphic function, and $k(\geq 2)$ be a positive integer. Suppose that $a(\not\equiv 0,\infty)$ is a small function of $f$ and $P[f]$, as defined in Theorem 1.11, is non-constant. If $\overline{N}(r,0;(f^d)^{(2)})+\overline{N}_{(2}(r,\infty;f)=S(r,f)$ and $f^d-a$, $P[f]-a$ share 0 CM, then $\overline{N}(r,\infty,;f)=S(r,f)$.*

**Proof:** If $f^d=\frac{-27}{2c(z+3d)}+q_1(z)$, then $a(z)$ becomes constant. Therefore, clearly $f^d-a$ and $P[f]-a$ can not share 0 CM. So from Lemma 2.1, we get $\overline{N}(r,\infty;f)=S(r,f)$. □

**Lemma 2.3** *[18]. Let $f$ be a non-constant meromorphic function and $P[f]$ be same as in definition 3. Then*

$$(i)\ T(r,P[f])\leq dT(r,f)+Q\overline{N}(r,\infty;f)+S(r,f).$$

$$\begin{aligned}(ii)\ N(r,0;P)&\leq T(r,P)-dT(r,f)+dN(r,0;f)+S(r,f)\\ &\leq Q\overline{N}(r,\infty;f)+dN(r,0;f)+S(r,f).\end{aligned}$$

**Lemma 2.4** *Let $g$ be a non-constant meromorphic function, and $a_1$, $a_2$, $a_3$ be distinct meromorphic function small functions of $g$. Then*

$$T(r,g) \leq \sum_{i=1}^{3} \overline{N}(r,0;g-a_i) + S(r,g).$$

**Lemma 2.5** *[10]. Given a transcendental meromorphic function $g$, and a constant $\Gamma > 1$. Then there exists a set $S(\Gamma)$ whose upper logarithmic density is at most*

$$\delta(\Gamma) = min\left\{(2e^{\Gamma-1})^{-1}, (1+e(\Gamma-1))exp(e(1-\Gamma))\right\}$$

*such that for every positive integer $k$,*

$$\limsup_{r\to\infty, r\in S(\Gamma)} \frac{T(r,g)}{T(r,g^{(k)})} \leq 3e\Gamma.$$

## 3. Proof of the main Results

We prove here Theorem 1.11 only, as proof of Theorem 1.10 is similar to the proof of Theorem 1.11.

**Proof:** Proof of Theorem 1.11. Here $P[f]$ is homogeneous differential polynomial of degree $d$ with $\overline{d}(P) = \underline{d}(P) = d$. If $f$ is not a transcendental, since by Lemma 2.2, we have $\overline{N}(r,\infty;f) = S(r,f)$, then $f$ must a polynomial. Let $deg(f) = n$. Thus we see that $P[f]$ would be a polynomial. If $deg(f) \geq \frac{[\Gamma_P]}{[d]} + 1$, then $deg(P[f]) = d\ deg(f) + \Gamma_P$. If $deg(f) \leq \frac{[\Gamma_P]}{[d]}$, then $deg(P[f]) = 0$, which is impossible as $P[f]$ is non-constant. Since in this case $a$ is a constant, we see that $f^d - a$ and $P[f] - a$ can not share the value 0 CM, which contradicts our assumption. Thus the function $f$ must be a transcendental meromorphic function.

