Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–24.
ISSN-0037-8712
doi:10.5269/bspm.65603

Boundary value problems for hybrid differential equations with fractional order

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ABSTRACT: This paper is motivated by some papers treating the fractional hybrid differential equations involving Caputo differential operators of order $0 < \alpha < 1$. An existence theorem for this equation is proved under mixed Lipschitz and Caratheodory conditions. Some fundamental fractional differential inequalities which are used to prove the existence of extremal solutions are also established. Necessary tools are considered and the comparison principle is proved, which will be useful for more study of qualitative behavior of solutions.

Key Words: Hybrid differential equation, Caputo fractional derivative, Maximal and minimal solutions.

Contents

1. Introduction

During recent years, fractional differential equations have attracted many authors, we refer the readers to the articles [1,2,3,4,5,6,7,8,9] . The differential equations involving fractional derivatives in time, are more realistic to describe many phenomena in nature (for example, to describe the memory and hereditary properties of various materials and processes), compared with those of integer order in time, the study of such equations has become an object of extensive study during the past decades.
The quadratic perturbations of nonlinear differential equations have attracted much attention. We call them fractional hybrid differential equations. There have been many works on the theory of hybrid differential equations, (see [9,10,11,12,13,14]).

Dhage and Lakshmikantham [13] discussed the following first order hybrid differential equation:

$$\begin{cases} \frac{d}{dt}\left[\frac{x(t)}{f(t,x(t))}\right] = g(t,x(t)) \quad \text{a.e. } t \in J = [0,T] \\ x\left(t_0\right) = x_0 \in \mathbb{R} \end{cases}$$

Where$f \in C(J \times \mathbb{R}, \mathbb{R}\backslash\{0\})$ and $g \in \mathcal{C}(x \times \mathbb{R}, \mathbb{R})$.They established the existence, uniqueness results and some fundamental differential inequalities for hybrid differential equations initiating the study of

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Submitted October 29, 2022. Published December 29, 2024
2010 *Mathematics Subject Classification:* 34A08, 34B15, 26A33.

Typeset by $\mathcal{B}^{S}\mathcal{P}_{\mathcal{M}}$ style.
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theory of such systems and proved utilizing the theory of inequalities, its existence of extremal solutions and comparison results.

Zhao et al. [15] have discussed the following fractional hybrid differential equations involving Riemann-Liouville differential operators:

$$\begin{cases} D^q \left[\frac{x(t)}{f(t,x(t))}\right] = g(t,x(t)) \quad \text{a.e. } t \in J = [0,T] \\ x(0) = 0 \end{cases}$$

Where$f \in C(J \times \mathbb{R}, \mathbb{R}\backslash\{0\})$ and $g \in \mathcal{C}(J \times \mathbb{R}, \mathbb{R})$.The authors of [15] established the existence theorem for fractional hybrid differential equations and some fundamental differential inequalities, they also established the existence of extremal solutions.
Benchohra et al. [16] discussed the following boundary value problems for differential equations with fractional order:

$$\begin{cases} {}^cD^\alpha y(t) = f(t,y(t)) \quad \text{for each } t \in J = [0,T], 0 < \alpha < 1, \\ ay(0) + by(T) = c, \end{cases}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, a, b, c are real constants with $a + b \neq 0$.

Hilal and Kajouni [9] have studied boundary fractional hybrid differential equations involving Caputo differential operators of order $0 < \alpha < 1$ as follows:

$$\begin{cases} D^\alpha \left(\frac{x(t)}{f(t,x(t))}\right) = g(t,x(t)) \text{ a.e. } t \in J = [0,T] \\ a\frac{x(0)}{f(0,x(0))} + b\frac{x(T)}{f(T,x(T))} = c, \end{cases}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R}\backslash\{0\})$, $g \in \mathcal{C}(J \times \mathbb{R}, \mathbb{R})$ and a, b, and c are real constants with $a + b \neq 0$. They proved the existence result for boundary fractional hybrid differential equations under mixed Lipschitz and Caratheodory conditions. Some fundamental fractional differential inequalities are also established which are utilized to prove the existence of extremal solutions.

From the above works, we develop the theory of boundary fractional hybrid differential equations involving Caputo differential operators of order $0 < \alpha < 1$.An existence theorem for boundary fractional hybrid differential equations is proved under mixed Lipschitz and Caratheodory conditions. Some fundamental fractional differential inequalities which are utilized to prove the existence of extremal solutions are also established. Essential tools are considered and the comparison principle is proved, which will be useful for more study of qualitative behavior of solutions.

2. Boundary value problems for hybrid differential equations with fractional order

Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.
By $X = C(J, \mathbb{R})$ we denote the Banach space of all continuous functions from $J = [0,T], T > 0$ into $\mathbb{R}$ with the norm

$$\|y\| = \sup\{|y(t)|, t \in J\}$$

and we denote by $\mathcal{C}ar(J \times \mathbb{R}, \mathbb{R})$ the class of functions $g : J \times \mathbb{R} \to \mathbb{R}$ such that:

(i) the map $t \mapsto g(t,x)$ is measurable for each $x \in \mathbb{R}$, and

(ii) the map $x \mapsto g(t,x)$ is continuous for each $t \in J$.

The class $\mathcal{C}ar(J \times \mathbb{R}, \mathbb{R})$ is called the Caratheodory class of functions on $J \times \mathbb{R}$ which are Lebesgue integrable when bounded by a Lebesgue integrable function on J.
Furthermore, if $g \in \mathcal{C}ar(J \times \mathbb{R}, \mathbb{R})$ and

(iii) For all $r > 0$, there exist a function $\phi_r \in L^1(J; \mathbb{R}^+)$ such that for all $u \in \mathbb{R}$ with $|u| \leq r$,

$$\|g(t, u)\| \leq \phi_r(t) \quad a.e \quad t \in J$$

Then the mapping g is called $L^1 - Caratheodory$.

By $L^1(J; \mathbb{R})$ denote the space of Lebesgue integrable real-valued functions on J equipped with the norm $\| \cdot \|_{L^1}$ defined by:

$$\|x\|_{L^1} = \int_0^T |x(s)|ds.$$

Also, we denote by $AC(J, \mathbb{R})$ the space of functions absolutely continuous from J into $\mathbb{R}$.

2.0.1. Special functions of the fractional calculus.

Definition 2.1 *[17]*
let $x \in \mathbb{R}_*^+$, The Gamma function is defined by the integral:

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt$$

Which converges in the right half of the complex plane $\mathcal{R}_e(z) > 0$.

Definition 2.2 *[17] The Beta function is usually defined by:*

$$\beta(p, q) = \int_0^1 t^{p-1}(1-t)^{q-1} dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

Definition 2.3 *[20] The exponential function, e^x, plays a very important role in the theory of integer-order differential equation. its one-parameter generalization.The function which is now denoted by:*

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}.$$

A two-parameter function of the Mittag-Lefler type is defined by the series expansion:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad (\alpha > 0, \beta > 0).$$

2.0.2. Riemann-Liouville Fractional Integral.

Definition 2.4 *[17] The fractional integral of the function $h \in L^1([a, b], \mathbb{R}_+)$ of order $\alpha \in \mathbb{R}$ is defined by:*

$$I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds$$

where Γ is the gamma function.

2.0.3. Caputo's Fractional Derivative.

Definition 2.5 *[17] For a function h given on the interval $[a, b]$, the Caputo fractional order derivative of h, is defined by:*

$$\left({}^c D_{a^+}^\alpha h\right)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s) ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

2.0.4. Laplace transform. **Basic facts on the Laplace transform:**

- If there exist positive constants M and T such that $|f(t)| \leq Me^{\alpha}t$ for all $t > T$ (i-e the function f(t) is of exponential order α), Then the function F of the complex variable s defined by:
$$F(s) = L(f(t))(s) = \int_0^{+\infty} e^{-st} f(t)dt$$
is called the Laplace transform of the function f.
- The original f(t) can be restored from the Laplace transform F(s) with help of the inverse Laplace transform,
$$f(t) = L^{-1}\{F(s); t\} = \int_{c-i\infty}^{c+i\infty} e^{st} F(s)ds; \quad c = \mathcal{R}_e(s)$$
- The Laplace transform of the convolution of the two functions f(t) and g(t) is equal to the product of the Laplace transform of those functions,
$$L\{f(t) * g(t); s\} = F(s)G(s)$$
under the assumption that both F(s) and G(s) exist.
- The Laplace transform of the derivative of an integer order n of the function $f(t)$ is:
$$L\{f^{(n)}(t); s\} = s^n F(s) - \sum_{k=0}^{n} s^{n-k} f^{(k-1)}(0).$$

Definition 2.6 *We can write the fractional integral of order $\alpha > 0$ as a convolution of the functions $\phi(t) = t^{\alpha} - 1$ and $f(t)$ as follows:*
$$I_a^{\alpha} f(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds = t^{\alpha-1} * f(t).$$
The Laplace transform of the function t^{p-1} is:
$$\Phi(s) = L(t^{\alpha} - 1; s) = \Gamma(\alpha)s^{-\alpha}$$
Therfore, using the formula for the Laplace transform of the convolution we obtain the Laplace transform of the fractional integral
$$L\{I_a^{\alpha} f(t); s\} = s^{-\alpha} F(s).$$