We set $h = \frac{f^d - a}{P[f]-a}$. Let if possible $f$ has a pole at $z_0$ of order $q$. Then, an elementary calculation shows that $z_0$ is a zero of $h$ of multiplicity $\Gamma_P$. Again it is clear that $h$ is an entire function. Then by the hypothesis and Lemma 2.2, we have

$$\overline{N}(r,0;h) \leq \overline{N}(r,\infty;f) = S(r,f). \tag{3.1}$$

Differentiating $f^d - a = hP[f] - ha$ twice we get that

$$\left(f^d\right)^{(2)} - a^{(2)} = (hP[f])^{(2)} - (ha)^{(2)} \tag{3.2}$$

**Case 1.** We suppose that $a^{(2)} \not\equiv 0$. We set

$$W = \frac{(hP[f])^{(2)}}{h\left(f^d\right)^{(2)}} - \frac{(ha)^{(2)}}{ha^{(2)}}. \tag{3.3}$$

**Subcase 1.1.** Suppose that $W \not\equiv 0$.
Let $z_1$ be a zero of $\left(f^d\right)^{(2)} - a^{(2)}$ and $a^{(2)} \not\equiv 0, \infty$. Then from (3.2), we see that $z_1$ be a zero of $(hP[f])^{(2)} - (ha)^{(2)}$, and hence $W(z_1) = 0$
Therefore, we see that

$$\begin{aligned}
m(r,W) &\leq m\left(r, \frac{(hP[f])^{(2)}}{h\left(f^d\right)^{(2)}}\right) + m\left(r, \frac{(ha)^{(2)}}{ha^{(2)}}\right) \\
&\leq m\left(r, \frac{(hP[f])^{(2)}}{hP[f]}\right) + m\left(r, \frac{P[f]}{\left(f^d\right)^{(2)}}\right) + m\left(r, \frac{(ha)^{(2)}}{ha}\right) + m\left(r, \frac{a}{a^{(2)}}\right) \\
&= S(r,f).
\end{aligned}$$

Therefore

$$\begin{aligned}\overline{N}\left(r,0;\left(f^d\right)^{(k)}-a^{(k)}\right) &\le N(r,0;W)+S(r,f)\\ &\le T(r,W)+S(r,W)\\ &= N(r,W)+m(r,W)+S(r,W)\\ &= N(r,W)+S(r,f).\end{aligned} \tag{3.4}$$

Let $z_2$ is a zero of $f$ of multiplicity $p$, such that $a(z_2)\neq 0,\infty$ and $a^{(2)}(z_2)\neq 0$. Then $z_2$ is a pole of $h$ of multiplicity $\Gamma_P$ . Hence $z_2$ is a pole of $(hP[f])^{(2)}$ with multiplicity $(pd+\Gamma_P-\Gamma_P)+2=pd+2$. Also, $z_2$ is a pole of $\frac{(hP[f])^{(2)}}{h(f^d)}$ of multiplicity $(pd+2)-(pd+2-2)=2\le k$. Then $z_2$ is a pole of $W$ with multiplicity at most $k$. Let $z_3$ be a zero of $\left(f^d\right)^{(2)}$ such that $a(z_3)\neq 0,\infty$, $a^{(2)}(z_2)\neq 0$. If $q>\Gamma_P$ , then $z_3$ be a zero of $hP[f]$ with multiplicity $q-\Gamma_P+2$, so $z_3$ is a zero of $(hP[f])^{(2)}$ with multiplicity $(q-\Gamma_P+2)-2=q-\Gamma_P$. Hence $z_3$ be a zero of $W$ with multiplicity at most $q-(q-\Gamma_P)=\Gamma_P$. We have

$$\begin{aligned}N(r,W) &\le \Gamma_P\overline{N}(r,\infty;f)+N_{\Gamma_P}\left(r,0;\left(f^d\right)^{(2)}\right)+\overline{N}\left(r,0;\left(f^d\right)^{(2)}\right)+S(r,f)\\ &= S(r,f)\end{aligned} \tag{3.5}$$

By (3.4) and (3.5), we get $\overline{N}\left(r,0;\left(f^d\right)^{(2)}-a^{(2)}\right)=S(r,f)$ and

$$\begin{aligned}T\left(r,\left(f^d\right)^{(2)}\right) &\le \overline{N}\left(r,\infty;\left(f^d\right)^{(2)}\right)+\overline{N}\left(r,0;\left(f^d\right)^{(2)}\right)+\overline{N}\left(r,0;\left(f^d\right)^{(k)}-a^{(k)}\right)+S\left(r,f^{(2)}\right)\\ &= S(r,f).\end{aligned} \tag{3.6}$$