Definition 2.7 *We can write the caputo derivative in the form:*
$${}_aD_c^{\alpha} f(t) = {}_aI_t^{n-\alpha} \left(\frac{d^n}{dt^n} f(t) \right) = {}_aI_t^{n-\alpha} g(t) \quad (n-1 < \alpha \leq n)$$
Where
$$\frac{d^n}{dt^n} f(t) = g(t)$$
Using the formula for the Laplace transform of the fractional integral gives:
$$L\{{}_aD_c^{\alpha} f(t); s\} = s^{-(n-\alpha)} G(s) \tag{2.1}$$
Where
$$G(s) = s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^k(0) = s^n F(s) - \sum_{k=0}^{n-1} s^k f^{n-k-1}(0) \tag{2.2}$$
Introducing (2.2) into (2.1) we arrive at the Laplace transform formula for the Caputo fractional derivative:
$$L\{{}_aD_c^{\alpha} f(t); s\} = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^k(0)$$

Lemma 2.1 *[19] Let $0<\alpha<1$ and $\lambda\in\mathbb{R}$ .For all $t\in[0,T]$, we have:*

(1) Let $0<\alpha<1$ and $K,U\in\mathbb{R}^{n\times n}$. Then for all $t\in[0,T]$ we have:

$$\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-K(t-\tau)^\alpha)U(D^\alpha x)(\tau)d\tau = Ux(t)-E_\alpha(-Kt^\alpha)Ux(0)$$

$$-K\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-K(t-s)^\alpha)Ux(s)ds$$

(2)

$$D_0^\alpha\left[\int_0^t f(t-s)g(s)ds\right](t)=\int_0^t D_0^\alpha\left[f(t)\right](s)g(t-s)ds+g(t)\lim_{t\to0^+}\left[{}_tI_{0^+}^{1-\alpha}f\right](t).$$

Lemma 2.2 *[21]*

(a) $D^\alpha_{a+}E_\alpha\left[\lambda(t-a)^\alpha\right](x)=\lambda E_\alpha\left[\lambda(x-a)^\alpha\right],\ (\mathcal{R}_e(\alpha)>0,\lambda\in\mathbb{C})$

(b) $\left(I^{\alpha'}_{a+}(t-a)^{\beta-1}E_{\mu,\beta}\left[\lambda(t-a)^\mu\right]\right)(x)=(x-a)^{\alpha'+\beta-1}E_{\mu,\alpha'+\beta}\left[\lambda(x-a)^\mu\right]$

$$\textit{with}\qquad \alpha'>0,\quad \beta>0,\quad \mu>0.$$

(c) $\int_0^z t^{\beta-1}E_{\alpha,\beta}\left(\lambda t^\alpha\right)dt=z^\beta E_{\alpha,\beta+1}\left(\lambda z^\alpha\right)$

(d) $|E_{\alpha,\beta}(z)|\le C_1\exp\left(\sigma|z|^\rho\right)$ *for each* $\sigma>1$ *and* $\rho=\dfrac{1}{\mathcal{R}_e(\alpha)}$

In this paper we consider the boundary value problems for hybrid differential equations with fractional order (BVPHDEF for short) involving Caputo differential operators of order $0<\alpha<1$.

$$\begin{cases} D^\alpha\left(\frac{x(t)}{f(t,x(t))}\right)-\lambda\left(\frac{x(t)}{f(t,x(t))}\right)=g(t,x(t)) \quad \text{a.e. } t\in J=[0,T],\\ a\frac{x(0)}{f(0,x(0))}+b\frac{x(T)}{f(T,x(T))}=c, \end{cases} \tag{2.3}$$

Where $f\in C(J\times\mathbb{R},\mathbb{R}\backslash\{0\})$ and $g:J\times\mathbb{R}\longmapsto\mathbb{R}$ is $\mathcal{L}^1$ -Caratheodory function ,$\lambda>0$, and a, b, c are real constants with $a+b\neq0$.
By a solution of BVPHDEF (2.3) we mean a function $x\in\mathcal{C}(J,\mathbb{R})$ such that:

(i) the function $t\mapsto\frac{x}{f(t,x)}$ is continuous for each $x\in\mathbb{R}$, and

(ii) x satisfies the equations in (2.3).

The theory of strict and nonstrict differential inequalities related to the ODEs and hybrid differential equations is available in the articles (see [2,13]). It is known that differential inequalities are useful for proving the existence of extremal solutions of the ODEs and hybrid differential equations defined on J.

Fractional Differential Equation:

we consider the following linear fractional differential equation with constant coefficients involving caputo fractional derivative of order α , such that: $0<\alpha\le1$, and $\lambda\in\mathbb{R}$:

$$D^\alpha x(t)-\lambda x(t)=g(t) \tag{2.4}$$

Appliying the Laplace transform of Caputo fractional operator of order α , we obtain:

$$s^\alpha X(s)-s^{\alpha-1}x(0)-\lambda X(s)=G(s),$$

we can also write it as follows:

$$X(s)=\frac{1}{s^\alpha-\lambda}G(s)+\frac{s^{\alpha-1}}{s^\alpha-\lambda}x(0)$$

as: $\mathcal{L}^{-1}\{\frac{1}{s^\alpha-\lambda}\} = t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)$ *and* $\mathcal{L}^{-1}\{\frac{s^{\alpha-1}}{s^\alpha-\lambda}\} = E_\alpha(\lambda t^\alpha)$ By convolution we obtain the integral solution x(t) as follows:

$$x(t) = x(0)E_\alpha(\lambda t^\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha)g(s)ds$$

3. Existence result

In this section, we prove the existence results for the boundary value problems for hybrid differential equations with fractional order (2.3) on the closed and bounded interval $J = [0,T]$ under mixed Lipschitz and Caratheodory conditions on the nonlinearities involved in it.

We define the multiplication in X by:

$$(xy)(t) = x(t)y(t) \quad for \quad x, y \in X.$$

Clearly, $X = C(J;\mathbb{R})$ is a Banach algebra with respect to the above norm and multiplication in it.

Lemma 3.1 *[22] In $(C(J;\mathbb{R});\leq;\|\|)$, $\|\|$ and $\leq$ are compatible in each subset partially compact of $C(J;\mathbb{R})$*

Definition 3.1 *A non-empty subset C of X is called chain or totally ordered if all elements of C are comparable.*

Definition 3.2 *A function $u \in C^1(J,\mathbb{R})$ is called a lower solution of BVPHDEF (2.3) if the function $t \longrightarrow \frac{u(t)}{f(t,u(t))}$ is continuously differentiable and satisfies*

$$\begin{cases} D^\alpha\left(\frac{x(t)}{f(t,x(t))}\right) - \lambda\left(\frac{x(t)}{f(t,x(t))}\right) \leq g(t,x(t)) & a.e.\ t \in J = [0,T], \\ a\frac{x(0)}{f(0,x(0))} + b\frac{x(T)}{f(T,x(T))} \leq c, \end{cases}$$

Similarly, a function $\upsilon \in C^1(J,\mathbb{R})$ is called an upper solution of BVPHDEF (2.3) if it satisfies the above property with inverse inequalities.

Theorem 3.1 *[22] Let $(X,\leqslant,\|\|)$ be a complete normal vector space,partially ordered, suth that the order relation $\leq$ and the norm $\|\|$ are compatible, and let $\mathcal{A}: X \mapsto X$ and $\mathcal{B}: X \mapsto X$ be two increasing operators such that:*

(a) $\mathcal{A}$ is partially bounded and $\mathcal{D}$-Lipschitzian with a $\mathcal{D}$- function $\psi_{\mathcal{A}}$,

(b) $\mathcal{B}$ is partially continuous, and uniformly partially compact.

(c) $M\psi_{\mathcal{A}} < r, r > 0$ where $M = \|\mathcal{B}(C)\| = sup\|\mathcal{B}x\|$; $C \in P_{ch}(X)$

(d) There exist an element $x_0 \in X$ such that $x_0 \geq \mathcal{A}x_0\mathcal{B}x_0$ or $x_0 \leq \mathcal{A}x_0\mathcal{B}x_0$,

Then the operator equation $\mathcal{A}x\mathcal{B}x = x$ has at least one positive solution x^* in X. and the sequence of successive iterations defined by $x_{n+1} = \mathcal{A}x_n\mathcal{B}x_n$, $n = 0, 1, ...;$ converges in a monotonic way into x^*.

We denote by $P_{ch}(X)$ the set of all subsets of X.
In order to establish the existence result we make the following assumptions:

(A_0) The function $x \mapsto \frac{x}{f(t,x)}$ is increasing in $\mathbb{R}$ almost everywhere for $t \in J$

(A_1) There exist a constant $M_f > 0$ such that $0 < f(t,x) < M_f$ for all $t \in J$ and $x \in \mathbb{R}$

(A_2) There exist a $\mathcal{D}$-function ϕ such that $0 \leq f(t,x) - f(t,y) \leq \phi(x-y)$ for all $t \in J$ and $x, y \in \mathbb{R}$ with $x \geq y$

(B_1) There exist a function $h \in L^1(J, \mathbb{R}^+)$ such that $|g(t,x)| \leq h(t)$ a.e $t \in J$ and for all $x \in \mathbb{R}$

(B_2) The function $g(t,x)$ is increasing with respect to x for all $t \in J$.

(B_3) The problem (2.3) has a lower solution $u \in \mathcal{C}^1(J, \mathbb{R})$

Lemma 3.2 *Assume that hypothesis (A_0) holds and $E_\alpha(\lambda T^\alpha) = 1$,and a, b, c are real constants with $a + b \neq 0$. Then, for any $g \in L^1(J; \mathbb{R}^+)$, the function $x \in C(J; \mathbb{R})$ is a solution of the BVPHDEF*

$$\begin{cases} D^\alpha\left(\frac{x(t)}{f(t,x(t))}\right) - \lambda\left(\frac{x(t)}{f(t,x(t))}\right) = g(t) \quad a.e.\ t \in J = [0,T], \\ a\frac{x(0)}{f(0,x(0))} + b\frac{x(T)}{f(T,x(T))} = c, \end{cases} \tag{3.1}$$

if and only if x satisfies the hybrid integral equation

$$\begin{aligned} x(t) = [f(t,x(t)]\Bigg(&\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha)g(s)ds \\ &- \frac{E_\alpha(\lambda t^\alpha)}{a+b}(b\int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(T-s)^\alpha)g(s)ds - c\Bigg) \end{aligned} \tag{3.2}$$

Proof: We consider the following differential fractional equation:

$$D^\alpha\left(\frac{x(t)}{f(t,x(t))}\right) - \lambda\left(\frac{x(t)}{f(t,x(t))}\right) = g(t)$$

By using the solution of the equation (2.4) we obtain:

$$\frac{x(t)}{f(t,x(t))} = \frac{x(0)}{f(0,x(0))}E_\alpha(\lambda t^\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha)g(s)ds$$

Then, we can write

$$\frac{x(T)}{f(T,x(T))} = \frac{x(0)}{f(0,x(0))}E_\alpha(\lambda T^\alpha) + \int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(T-s)^\alpha)g(s)ds$$

As: $E_\alpha(\lambda T^\alpha) = 1$, we get

$$a\frac{x(0)}{f(0,x(0))} + b\frac{x(T)}{f(T,x(T))} = (a+b)\frac{x(0)}{f(0,x(0))} + b\int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(T-s)^\alpha)g(s)ds$$

By using the boundary condition, we obtain

$$\frac{x(0)}{f(0,x(0))} = \frac{c}{a+b} - \frac{b}{a+b}\int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(T-s)^\alpha)g(s)ds$$

This implies that

$$\begin{aligned} \frac{x(t)}{f(t,x(t))} = &\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha)g(s)ds \\ &- \frac{E_\alpha(\lambda t^\alpha)}{a+b}(b\int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(T-s)^\alpha)g(s)ds - c) \end{aligned}$$

Then

$$x(t) = [f(t,x(t)](\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha) g(s)ds$$
$$- \frac{E_\alpha(\lambda t^\alpha)}{a+b} (b \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(T-s)^\alpha) g(s)ds - c)$$

Convesely, if we have

$$x(t) = [f(t,x(t)] \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha) g(s)ds \right.$$
$$\left. - \frac{E_\alpha(\lambda t^\alpha)}{a+b} (b \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(T-s)^\alpha) g(s)ds - c \right)$$

By Applying the Caputo fractional derivative of order α we obtain

$$D^\alpha \left(\frac{x(t)}{f(t,x(t))} \right) = D^\alpha \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha) g(s)ds \right)$$
$$- D^\alpha \left(\frac{E_\alpha(\lambda t^\alpha)}{a+b} (b \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(T-s)^\alpha) g(s)ds - c) \right)$$

We set

$$F(u) := u^{\alpha-1} E_{\alpha,\alpha}(\lambda u^\alpha)$$

Then, by using relation (2) of the lemma (2.1) we get

$$D^\alpha \left[\int_0^t F(t-s) g(s) ds) \right](t) = \int_0^t D^\alpha \left[F(t) \right](s) g(t-s) ds + g(t) \lim_{t \longrightarrow 0^+} \left[{}_t I_{0^+}^{1-\alpha} F \right](t)$$

$$= \int_0^t D^\alpha \left[s^{\alpha-1} E_{\alpha,\alpha}(\lambda s^\alpha) g(t-s) \right] ds + g(t) \lim_{t \to 0^+} \left[{}_t I_{0^+}^{1-\alpha} F \right](t)$$

$$= \int_0^t \lambda s^{\alpha-1} E_{\alpha,\alpha}(\lambda s^\alpha) g(t-s) ds + g(t) \lim_{t \to 0^+} I_{0^+}^{1-\alpha} F(t)$$

we set $t - s = u$ then

$$D^\alpha \left[\int_0^t F(t-s) g(s) ds) \right](t) = \int_0^t \lambda (t-u)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-u)^\alpha) g(u) du + g(t) \lim_{t \to 0^+} I_{0^+}^{1-\alpha} F(t)$$

Also by substituting $\alpha' = 1-\alpha$,$\mu = \beta = \alpha$, $a = 0$, and $t = x$ in relation (b) of the lemma (2.2) we obtain

$$I^{1-\alpha} t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^\alpha) = E_{\alpha,1}(\lambda t^\alpha)$$

Then

$$g(t) \lim_{t \to 0^+} I_{0^+}^{1-\alpha} F(t) = g(t) \lim_{t \to 0^+} E_{\alpha,1}(\lambda t^\alpha) = g(t)$$

Thus

$$D^\alpha \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha) g(s)ds \right) = \int_0^t \lambda (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha) g(s)ds + g(t)$$

On the other hand

$$D^\alpha \left(\frac{E_\alpha(\lambda t^\alpha)}{a+b} (b \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(T-s)^\alpha) g(s)ds - c) \right) =$$

$$\frac{b}{a+b}\int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(T-s)^\alpha)g(s)dsD^\alpha\left[E_\alpha(\lambda t^\alpha)\right]-\frac{c}{a+b}D^\alpha\left[E_\alpha(\lambda t^\alpha)\right]$$

According to the relation (a) of lemma (2.1) we obtain: for $a=0$

$$D^\alpha E_\alpha(\lambda t^\alpha)=\lambda E_\alpha(\lambda t^\alpha)$$

Then

$$D^\alpha\left(\frac{E_\alpha(\lambda t^\alpha)}{a+b}(b\int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(T-s)^\alpha)g(s)ds-c)\right)$$
$$=\frac{\lambda}{a+b}E_\alpha(\lambda t^\alpha)\left(b\int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(T-s)^\alpha)g(s)ds-c\right)$$

Finally

$$D^\alpha\left(\frac{x(t)}{f(t,x(t))}\right)=\int_0^t \lambda(t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha)g(s)ds+g(t)$$
$$-\frac{\lambda}{a+b}E_\alpha(\lambda t^\alpha)\left(b\int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(T-s)^\alpha)g(s)ds-c\right)$$

i.e

$$D^\alpha\left(\frac{x(t)}{f(t,x(t))}\right)=\lambda\left[\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha)g(s)ds\right.$$
$$\left.-\frac{E_\alpha(\lambda t^\alpha)}{a+b}\left(b\int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(T-s)^\alpha)g(s)ds-c\right)\right]+g(t)$$

i.e

$$D^\alpha\left(\frac{x(t)}{f(t,x(t))}\right)-\lambda\left(\frac{x(t)}{f(t,x(t))}\right)=g(t)$$

Again, substituting $t=0$ and $t=T$ in (3.2) we have

$$\frac{x(0)}{f(0,x(0))}=\frac{-b}{a+b}\int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(T-s)^\alpha)g(s)ds+\frac{c}{a+b}$$

$$\frac{x(T)}{f(T,x(T))}=\left(1-\frac{bE_\alpha(\lambda T^\alpha)}{a+b}\right)\int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(T-s)^\alpha)g(s)ds+\frac{E_\alpha(\lambda T^\alpha)}{a+b}c$$

Then

$$a\frac{x(0)}{f(0,x(0))}+b\frac{x(T)}{f(T,x(T))}=\left(\frac{-ab+ab+b^2-b^2E_\alpha(\lambda T^\alpha)}{a+b}\right)+\frac{a+bE_\alpha(\lambda T^\alpha)}{a+b}c$$

As

$$E_\alpha(\lambda T^\alpha)=1$$

this implies that

$$a\frac{x(0)}{f(0,x(0))}+b\frac{x(T)}{f(T,x(T))}=c$$

□

Theorem 3.2 *Assume that* $(A_0)-(A_2)$ *and* $(B_1)-(B_3)$ *hold and a,b,c are real constants such that* $a+b>0$ *with* $b\leq 0$ *and* $E_\alpha(\lambda T^\alpha)=1$*.Further,if*

$$\left[C_1\exp\left(2\lambda^\rho\right)\left(\left(1+\frac{|b|}{|a+b|}\right)T^{\alpha-1}\|h\|_{L^1}+\frac{|c|}{|a+b|}\right)\right]\phi(r)<r,\quad r>0 \tag{3.3}$$

then the hybrid fractional-order differential equation (2.3) has a positive solution x^* defined on J and the sequence $\{x_n\}_{n=1}^{\infty}$ of successive approximations defined by

$$\begin{aligned} x_{n+1}(t) = f(t, x_n(t)\Bigg[\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha) g(s, x_n(s)) ds \\ - \frac{E_\alpha(\lambda t^\alpha)}{a+b}\left(b\int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(T-s)^\alpha) g(s, x_n(s)) ds - c\right)\Bigg]. \end{aligned} \tag{3.4}$$

for $t \in \mathbb{R}$ converges into x^*.