Let $S(\Gamma)$ be defined as in Lemma 2.5. Then by (3.6), there exists a sequence $r_n\to\infty$, $r_n\notin S(\Gamma)$ such that $\frac{T\left(r_n\left(f^d\right)^{(2)}\right)}{T(r_n,f)}\to 0$ as $n\to\infty$. This contradicts the Lemma 2.5. Thus we have $W\equiv 0$, and so from the equation (3.2) and (3.3), we get

$$\left(\left(f^d\right)^{(2)}-a^{(2)}\right)a^{(2)}=(ha)^{(2)}\left(\left(f^d\right)^{(2)}-a^{(2)}\right).$$

Since $\left(f^d\right)^{(2)}\not\equiv a^{(2)}$, we obtained that $(ha)^{(2)}=a^{(2)}$. On integration, we obtained that $ha=a+d_1z+d_0$, $d_1$, $d_0\in\mathbb{C}$ and so $h=1+\frac{d_1z+d_0}{a}$.
We again note that $h$ is an entire and the zeros of $h$ are precisely the poles of $f$ . Also we note that zeros of $h$ is of multiplicity $\Gamma_P$. Let $d_1\neq 0$. Then, we have $T(r,h)=T(r,a)+O(logr)$. Also $\overline{N}(r,1;h)=\overline{N}(r,\infty;a)+O(logr)$ and $\overline{N}(r,0;h)=\frac{1}{\Gamma_P}N(r,0;h)$. Thus by the Second Fundamental Theorem, we have

$$\begin{aligned}T(r,h) &\le \overline{N}(r,1;h)+\overline{N}(r,0;h)+\overline{N}(r,\infty;h)+S(r,h)\\ &= \overline{N}(r,\infty;a)+\frac{1}{\Gamma_P}N(r,0;h)+O(logr)+S(r,h)\\ &\le \lambda T(r,a)+\frac{1}{\Gamma_P}T(r,h)+O(logr)+S(r,h)\\ &= \left(\lambda+\frac{1}{\Gamma_P}\right)T(r,h)+O(logr)+S(r,h),\end{aligned}$$

and so we get by our assumption that $T(r,h)=O(logr)+S(r,h)$. This implies that $h-1$ is a polynomial, say $P(z)$.
If $P(z)\equiv 0$, then $h\equiv 1$, and we get the result. We suppose that $P(z)\not\equiv 0$. Then $h=1+\frac{d_1z+d_0}{a}$, implies that $a=\frac{d_1z+d_0}{P(z)}$.

We suppose that $d_1z+d_0$ is a factor of $P(z)$. Then $a=\frac{1}{Q(z)}$, where $P(z)=(d_1z+d_0)Q(z)$. This

implies that $T(r,a) = (deg(Q)logr + O(1)) = N(r,\infty;a) + O(1)$, a contradiction. So $d_1z + d_0$ is not a factor of $P(z)$. Then $T(r,a) = max\{deg(P),1\}logr + O(1)$ and $N(r,\infty;a) = (deg(P))logr + O(1)$. Therefore by the hypothesis, we obtained that $deg(P) \leq \lambda max\{deg(P),1\}$. This implies that $deg(P) = 0$, and so $a = \frac{d_1z+d_0}{d}$, where $d(\neq 0)$ a constant.
Let $d_1 = 0$. Then $h = \frac{a+d_0}{a}$. Since $h$ is entire and each zero of $h$ is of multiplicity $\Gamma_P$, we have $\overline{N}(r,0;a) = 0$ and $\overline{N}(r,0;a+d_0) \leq \frac{1}{\Gamma_P}N(r,0;a+d_0)$. Therefore if $d_0 \neq 0$, we get by the Second Fundamental Theorem,

$$\begin{aligned} T(r,a) &= \overline{N}(r,\infty;a) + \overline{N}(r,0;a) + \overline{N}(r,0;a+d_0) + S(r,f) \\ &\leq \left(\lambda + \frac{1}{\Gamma_P}\right)T(r,a) + S(r,a), \end{aligned}$$

which contradicts $0 < \lambda < 1 - \frac{1}{\Gamma_P}$. So $d_0 = 0$ and hence $h = 1$.