Proof: Let $X = \mathcal{C}(J, \mathbb{R})$.
By application of lemma (3.2), equation (2.3) is equivalent to the nonlinear hybrid integral equation

$$\begin{aligned} x(t) = [f(t, x(t))]\Bigg(\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha) g(s) ds \\ - \frac{E_\alpha(\lambda t^\alpha)}{a+b}(b\int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(T-s)^\alpha) g(s) ds - c\Bigg) \end{aligned} \tag{3.5}$$

We define two operators $\mathcal{A}$ and $\mathcal{B}$ on X by:

$$\mathcal{A}x(t) = f(t, x(t))$$

and

$$\mathcal{B}x(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha) g(s) ds - \frac{E_\alpha(\lambda t^\alpha)}{a+b}\left(b\int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(T-s)^\alpha) g(s) ds - c\right)$$

Then the hybrid integral equation (3.5) is transformed into the operator equation as,

$$x(t) = \mathcal{A}x(t)\mathcal{B}x(t)$$

We shall show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem (3.1)
Claim 1: $\mathcal{A}$ and $\mathcal{B}$ are two increasing operators on X.
Let $x, y \in X$ such that $x \geq y$.
From the hypothesis (A_2), we get for each $t \in J$

$$\mathcal{A}x(t) = f(t, x(t)) \geq f(t, y(t)) = \mathcal{A}y(t)$$

This implies that the operator $\mathcal{A}$ is increasing on X.
In the same way, for each $t \in J$, $x(t) \geq y(t)$. From the hypothesis (B_2) we have:

$$g(t, x(t)) \geq g(t, y(t))$$

Then

$$\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha) g(s, x(s)) ds \geq \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha) g(s, y(s)) ds$$

Since $\dfrac{b}{a+b} \leq 0$ we have

$$\begin{aligned} &-\frac{E_\alpha(\lambda t^\alpha)}{a+b}\left(b\int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(T-s)^\alpha) g(s, x(s)) ds - c\right) \\ &\qquad \geq -\frac{E_\alpha(\lambda t^\alpha)}{a+b}\left(b\int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(T-s)^\alpha) g(s, y(s)) ds - c\right) \end{aligned}$$

Hence

$$\mathcal{B}x(t) \geq \mathcal{B}y(t), \quad for \quad each \quad t \in J$$

Then the operator $\mathcal{B}$ is increasing on X.
Consequently,$\mathcal{A}$ and $\mathcal{B}$ are increasing on X in itself.
Claim 2:$\mathcal{A}$ is partially bounded and partially $D-Lipschitzian$ on X.
Let $x \in X$. By using the hypothesis (A_1), we have:

$$|\mathcal{A}x(t)| \leq f(t, x(t)) \leq M_f, \quad for \quad each \quad t \in J.$$

By taking the supremum on t we obtain $\|Ax\| \leq M_f$, Then $\mathcal{A}$ is bounded on X.
This implies that $\mathcal{A}$ is partially bounded on X.
Let $x, y \in X$ such that $x \geq y$, then for each $t \in J$, we have

$$|\mathcal{A}x(t) - \mathcal{A}y(t)| = f(t, x(t)) - f(t, y(t)) \leq \phi(|x(t) - y(t)|) \leq \phi(\|x - y\|)$$

By taking the supremum on t , we get

$$\|\mathcal{A}x - \mathcal{A}y\| \leq \phi(\|x - y\|)$$

Conseguently, $\mathcal{A}$ is partially $D-Lipschitzian$ on X.
This implies that $\mathcal{A}$ is partially continuous on X.
Claim 3: $\mathcal{B}$ is partially continuous on X.
Let $\{x_n\}_{n\in\mathbb{N}}$ a sequence in a chain C of X, such that $x_n \longrightarrow x$. by lebesgue's dominated convergence theorem we have for each $t \in J$,

$$\begin{aligned}
\lim_{n\to\infty} (Bx_n)(t) &= \lim_{n\to\infty} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha) g(s, x_n(s)) ds \\
&- \lim_{n\to\infty} \frac{E_\alpha(\lambda t^\alpha)}{a+b} \left(b \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(T-s)^\alpha) g(s, x_n(s)) ds - c \right) \\
&= \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha) \lim_{n\to\infty} g(s, x_n(s)) ds \\
&- \frac{E_\alpha(\lambda t^\alpha)}{a+b} \left(b \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(T-s)^\alpha) \lim_{n\to\infty} g(s, x_n(s)) ds - c \right) \\
&= \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha) g(s, x(s)) ds \\
&- \frac{E_\alpha(\lambda t^\alpha)}{a+b} \left(b \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(T-s)^\alpha) g(s, x(s)) ds - c \right) \\
&= (\mathcal{B}x)(t)
\end{aligned}$$

Then Bx_n converges monotonously into Bx at any point of J.
then we show that $\{\mathcal{B}x_n\}$ is a sequence of equicontinuous functions on x.

Let $t_1, t_2 \in J$ such that $t_1 < t_2$. We set $\psi(t) = \int_0^t h(s)ds$, then:

$$\begin{aligned}
|Bx_n(t_2) - Bx_n(t_1)| &= \Big| \int_0^{t_2} (t_2-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t_2-s)^\alpha) g(s,x_n(s))\,ds \\
&- \frac{b}{a+b} E_\alpha(\lambda t_2^\alpha) \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(T-s)^\alpha) g(s,x_n(s))ds + \frac{c}{a+b} E_\alpha(\lambda t_2^\alpha) \\
&- \int_0^{t_1} (t_1-s)^{a-1} E_{\alpha,\alpha}(\lambda(t_1-s)^\alpha) g(s,x_n(s))\,ds \\
&+ \frac{b}{a+b} E_\alpha(\lambda t_1^\alpha) \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(T-s)^\alpha) g(s,x_n(s))ds - \frac{c}{a+b} E_\alpha(\lambda t_1^\alpha) \Big| \\
&\leq \Big| \int_0^{t_2} (t_2-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t_2-s)^\alpha) g(s,x_n(s))\,ds \\
&- \int_0^{t_1} (t_1-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t_1-s)^\alpha) g(s,x_n(s))\,ds \Big| + \left| \frac{c}{a+b} \right| |E_\alpha(\lambda t_2^\alpha) - E_\alpha(\lambda t_1^\alpha)| \\
&+ \left| \frac{b}{a+b} \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(T-s)^\alpha) g(s,x_n(s))ds \right| |E_\alpha(\lambda t_2^\alpha) - E_\alpha(\lambda t_1^\alpha)| \\
&\leq \left| \int_0^{t_1} \left[(t_2-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t_2-s)^\alpha) - (t_1-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t_1-s)^\alpha)\,ds \right] g(s,x_n(s))\,ds \right| \\
&+ \left| \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t_2-s)^\alpha) g(s,x_n(s))\,ds \right| + \frac{|c|}{|a+b|} |E_\alpha(\lambda t_2^\alpha) - E_\alpha(\lambda t_1^\alpha)| \\
&+ \frac{|b|}{|a+b|} \left| \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(T-s)^\alpha) g(s,x_n(s))ds \right| |E_\alpha(\lambda t_2^\alpha) - E_\alpha(\lambda t_1^\alpha)| \\
&\leq \|h\|_{L^1} \left(\int_0^{t_1} (t_2-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t_2-s)^\alpha) - (t_1-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t_1-s)^\alpha)\,ds \right) \\
&+ C_1 \exp(2\lambda^\rho) T^{\alpha-1} \int_{t_1}^{t_2} h(s)ds + \frac{|c|}{|a+b|} |E_\alpha(\lambda t_2^\alpha) - E_\alpha(\lambda t_1^\alpha)| \\
&+ \frac{|b|}{|a+b|} \left| \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(T-s)^\alpha) g(s,x_n(s))ds \right| |E_\alpha(\lambda t_2^\alpha) - E_\alpha(\lambda t_1^\alpha)|
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
\int_0^{t_1} (t_2-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t_2-s)^\alpha)\,ds &= -\int_{t_2}^{t_2-t_1} u^{\alpha-1} E_{\alpha,\alpha}(\lambda(u)^\alpha)\,du = \int_{t_2-t_1}^{t_2} u^{\alpha-1} E_{\alpha,\alpha}(\lambda(u)^\alpha)\,du \\
&= \int_0^{t_2} u^{\alpha-1} E_{\alpha,\alpha}(\lambda(u)^\alpha)\,du - \int_0^{t_2-t_1} u^{\alpha-1} E_{\alpha,\alpha}(\lambda(u)^\alpha)\,du
\end{aligned}$$

According to the relation (c) of lemma (2.2) , with $\beta = \alpha$ we get

$$\int_0^{t_1} (t_2-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t_2-s)^\alpha)\,ds = t_2^\alpha E_{\alpha,\alpha+1}(\lambda t_2^\alpha) - (t_2-t_1)^\alpha E_{\alpha,\alpha+1}(\lambda(t_2-t_1)^\alpha) \tag{3.6}$$

In a similar way we have

$$\int_0^{t_1} (t_1-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t_1-s)^\alpha)\,ds = t_1^\alpha E_{\alpha,\alpha+1}(\lambda t_1^\alpha) \tag{3.7}$$

Finally

$$\begin{aligned}
|Bx_n(t_2) - Bx_n(t_1)| &\leq \|h\|_{L^1} \left[t_2^\alpha E_{\alpha,\alpha+1}(\lambda t_2^\alpha) - t_1^\alpha E_{\alpha,\alpha+1}(\lambda t_1^\alpha) + (t_2-t_1)^\alpha E_{\alpha,\alpha+1}(\lambda(t_2-t_1)^\alpha) \right] \\
&+ C_1 \exp(2\lambda^\rho) T^{\alpha-1} |\psi(t_2) - \psi(t_1)| + \frac{|c|}{|a+b|} |E_\alpha(\lambda t_2^\alpha) - E_\alpha(\lambda t_1^\alpha)| \\
&+ \frac{|b|}{|a+b|} T^{\alpha-1} C_1 \exp(2\lambda^\rho) \|h\|_{L^1} |E_\alpha(\lambda t_2^\alpha) - E_\alpha(\lambda t_1^\alpha)|
\end{aligned}$$

Then

$$|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| \to 0 \quad When \quad t_2 - t_1 \to 0$$

uniformly for each $n \in \mathbb{N}$.
This shows that the convergence $\mathcal{B}x_n \to \mathcal{B}x$ is uniform and therefore $\mathcal{B}$ is partially continuous on X.
Claim 4: $\mathcal{B}$ is a partially uniformaly compact on X.
Let C be an arbitrary chain in X. we propose to show that $\mathcal{B}$ is a uniformly bounded and equicontinuous set in x. First $\mathcal{B}(C)$ is uniformly bounded.Indeed,
For each $y \in \mathcal{B}(C)$ there exist an element $x \in C$ such that $y = \mathcal{B}x$.
According to the hypotesis (B_1), for each $t \in J$, we have

$$\begin{aligned}
|y(t)| &= \Bigg| \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha) g(s,x(s)) ds \\
&\quad - \frac{bE_\alpha(\lambda t^\alpha)}{a+b} \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(T-s)^\alpha) g(s,x(s)) ds + c\frac{E_\alpha(\lambda t^\alpha)}{a+b} \Bigg| \\
&\le \left| \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha) g(s,x(s)) ds \right| \\
&\quad + \frac{|b|}{|a+b|} |E_\alpha(\lambda t^\alpha)| \left| \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(T-s)^\alpha) g(s,x(s)) ds \right| + \frac{|c|}{|a+b|} |E_\alpha(\lambda t^\alpha)| \\
&\le C_1 \exp(2\lambda^\rho) T^{\alpha-1} \int_0^T h(s) ds + \frac{|b|}{|a+b|} C_1 \exp(2\lambda^\rho) \times T^{\alpha-1} \int_0^T h(s) ds + \frac{|c|}{|a+b|} C_1 \exp(2\lambda^\rho) \\
&\le C_1 \exp(2\lambda^\rho) T^{\alpha-1} \|h\|_{L^1} + \frac{|b|}{|a+b|} C_1 \exp(2\lambda^\rho) \times T^{\alpha-1} \|h\|_{L^1} + \frac{|c|}{|a+b|} C_1 \exp(2\lambda^\rho)
\end{aligned}$$

Then

$$|y(t)| \le C_1 \exp(2\lambda^\rho) \left[\left(1 + \frac{|b|}{|a+b|}\right) T^{\alpha-1} \|h\|_{L^1} + \frac{|c|}{|a+b|} \right] = M$$

By taking the supremum on t, we obtain

$$\|y\| = \|\mathcal{B}x\| \le M \quad for \quad each \quad y \in \mathcal{B}(C)$$

Therefore $\mathcal{B}(C)$ is a uniformly bounded subset of X.
Thus, $\|\mathcal{B}(C)\| \le M$ for each chain C in X.
Then we show that $\mathcal{B}(C)$ is an equicontinuous set of X.
Let $t_1, t_2 \in J$ such that $t_1 < t_2$, and $\psi(t) = \int_0^t h(s)ds$, then for each $y \in \mathcal{B}(C)$, we have

$$
\begin{aligned}
|y(t_2)-y(t_1)| &= \Big|\int_0^{t_2}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t_2-s)^{\alpha}\right)g(s,x(s))\,ds \\
&-\frac{b}{a+b}E_\alpha(\lambda t_2^\alpha)\int_0^T(T-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(T-s)^\alpha)g(s,x(s))ds+\frac{c}{a+b}E_\alpha(\lambda t_2^\alpha) \\
&-\int_0^{t_1}(t_1-s)^{a-1}E_{\alpha,\alpha}\left(\lambda(t_1-s)^{\alpha}\right)g(s,x(s))\,ds \\
&+\frac{b}{a+b}E_\alpha(\lambda t_1^\alpha)\int_0^T(T-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(T-s)^\alpha)g(s,x(s))ds-\frac{c}{a+b}E_\alpha(\lambda t_1^\alpha)\Big| \\
&\leq \Big|\int_0^{t_2}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t_2-s)^{\alpha}\right)g(s,x(s))\,ds \\
&-\int_0^{t_1}(t_1-s)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t_1-s)^{\alpha}\right)g(s,x(s))\,ds\Big|+\left|\frac{c}{a+b}\right|\left|E_\alpha\left(\lambda t_2^\alpha\right)-E_\alpha\left(\lambda t_1^\alpha\right)\right| \\
&+\left|\frac{b}{a+b}\int_0^T(T-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(T-s)^\alpha)g(s,x(s))ds\right|\left|E_\alpha\left(\lambda t_2^\alpha\right)-E_\alpha\left(\lambda t_1^\alpha\right)\right| \\
&\leq \left|\int_0^{t_1}\left[(t_2-s)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t_2-s)^{\alpha}\right)-(t_1-s)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t_1-s)^{\alpha}\right)ds\right]g(s,x(s))\,ds\right| \\
&+\left|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t_2-s)^{\alpha}\right)g(s,x(s))\,ds\right|+\frac{|c|}{|a+b|}\left|E_\alpha\left(\lambda t_2^\alpha\right)-E_\alpha\left(\lambda t_1^\alpha\right)\right| \\
&+\frac{|b|}{|a+b|}\left|\int_0^T(T-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(T-s)^\alpha)g(s,x(s))ds\right|\left|E_\alpha\left(\lambda t_2^\alpha\right)-E_\alpha\left(\lambda t_1^\alpha\right)\right| \\
&\leq \|h\|_{L^1}\left(\int_0^{t_1}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t_2-s)^{\alpha}\right)-(t_1-s)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t_1-s)^{\alpha}\right)ds\right) \\
&+C_1\exp\left(2\lambda^\rho\right)T^{\alpha-1}\int_{t_1}^{t_2}h(s)ds+\frac{|c|}{|a+b|}\left|E_\alpha\left(\lambda t_2^\alpha\right)-E_\alpha\left(\lambda t_1^\alpha\right)\right| \\
&+\frac{|b|}{|a+b|}\left|\int_0^T(T-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(T-s)^\alpha)g(s,x(s))ds\right|\left|E_\alpha\left(\lambda t_2^\alpha\right)-E_\alpha\left(\lambda t_1^\alpha\right)\right| \\
&\leq \|h\|_{L^1}\left(\int_0^{t_1}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t_2-s)^{\alpha}\right)-(t_1-s)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t_1-s)^{\alpha}\right)ds\right) \\
&+C_1\exp\left(2\lambda^\rho\right)T^{\alpha-1}\int_{t_1}^{t_2}h(s)ds+\frac{|c|}{|a+b|}\left|E_\alpha\left(\lambda t_2^\alpha\right)-E_\alpha\left(\lambda t_1^\alpha\right)\right| \\
&+\frac{|b|}{|a+b|}T^{\alpha-1}\|h\|_{L^1}\left|E_\alpha\left(\lambda t_2^\alpha\right)-E_\alpha\left(\lambda t_1^\alpha\right)\right|
\end{aligned}
$$

Finally, according to relations (3.6) and (3.7), we have

$$
\begin{aligned}
|y(t_2)-y(t_1)| \leq{}& \left|E_\alpha\left(\lambda t_2^\alpha\right)-E_\alpha\left(\lambda t_1^\alpha\right)\right|\left(\frac{|c|}{|a+b|}+\frac{|b|}{|a+b|}T^{\alpha-1}\|h\|_{L^1}\right) \\
&+C_1\exp\left(2\lambda^\rho\right)T^{\alpha-1}\left|\psi(t_2)-\psi(t_1)\right| \\
&+\|h\|_{L^1}\left[t_2^\alpha E_{\alpha,\alpha+1}\left(\lambda t_2^\alpha\right)-t_1^\alpha E_{\alpha,\alpha+1}\left(\lambda t_1^\alpha\right)+(t_2-t_1)^\alpha E_{\alpha,\alpha+1}\left(\lambda(t_2-t_1)^\alpha\right)\right]
\end{aligned}
$$

Then, $|y(t_2)-y(t_1)| \to 0$ When $t_2-t_1 \to 0$, uniformly for each $y \in \mathcal{B}(C)$, consequently $\mathcal{B}(C)$ is an equicontinuous subset of X.

$\mathcal{B}(C)$ uniformly bounded and equicontinuous in X, therefore it is compact.

Hence, $\mathcal{B}$ is a uniformly partially compact operator of X in itself.

Claim 5: The lower-solution u satisfies the inequality $u \leq \mathcal{A}u\mathcal{B}u$.