**Case 2.** Let $a^{(2)} \equiv 0$. Then $a(z) = a_1z + a_0$, where $a_0, a_1 \in \mathbb{C}$. Then from (3.2), we see that

$$\left(f^d\right)^{(2)} = (hP[f])^{(2)} - (ah)^{(2)},$$

i.e.,

$$\frac{1}{h} = \frac{(hP[f])^{(2)}}{h\left(f^d\right)^{(2)}} - \frac{(ah)^{(2)}}{h\left(f^d\right)^{(2)}}. \tag{3.7}$$

We set $F = \left(f^d\right)^{(2)}$, $G = \frac{(hP[f])^{(2)}}{h(f^d)^{(2)}}$ and $b = \frac{(ah)^{(2)}}{h}$. Therefore, from (3.7), we have $\frac{1}{h} = G - \frac{b}{F}$. On differentiating, we have

$$-\frac{1}{h}\frac{h'}{h} = G' - \frac{b'}{F} + \frac{b}{F}\frac{F'}{F}, \tag{3.8}$$

From (3.7) and (3.8), we have

$$\frac{A}{F} = G' + G\frac{h'}{h}, \tag{3.9}$$

where $A = b\frac{h'}{h} + b' - b\frac{F'}{F}$
We now discuss the following cases.
**Subcase 2.1.** Let $G \equiv 0$. i.e., $(hP[f])^{(2)} = 0$. On integration, we get $hP[f] = b_1(z) + b_0,\ b_0,\ b_1 \in \mathbb{C}$. Putting $h = \frac{f^d - a}{P[f]-a}$, we get

$$\left(f^d - a\right)P[f] = (P[f] - a)(b_1(z) + b_0). \tag{3.10}$$

Since $a$ is a polynomial, so from (3.10), we see that $f$ is an entire function. Therefore, $h$ is an entire function having no zero. We set $h = ce^{\alpha}$, where $c \neq 0$, $\alpha$ is an entire function.

Thus we see that $f^d = a + (b_1z + b_0) - ace^{\alpha}$ and $P[f] = c(b_1z + b_0)e^{-\alpha}$. An elementary calculation shows that $P[f] = aP(\alpha, \alpha', \ldots, \alpha^{(k)})e^{\alpha}$, where $P(\alpha, \alpha', \ldots, \alpha^{(k)})$ is a differential monomial in $\alpha, \alpha', \ldots, \alpha^{(k)}$. This shows that

$$\begin{aligned} 2T\left(r, e^{\alpha}\right) &= T\left(r, e^{2\alpha}\right) \\ &= T\left(r, \frac{c(b_1z + b_0)}{aP(\alpha, \alpha', \ldots, \alpha^{(k)})}\right) \\ &= S(r,f) \end{aligned}$$

which is not possible.
**Subcase 2.2.** Let $G \not\equiv 0$.
Now we have the following two possibilities.
**Subcase 2.2.1.** If $h$ is constant, then we get our result.
**Subcase 2.2.2.** If $h$ is non-constant. Suppose that $b = 0$ i.e., $(ah)^{(2)} = 0$. Then on integration, we have

$ah=e_1z+e_0$, where $e_1,\ e_0\in\mathbb{C}$, i.e., $h=\frac{e_1z+e_0}{a}$. Since $h$ is entire, and $a$ is a polynomial of degree 1, thus it is clear that $a$ is a factor of the polynomial $e_1z+e_0$, and hence

$$h=Q_1 \tag{3.11}$$

where $Q_1\equiv Q_1(z)$ is a polynomial of degree atmost 1. Since each pole of $f$ is a zero of $h$ of multiplicity $\Gamma_P(\geq 2)$, by (3.11) we see that $f$ must be an entire function. So, $h$ is an entire function having no zero and which by (3.11) implies that $h$ must be a constant, a contradiction. Thus we have, $b\equiv 0$.

**Subcase 2.2.3.** Suppose $A\equiv 0$.

Then from (3.9), we obtained $\frac{G\prime}{G}+\frac{h'}{h}=0$, on integration, we get $Gh=D$ such that

$$(hP[f])^{(2)}\equiv \mathbf{B}\left(f^d\right)^{(2)}, \tag{3.12}$$

where $B$ is an arbitrary constant of integration. Again since,

$$\frac{A}{b}=\frac{h'}{h}+\frac{b'}{b}-\frac{F'}{F},$$

so on integration, we get $hb=DF$, and so

$$(ah)^{(2)}=D\left(f^d\right)^{(2)}, \tag{3.13}$$

where $D$ is an arbitrary constant of integration.

Since $a$ is a polynomial, $h$ is an entire function, then from (3.13), we see that $f$ is an entire function and so $h=e^{\alpha}$ where $\alpha\equiv\alpha(z)$ is an entire function. Again integrating (3.12) twice we obtained

$$hP[f]=Bf^d+P_1, \tag{3.14}$$

where $P_1\equiv P_1(z)$ is a polynomial of degree atmost 1. Since $hP[f]=f^d-a+ah$, so from (3.14) we obtained

$$(1-B)f^d=a(1-e^{\alpha})+P_1. \tag{3.15}$$

If $B=1$, then from (3.15), we get $e^{\alpha}=1+\frac{P_1}{a}$, a contradiction. Hence, $B\neq 1$, and from (3.15) can be written as

$$f^d=\frac{ae^{\alpha}}{B-1}-\frac{a+P_1}{B-1} \tag{3.16}$$

From the definition of differential polynomial $P[f]$, we have

$$P[f]=b\ P(\alpha,\alpha',\ldots,\alpha^{(k)})e^{\alpha}, \tag{3.17}$$

where $P(\alpha,\alpha',\ldots,\alpha^{(k)})(\not\equiv 0)$ is a differential polynomial in $\alpha,\alpha',\ldots,\alpha^{(k)}$ with the coefficients as polynomials.

From (3.15) and (3.17), we have

$$P[f]=\frac{aB}{B-1}-\frac{(aB+P_1)}{B-1}e^{-a}. \tag{3.18}$$

Again from (3.17) and (3.18), we have

$$b\ P(\alpha,\alpha',\ldots,\alpha^{(k)})e^{2\alpha}=\frac{aBe^{\alpha}}{B-1}-\frac{(aB+P_1)}{B-1}, \tag{3.19}$$

which implies that

$$\begin{aligned}
2T(r,e^{\alpha}) &= T(r,e^{2\alpha})\\
&= T\left(r,\frac{aBe^{\alpha}}{(B-1)aP(\alpha,\alpha',\ldots,\alpha^{(k)})}-\frac{(aB+P_1)}{(B-1)aP(\alpha,\alpha',\ldots,\alpha^{(k)})}\right)\\
&\leq T\left(r,\frac{e^{\alpha}}{aP(\alpha,\alpha',\ldots,\alpha^{(k)})}\right)+T\left(r,\frac{aB+P_1}{aP(\alpha,\alpha',\ldots,\alpha^{(k)})}\right)\\
&= S(r,f)
\end{aligned}$$