According to the hypothesis (B_3) the BNVPHDEF (2.3) has a lower-solution u defined on J, then we have

$$a\frac{u(0)}{f(0,u(0))}+b\frac{u(T)}{f(T,u(T))}\leq c$$

and,

$$D^{\alpha}\left(\frac{u(t)}{f(t,u(t))}\right)-\lambda\left(\frac{u(t)}{f(t,u(t))}\right)\leq g(t,u(t)) \qquad \text{a.e. } t\in J=[0,T],$$

By multiplying the above inequality by the factor $E_{\alpha,\alpha}$ and integrating we obtain for each $t\in J$.

$$\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t-s)^{\alpha}\right)\left(D^{\alpha}\left(\frac{u(s)}{f(s,u(s))}\right)-\lambda\left(\frac{u(s)}{f(s,u(s))}\right)\right)ds$$
$$\leq \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t-s)^{\alpha}\right)g(s,u(s))ds$$

By using the relation (1) of lemma (2.1) with $n=1, u=1$ and $k=-\lambda$ we obtain

$$\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t-s)^{\alpha}\right)D^{\alpha}u(s)ds=u(t)-E_{\alpha}\left(\lambda t^{\alpha}\right)u(0)+\lambda\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t-s)^{\alpha}\right)u(s)ds$$

Then,

$$\frac{u(t)}{f(t,u(t))}-E_{\alpha}\left(\lambda t^{\alpha}\right)\frac{u(0)}{f(0,u(0))}+\lambda\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t-s)^{\alpha}\right)\frac{u(s)}{f(s,u(s))}ds$$
$$-\lambda\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t-s)^{\alpha}\right)\frac{u(s)}{f(s,u(s))}ds\leq\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t-s)^{\alpha}\right)g(s,u(s))ds$$

Then,

$$\frac{u(t)}{f(t,u(t))}\leq E_{\alpha}\left(\lambda t^{\alpha}\right)\frac{u(0)}{f(0,u(0))}+\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t-s)^{\alpha}\right)g(s,u(s))ds$$

For $t=T$, we have,

$$\frac{u(T)}{f(t,u(T))}\leq E_{\alpha}\left(\lambda T^{\alpha}\right)\frac{u(0)}{f(0,u(0))}+\int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(T-s)^{\alpha}\right)g(s,u(s))ds$$

As, $b<0$ and $E_{\alpha}\left(\lambda T^{\alpha}\right)=1$, we get

$$b\frac{u(T)}{f(t,u(T))}\geq b\frac{u(0)}{f(0,u(0))}+b\int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(T-s)^{\alpha}\right)g(s,u(s))ds$$

Thus,

$$a\frac{u(0)}{f(0,u(0))}+b\frac{u(T)}{f(t,u(T))}\geq(a+b)\frac{u(0)}{f(0,u(0))}+b\int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(T-s)^{\alpha}\right)g(s,u(s))ds$$

i.e

$$\frac{u(0)}{f(0,u(0))}\leq-\frac{1}{a+b}\left(b\int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(T-s)^{\alpha}\right)g(s,u(s))ds+c\right)$$

therfore

$$\frac{u(t)}{f(t,u(t))}\leq\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t-s)^{\alpha}\right)g(s,u(s))ds-$$
$$\frac{E_{\alpha}(\lambda t^{\alpha})}{a+b}\left(b\int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(T-s)^{\alpha})g(s)ds-c\right)$$

i.e

$$u(t) \leq \mathcal{A}u(t)\mathcal{B}u(t) \quad for \quad each \quad t \in J$$

consequently

$$u \leq \mathcal{A}u\mathcal{B}u$$

Claim 6: The D -function ϕ satisfies the condition $M\phi_A(r) < r, \quad r > 0$.
The D -function ϕ satisfies the condition giving by hypothesis (c) of the Dhage theorem.
According to the estiamtion giving in claim 4, we have for each $r > 0$

$$M\phi(r) = C_1 \exp\left(2\lambda^{\rho}\right)\left[\left(1 + \frac{|b|}{|a+b|}\right)T^{\alpha-1}\|h\|_{L^1} + \frac{|c|}{|a+b|}\right]\phi(r) < r$$

Then, $\mathcal{A}$ and $\mathcal{B}$ satisfy all conditions of Dhage theorem,and hence the operator equation $\mathcal{A}x\mathcal{B}x = x$ has a solution x^* in X.Consequently, x^* is a soltion of the integral equation (5.3). As a result, BVPHDEF (2.3) has a solution defined on J. Moreover the sequence $\{x_n\}_{n=1}^{\infty}$ of successive approximations defined by (3.4) conerges into x^*. This completes the proof.

□

4. Fractional hybrid differential inequalities

We discuss a fundamental result relative to strict inequalities for BVPHDEF (2.3). We begin with the definition of the class $C_p([0,T],\mathbb{R})$.

Definition 4.1 *$m \in C_p([0,T],\mathbb{R})$ means that $m \in C([0,T],\mathbb{R})$ and $t^p m(t) \in C([0,T],\mathbb{R})$*

Lemma 4.1 *[18] Let $m \in C_p([0,T],\mathbb{R})$. Suppose that for any $t_1 \in (0,+\infty)$ we have $m(t_1) = 0$ and $m(t) \leq 0$ for $0 \leq t \leq t_1$*

Then it follows that

$$D^q m(t_1) \geq 0$$

Theorem 4.1 *Assume that hypothesis (A_0) holds. Suppose that there exist functions $y, z \in C_p([0,T],\mathbb{R})$ such that*

$$D^{\alpha}\left(\frac{y(t)}{f(t,y(t))}\right) - \lambda\left(\frac{y(t)}{f(t,y(t))}\right) \leq g(t,y(t)) \quad a.e.\ t \in J \tag{4.1}$$

and

$$D^{\alpha}\left(\frac{z(t)}{f(t,z(t))}\right) - \lambda\left(\frac{z(t)}{f(t,z(t))}\right) \geq g(t,z(t)) \quad a.e.\ t \in J \tag{4.2}$$

$0 < t \leq T$, with one of the inequalities being strict. Then

$$y^0 < z^0$$

where $y^0 = t^{1-\alpha}y(t)\big|_{t=0}$ and $z^0 = t^{1-\alpha}z(t)\big|_{t=0}$ implies

$$y(t) < z(t)$$

for all $t \in J$

Proof: Suppose that inequality (4.2) holds.Assume that the claim is false.Then, since $y^0 < z^0$ and $t^{1-\alpha}y(t)$ and $t^{1-\alpha}z(t)$ are continuous functions, there exists t_1 such that $0 < t_1 \leq T$ with $y(t_1) = z(t_1)$ and $y(t) < z(t)$, $0 \leq t < t_1$.
Define

$$Y(t) = \frac{y(t)}{f(t,y(t))} \quad \text{and} \quad Z(t) = \frac{z(t)}{f(t,z(t))}$$

Then we have $Y(t_1) = Z(t_1)$ and according to the hypothesis (A_0), we get $Y(t) < Z(t)$ for all $0 \le t < t_1$. Setting $m(t) = Y(t) - Z(t), 0 \le t \le t_1$, we find that $m(t) < 0, 0 \le t < t_1$ and $m(t_1) = 0$ with $m \in C_p([0,T],\mathbb{R})$. Then, by Lemma (4.1), we have $D^q m(t_1) \ge 0$. By (4.1) and (4.2), we obtain

$$g(t_1, y(t_1)) \ge D^q Y(t_1) - \lambda Y(t_1) \ge D^q Z(t_1) - \lambda Z(t_1) > g(t_1, z(t_1))$$

This is a contradiction with $y(t_1) = z(t_1)$. Thus the conclusion of the theorem holds and the proof is complete. □

Theorem 4.2 *Assume that hypothesis (A_0) holds and a, b, c are real constants with $a+b \neq 0$. Suppose that there exist functions $y, z \in C_p([0,T],\mathbb{R})$ such that*

$$D^\alpha\left(\frac{y(t)}{f(t,y(t))}\right) - \lambda\left(\frac{y(t)}{f(t,y(t))}\right) \le g(t,y(t)) \quad a.e.\ t \in J \tag{4.3}$$

and

$$D^\alpha\left(\frac{z(t)}{f(t,z(t))}\right) - \lambda\left(\frac{z(t)}{f(t,z(t))}\right) \ge g(t,z(t)) \quad a.e.\ t \in J \tag{4.4}$$

one of the inequalities being strict, and if $a > 0$, $b < 0$ and $y(T) < z(T)$, then

$$a\frac{y(0)}{f(0,y(0))} + b\frac{y(T)}{f(T,y(T))} < a\frac{z(0)}{f(0,z(0))} + b\frac{z(T)}{f(T,z(T))}$$

implies

$$y(t) < z(t)$$

for all $t \in J$.

Proof: We have

$$a\frac{y(0)}{f(0,y(0))} + b\frac{y(T)}{f(T,y(T))} < a\frac{z(0)}{f(0,z(0))} + b\frac{z(T)}{f(T,z(T))}$$

This implies

$$a\left(\frac{y(0)}{f(0,y(0))} - \frac{z(0)}{f(0,z(0))}\right) < b\left(\frac{z(T)}{f(T,z(T))} - \frac{y(T)}{f(T,y(T))}\right)$$

Since $b < 0$ and $y(T) < z(T)$ by hypothesis (A_0) we have $\frac{z(T)}{f(T,z(T))} - \frac{y(T)}{f(T,y(T))} > 0$.
This shows that $\frac{y(0)}{f(0,y(0))} - \frac{z(0)}{f(0,z(0))} < 0$ since $a > 0$, and by hypothesis (A_0) we have $y(0) < z(0)$ Hence the application of Theorem (4.1) yields that $y(t) < z(t)$.