and this is absurd.
Therefore $A \not\equiv 0$. Again since $A = b\left(\frac{h'}{h} + \frac{b'}{b} - \frac{F'}{F}\right)$, clearly we have $m(r, A) = S(r, f)$. Also we note that the poles of $A$ are coming from (i) the poles of $b = \frac{(ah)^{(2)}}{h}$, (ii) the poles of $\frac{h'}{h}$ and (iii) the poles of $\frac{F'}{F} = \frac{\left(f^d\right)^{(3)}}{(f^d)^{(2)}}$. Since $h$ is an entire function and the zeros of $h$ are precisely the poles of $f$, and each zero of $h$ is of multiplicity $\Gamma_P$, we get by the hypothesis and Lemma 2.2

$$\begin{aligned}N(r, A) &\leq (\Gamma_P + 1)\overline{N}(r, \infty; f) + \overline{N}\left(r, 0; \left(f^d\right)^{(2)}\right) + S(r, f)\\&= S(r, f).\end{aligned}$$

Therefore, we have $T(r, A) = m(r, A) + N(r, A) = S(r, f)$.
Next by (3.9), we obtained that

$$\begin{aligned}m\left(r, \frac{1}{F}\right) &\leq m\left(r, \frac{1}{A}\right) + m\left(r, G' + G\frac{h'}{h}\right)\\&\leq T(r, A) + m(r, G) + m\left(r, \frac{G'}{G} + \frac{h'}{h}\right)\\&= m(r, G) + S(r, h)\\&= m\left(r, \frac{(hP[f])^{(2)}}{hP[f]} . \frac{P[f]}{\left(f^d\right)^{(2)}}\right) + S(r, f)\\&\leq m\left(r, \frac{(hP[f])^{(2)}}{hP[f]}\right) + m\left(r, \frac{P[f]}{\left(f^d\right)^{(2)}}\right)\\&= S(r, f).\end{aligned} \tag{3.20}$$

In view of (3.1), we get that

$$\begin{aligned}T(r, f) &= N(r, b) + S(r, f)\\&= N\left(r, \frac{(ah)^{(2)}}{h}\right) + S(r, f)\\&\leq 2\overline{N}(r, 0; h) + S(r, f)\\&= S(r, f).\end{aligned} \tag{3.21}$$

Let $z_4$ be a zero of $F = \left(f^d\right)^{(2)}$ such that $a(z_4) \neq 0$. Then $z_4$ will be a zero of $(hP[f])^{(2)}$ with multiplicity at least $q - (\Gamma_P - 2) - 2 = q - \Gamma_P$. So, $z_4$ is a zero of $FG = \frac{(hP[f])^{(2)}}{h}$ with multiplicity at least $q - \Gamma_P$. Hence $z_4$ is a zero of $b = FG - \frac{F}{h}$ with multiplicity $q - \Gamma_P$.
Therefore by (3.21), we get

$$\begin{aligned}N_{(\Gamma_P+1}\left(r, 0; \left(f^d\right)^{(2)}\right) &\leq N(r, 0; b) + \Gamma_P\overline{N}_{(\Gamma_P+1}\left(r, 0; \left(f^d\right)^{(2)}\right)\\&= \Gamma_P\left(r, 0; \left(f^d\right)^{(2)}\right).\end{aligned}$$

Hence we have

$$\begin{aligned}N\left(r, \frac{1}{F}\right) &= N\left(r, 0; \left(f^d\right)^{(2)}\right)\\&= N_{\Gamma_P)}\left(r, 0; \left(f^d\right)^{(2)}\right) + \Gamma_P\overline{N}_{(\Gamma_P+1}\left(r, 0; \left(f^d\right)^{(2)}\right) + S(r, f)\\&= \Gamma_P\overline{N}_{(\Gamma_P+1}\left(r, 0; \left(f^d\right)^{(2)}\right) + S(r, f)\\&= S(r, f).\end{aligned} \tag{3.22}$$

Then from (3.20) and (3.22), and by applying the First Fundamental Theorem, we get $T\left(r, \left(f^d\right)^{(2)}\right) = S(r, f)$ which is (3.6), and likewise we get a contradiction.
This completes the proof. □

**Acknowledgements.**

The authors would like to thank the referee for his /her helpful suggestions and comments.