□

Theorem 4.3 *Assume that the conditions of Theorem (4.2) hold with inequalities (4.1) and (4.2). Suppose that there exists a real number $M > 0$ such that*

$$g(t,x_1) - g(t,x_2) \le \frac{M}{1+t^\alpha}\left(\frac{x_1}{f(t,x_1)} - \frac{x_2}{f(t,x_2)}\right) \quad a.e.\ t \in J \tag{4.5}$$

for all $x_1, x_2 \in \mathbb{R}$ with $x_1 \ge x_2$. Then

$$a\frac{y(0)}{f(0,y(0))} + b\frac{y(T)}{f(T,y(T))} < a\frac{z(0)}{f(0,z(0))} + b\frac{z(T)}{f(T,z(T))}$$

implies, provided $M \le \Gamma(1+\alpha)$,

$$y(t) < z(t)$$

for all $t \in J$.

Proof: We set $\frac{z_\varepsilon(t)}{f(t,z_\varepsilon(t))} = \frac{z(t)}{f(t,z(t))} + \varepsilon(1+t^\alpha)$ for small $\varepsilon > 0$ and let $Z_\varepsilon(t) = \frac{z_\varepsilon(t)}{f(t,z_\varepsilon(t))}$ and $Z(t) = \frac{z(t)}{f(t,z(t))}$ for $t \in J$ So that we have

$$Z_\varepsilon(t) > Z(t) \quad \Longrightarrow \quad z_\varepsilon(t) > z(t).$$

Since $g(t,x_1) - g(t,x_2) \leq \frac{M}{1+t^\alpha}\left(\frac{x_1}{f(t,x_1)} - \frac{x_2}{f(t,x_2)}\right)$ and $D^\alpha\left(\frac{z(t)}{f(t,z(t))}\right) \geq g(t,z(t))$ for all $t \in J$, one has

$$\begin{aligned} D^\alpha Z_\varepsilon(t) &= D^\alpha Z(t) + \varepsilon D^\alpha t^\alpha \\ &\geq g(t,z(t)) + \varepsilon\Gamma(\alpha+1) \\ &\geq g(t,z_\varepsilon(t)) - \frac{M}{1+t^\alpha}(Z_\varepsilon - Z) + \varepsilon\Gamma(1+\alpha) \\ &\geq g(t,z_\varepsilon(t)) + \varepsilon(\Gamma(1+\alpha) - M) \\ &> g(t,z_\varepsilon(t)) \end{aligned}$$

provided $M \leq \Gamma(1+\alpha)$.
Also, we have $z_\varepsilon(0) > z(0) \geq y(0)$. Hence, the application of Theorem (4.1) yields that $y(t) < z_\varepsilon(t)$ for all $t \in J$.
By the arbitrariness of $\varepsilon > 0$, taking the limits as $\varepsilon \to 0$, we have $y(t) \leq z(t)$ for all $t \in J$. This completes the proof. □

Remark 4.1 Let $f(t,x) = 1$ and $g(t,x) = x$. We can easily verify that f and g satisfy condition (4.5)

5. Existence of maximal and minimal solutions

In this section, we have to prove the existence of maximal and minimal solutions for BVPHDEF (2.3) on $J = [0,T]$. We need the following definition in what follows.

Definition 5.1 *A solution r of BVPHDEF (2.3) is said to be maximal if for any other solution x to BVPHDEF (2.3) one has $x(t) \leq r(t)$ for all $t \in J$. Similarly, a solution ρ of BVPHDEF (2.3) is said to be minimal if $\rho(t) \leq x(t)$ for all $t \in J$, where x is any solution of BVPHDEF (2.3) on J.*

We treat the case of maximal solution only, as the case of minimal solution is similar and can be geted with the same arguments with appropriate modifications. Given an arbitrarily small real number $\epsilon > 0$, consider the following boundary value problem of BVPHDEF of order $0 < \alpha < 1$:

$$\begin{cases} D^\alpha\left(\frac{x(t)}{f(t,x(t))}\right) - \lambda\left(\frac{x(t)}{f(t,x(t))}\right) = g(t,x(t)) + \epsilon \quad \text{a.e. } t \in J = [0,T], \\ a\frac{x(0)}{f(0,x(0))} + b\frac{x(T)}{f(T,x(T))} = c + \epsilon, \end{cases} \tag{5.1}$$

Where $f \in C(J \times \mathbb{R}, \mathbb{R}\backslash\{0\})$ and $g : J \times \mathbb{R} \longmapsto \mathbb{R}$ is $\mathcal{L}^1$ -Caratheodory function ,$\lambda > 0$, and a, b, c are real constants with $a + b \neq 0$.

Theorem 5.1 *Assume that $(A_0) - (A_2)$ and $(B_1) - (B_3)$ hold and a,b,c are real constants such that $a + b > 0$ with $b < 0$ and $E_\alpha(\lambda T^\alpha) = 1$. Suppose that inequality (3.3) holds. Then, for every small number $\epsilon > 0$, BVPHDEF (5.1) has a solution defined on J.*

Proof: By hypothesis, since

$$\left[C_1 \exp(2\lambda^\rho)\left(\left(1 + \frac{|b|}{|a+b|}\right)T^{\alpha-1}\|h\|_{L^1} + \frac{|c|}{|a+b|}\right)\right]\phi(r) < r, \quad r > 0$$

there exists $\epsilon > 0$ such that

$$C_1 \exp(2\lambda^\rho)\left[\left(1 + \frac{|b|}{|a+b|}\right)T^{\alpha-1}(\|h\|_{L^1} + \epsilon) + \frac{|c|+\epsilon}{|a+b|}\right]\phi(r) < r, \quad r > 0$$

for all $0 < \epsilon \leq \epsilon_0$. Now the rest of the proof is similar to Theorem (3.2). □

The main existence theorem for maximal solution for BVPHDEF (2.3) is the following

Theorem 5.2 *Assume that hypotheses* $(A_0)-(A_2)$ *and*$(B_1)-(B_3)$ *hold with the conditions of Theorem* (4.2) *and* a, b, c *are real constants with* $a+b>0$ *and* $b<0$ *Furthermore, if condition* (3.3) *holds, then BVPHDEF* (2.3) *has a maximal solution defined on J.*

Proof: Let $\{\epsilon_n\}_0^\infty$ be a decreasing sequence of positive real numbers such that $\lim_{t\to 0^+}\epsilon_n=0$, where ϵ_0 is a positive real number satisfying the inequality

$$C_1\exp\left(2\lambda^\rho\right)\left[\left(1+\frac{|b|}{|a+b|}\right)T^{\alpha-1}\left(\|h\|_{L^1}+\epsilon\right)+\frac{|c|+\epsilon}{|a+b|}\right]\phi(r)<r,\quad r>0$$

The number ϵ_0 exists in view of inequality (3.3). By Theorem (5.1), there exists a solution $r(t,\epsilon_n)$ defined on J of the BVPHDEF

$$\begin{cases} D^\alpha\left(\frac{x(t)}{f(t,x(t))}\right)-\lambda\left(\frac{x(t)}{f(t,x(t))}\right)=g(t,x(t))+\epsilon_n & \text{a.e. } t\in J=[0,T],\\ a\frac{x(0)}{f(0,x(0))}+b\frac{x(T)}{f(T,x(T))}=c+\epsilon_n, \end{cases}\tag{5.2}$$

Then a solution u of BVPHDEF (2.3) satisfies

$$D^\alpha\left(\frac{u(t)}{f(t,u(t))}\right)-\lambda\left(\frac{u(t)}{f(t,u(t))}\right)\le g(t,u(t))\quad \text{a.e. } t\in J$$

and any solution of auxiliary problem (5.2) satisfies

$$D^\alpha\left(\frac{r(t,\epsilon_n)}{f(t,r(t,\epsilon_n))}\right)-\lambda\left(\frac{r(t,\epsilon_n)}{f(t,u(t))}\right)=g(t,r(t,\epsilon_n))+\epsilon_n>g(t,r(t,\epsilon_n))$$

where $a\frac{u(0)}{f(0,u(0))}+b\frac{u(T)}{f(T,u(T))}=c\le c+\epsilon_n=a\frac{r(0,\epsilon_n)}{f(0,r(t,\epsilon_n))}+b\frac{r(T,\epsilon_n)}{f(T,r(T,\epsilon_n))}$. By theorem (4.2), we infer that

$$u(t)\le r(t,\epsilon_n)\quad \text{for all}\quad t\in J\quad \text{and}\quad n\in\mathbb{N}\tag{5.3}$$