**References**

1. M. B. Ahamed, *Uniqueness of two differential polynomials of a meromorphic function sharing a set.* Commun. Korean Math. Soc. **33** , no. 4, 1181–1203, (2018).
2. A. H. H. Al-Khaladi, *On entire functions which share one small function CM with their first derivative.* Kodai Math. J. **27** , no. 3, 201–205, (2004).
3. A. H. H. Al-Khaladi, *Meromorphic functions that share one small function with their k-th derivative.* Analysis (Munich) **31**, no. 4, 341–354, (2011).
4. A. Banerjee and M. B. Ahamed, *Meromorphic function sharing a small function with its differential polynomial.* Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math. **54**, no. 1, 33–45, (2015).
5. A. Banerjee and M. B. Ahamed, *Uniqueness of a polynomial and a differential monomial sharing a small function.* An. Univ. Vest Timiş. Ser. Mat.-Inform. **54**, no. 1, 55–71, (2016).
6. A. Banerjee and M. B. Ahamed, *Polynomial of a meromorphic function and its k-th derivative sharing a set.* Rend. Circ. Mat. Palermo (2) **67**, no. 3, 581–598, (2018).
7. R. Brück,*On entire functions which share one value CM with their first derivative.* Results Math. **30**, no. 1-2, 21–24, (1996).
8. J.-F. Chen and G.-R. Wu, *On an entire function sharing one small function CM.* Southeast Asian Bull. Math. **34**, no. 1, 51–57, (2010).
9. W. K. Hayman, *Meromorphic functions*, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1964.
10. W. K. Hayman and J. Miles, *On the growth of a meromorphic function and its derivatives.* Complex Variables Theory Appl. **12**, no. 1-4, 245–260, (1989).
11. I. Lahiri and B. Pal, J. Contemp. Math. Anal. **52** (2017), no. 1, 54–60; translated from Izv. Nats. Akad. Nauk Armenii Mat. **52**, no. 1, 68–77, (2017).
12. H. L. Qiu, *On meromorphic functions sharing one value.* J. Nanjing Norm. Univ. Nat. Sci. Ed. **25**, no. 2, 97–99, 104, (2002).
13. L.-Z. Yang, *Solution of a differential equation and its applications.* Kodai Math. J. **22**, no. 3, 458–464, (1999).
14. M. B. Ahamed and S. Linkha, *Brück conjecture and certain solution of some differential equation.* Tamkang J. Math. **51**, no. 3, 245–259, (2020).
15. Dilip Chandra Pramanik, Jayanta Roy, *Uniqueness of p(f) and P[f] concerning weakly weighted-sharing.* General Mathematics Vol. 28, No. 1, 11–24, (2020).
16. H. P. Waghamore and Husna V., *Uniqueness results related to polynomial and differential polynomial.* Jordan J. Math. Stat. **10**, no. 1, 53–71, (2017).
17. H. P. Waghamore and V. Husna, *On uniqueness of meromorphic function that share one small function with their derivatives.* Int. Electron. J. Pure Appl. Math. **10**, no. 1, 63–75, (2016).
18. I. Lahiri and B. Pal, *Uniqueness of meromorphic functions with their homogeneous and linear differential polynomials sharing a small function.* Bull. Korean Math. Soc. **54**, no. 3, 825–838, (2017).

*Husna V.,*

*Department of Mathematics,*

*Presidency University,*

*Itgalpura, Rajanukunte, Yelahanka Bengaluru-560 064,*

*India,*

*E-mail address:husna43@gmail.com*

*and*

*Veena*$^{*}$,
*Department of Mathematics,*
*Government First Grade College, Devanahalli,Bangalore Rural-562110 ,*
*India,* $^{*}$ *Corresponding author,*
*E-mail address:* manjveena@gmail.com