Since

$$\begin{aligned} c+\epsilon_2&=a\frac{r(0,\epsilon_2)}{f(0,r(t,\epsilon_2))}+b\frac{r(T,\epsilon_2)}{f(T,r(T,\epsilon_2))}\\ &\le a\frac{r(0,\epsilon_1)}{f(0,r(t,\epsilon_1))}+b\frac{r(T,\epsilon_1)}{f(T,r(T,\epsilon_1))}=c+\epsilon_1, \end{aligned}$$

then by Theorem (4.2) we infer that $r(t,\epsilon_2)\le r(t,\epsilon_1)$. Therfore,$r(t,\epsilon_n)$ is a decreasing sequence of positive real numbers, and the limit

$$r(t)=\lim_{n\to\infty}r(t,\epsilon_n)\tag{5.4}$$

exists. We show that the convergence in (5.4) is uniform on J. To finish, it is enough to prove that the sequence $r(t,\epsilon_n)$ is equicontinuous in $C(J,\mathbb{R})$. Let $t_1,t_2\in J$ with $t_1<t_2$ be arbitrary. we set $\psi(t)=\int_0^t h(s)ds$. Then

$$\begin{aligned} |r(t_1,\varepsilon_n)-r(t_2,\varepsilon_n)|=\Bigg| & f(t_1,r(t_1,\varepsilon_n))\left(\int_0^{t_1}(t_1-s)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t_1-s)^\alpha\right)\left(g(s,r(s,\varepsilon_n))+\varepsilon_n\right)ds\right.\\ &+\frac{c+\epsilon}{a+b}E_\alpha\left(\lambda t_1^\alpha\right)\\ &\left.-\frac{b}{a+b}E_\alpha\left(\lambda t_1^\alpha\right)\left(\int_0^T(T-s)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(T-s)^\alpha\right)\left(g(s,r(s,\varepsilon_n))+\varepsilon_n\right)ds\right)\right)\\ &-f(t_2,r(t_2,\varepsilon_n))\left(\int_0^{t_2}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t_2-s)^\alpha\right)\left(g(s,r(s,\varepsilon_n))+\varepsilon_n\right)ds\right.\\ &+\frac{c+\epsilon}{a+b}E_\alpha\left(\lambda t_2^\alpha\right)\\ &\left.-\frac{b}{a+b}E_\alpha\left(\lambda t_2^\alpha\right)\left(\int_0^T(T-s)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(T-s)^\alpha\right)\left(g(s,r(s,\varepsilon_n))+\varepsilon_n\right)ds\right)\right)\Bigg| \end{aligned}$$

Thus

$$
\begin{aligned}
|r(t_1,\varepsilon_n) & -r(t_2,\varepsilon_n)| = \Big| f(t_1, r(t_1,\varepsilon_n)) \int_0^{t_1} (t_1-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t_1-s)^{\alpha}) (g(s,r(s,\varepsilon_n))+\varepsilon_n)\, ds \\
& + f(t_1, r(t_1,\varepsilon_n)) \frac{c+\epsilon}{a+b} E_{\alpha}(\lambda t_1^{\alpha}) \\
& - f(t_2, r(t_2,\varepsilon_n)) \int_0^{t_1} (t_1-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t_1-s)^{\alpha}) (g(s,r(s,\varepsilon_n))+\varepsilon_n)\, ds \\
& - f(t_1, r(t_1,\varepsilon_n)) \frac{b}{a+b} E_{\alpha}(\lambda t_1^{\alpha}) \int_0^{T} (T-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(T-s)^{\alpha}) (g(s,r(s,\varepsilon_n))+\varepsilon_n)\, ds \\
& + f(t_2, r(t_2,\varepsilon_n)) \int_0^{t_1} (t_1-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t_1-s)^{\alpha}) (g(s,r(s,\varepsilon_n))+\varepsilon_n)\, ds \\
& - f(t_2, r(t_2,\varepsilon_n)) \int_0^{t_2} (t_2-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t_2-s)^{\alpha}) (g(s,r(s,\varepsilon_n))+\varepsilon_n)\, ds \\
& - f(t_2, r(t_2,\varepsilon_n)) \frac{c+\epsilon}{a+b} E_{\alpha}(\lambda t_2^{\alpha}) \\
& + f(t_2, r(t_2,\varepsilon_n)) \frac{b}{a+b} E_{\alpha}(\lambda t_2^{\alpha}) \int_0^{T} (T-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(T-s)^{\alpha}) (g(s,r(s,\varepsilon_n))+\varepsilon_n)\, ds \Big| \\
& = \Big| [f(t_1, r(t_1,\varepsilon_n)) - f(t_2, r(t_2,\varepsilon_n))] \int_0^{t_1} (t_1-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t_1-s)^{\alpha}) (g(s,r(s,\varepsilon_n))+\varepsilon_n)\, ds \\
& + [f(t_1, r(t_1,\varepsilon_n)) E_{\alpha}(\lambda t_1^{\alpha}) - f(t_2, r(t_2,\varepsilon_n)) E_{\alpha}(\lambda t_2^{\alpha})] \frac{c+\epsilon}{a+b} \\
& - [f(t_1, r(t_1,\varepsilon_n)) E_{\alpha}(\lambda t_1^{\alpha}) - f(t_2, r(t_2,\varepsilon_n)) E_{\alpha}(\lambda t_2^{\alpha})] \frac{b}{a+b} \int_0^{T} (T-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(T-s)^{\alpha}) \\
& \times (g(s,r(s,\varepsilon_n))+\varepsilon_n)\, ds + f(t_2, r(t_2,\varepsilon_n)) \Big(\int_0^{t_1} (t_1-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t_1-s)^{\alpha}) (g(s,r(s,\varepsilon_n))+\varepsilon_n)\, ds \\
& - \int_0^{t_2} (t_2-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t_2-s)^{\alpha}) (g(s,r(s,\varepsilon_n))+\varepsilon_n)\, ds \Big) \Big|
\end{aligned}
$$

Then

$$
\begin{aligned}
|r(t_1,\varepsilon_n) - r(t_2,\varepsilon_n)| \leq & \, |f(t_1, r(t_1,\varepsilon_n)) - f(t_2, r(t_2,\varepsilon_n))| C_1 \exp(2\lambda^{\rho}) \times T^{\alpha-1} (\|h\|_{L^1} + \epsilon_n) \\
& + |f(t_1, r(t_1,\varepsilon_n)) - f(t_2, r(t_2,\varepsilon_n))| C_1 \exp(2\lambda^{\rho}) \frac{|c|+\epsilon_n}{|a+b|} \\
& + |f(t_1, r(t_1,\varepsilon_n)) - f(t_2, r(t_2,\varepsilon_n))| C_1 \exp(2\lambda^{\rho}) \times T^{\alpha-1} \frac{|b|}{|a+b|} (\|h\|_{L^1} + \epsilon_n) \\
& + |f(t_2, r(t_2,\varepsilon_n))| \Big| \int_0^{t_1} \big((t_1-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t_1-s)^{\alpha}) - (t_2-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t_2-s)^{\alpha}) \big) \\
& \times (g(s,r(s,\varepsilon_n))+\varepsilon_n)\, ds| \\
& + |f(t_2, r(t_2,\varepsilon_n))| \Big| \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t_2-s)^{\alpha}) (g(s,r(s,\varepsilon_n))+\varepsilon_n)\, ds \Big|
\end{aligned}
$$

According to relations (3.6) and (3.7) we have

$$
\begin{aligned}
|r(t_1,\varepsilon_n) - r(t_2,\varepsilon_n)| \leq & \, |f(t_1, r(t_1,\varepsilon_n)) - f(t_2, r(t_2,\varepsilon_n))| C_1 \exp(2\lambda^{\rho}) \times T^{\alpha-1} (\|h\|_{L^1} + \epsilon_n) \\
& + |f(t_1, r(t_1,\varepsilon_n)) - f(t_2, r(t_2,\varepsilon_n))| C_1 \exp(2\lambda^{\rho}) \frac{|c|+\epsilon_n}{|a+b|} \\
& + |f(t_1, r(t_1,\varepsilon_n)) - f(t_2, r(t_2,\varepsilon_n))| C_1 \exp(2\lambda^{\rho}) \times T^{\alpha-1} \frac{|b|}{|a+b|} (\|h\|_{L^1} + \epsilon_n) \\
& + F \left[t_2^{\alpha} E_{\alpha,\alpha+1}(\lambda t_2^{\alpha}) - t_1^{\alpha} E_{\alpha,\alpha+1}(\lambda t_1^{\alpha}) + (t_2-t_1)^{\alpha} E_{\alpha,\alpha+1}(\lambda (t_2-t_1)^{\alpha}) \right] (\|h\|_{L^1} + \epsilon_n) \\
& + F \times T^{\alpha-1} C_1 \exp(2\lambda^{\rho}) (|\psi(t_2) - \psi(t_1)| + \epsilon_n (t_2 - t_1)).
\end{aligned}
$$

Finally

$$\begin{aligned}|r(t_1,\varepsilon_n)-r(t_2,\varepsilon_n)| \le & |f(t_1,r(t_1,\varepsilon_n))-f(t_2,r(t_2,\varepsilon_n))|\, C_1\exp(2\lambda^\rho)\left[\left(1+\frac{|b|}{|a+b|}\right)T^{\alpha-1}\left(\|h\|_{L^1}+\epsilon_n\right)\right]\\ &+M_f\left[t_2^\alpha E_{\alpha,\alpha+1}(\lambda t_2^\alpha)-t_1^\alpha E_{\alpha,\alpha+1}(\lambda t_1^\alpha)+(t_2-t_1)^\alpha E_{\alpha,\alpha+1}(\lambda(t_2-t_1)^\alpha)\right]\left(\|h\|_{L^1}+\epsilon_n\right)\\ &+M_f\times T^{\alpha-1}C_1\exp(2\lambda^\rho)\left[|\psi(t_2)-\psi(t_1)|+\epsilon_n(t_2-t_1)\right].\end{aligned}$$

Since f is continuous on a compact set $J\times[-M_f,M_f]$, then it is uniformly continuous there. Thus,

$$|f(t_1,r(t_1,\varepsilon_n))-f(t_2,r(t_2,\varepsilon_n))|\to 0 \quad \text{as } t_1\to t_2$$

uniformly for all $n\in\mathbb{N}$. Therfore,

$$r(t,\varepsilon_n)\to r(t) \quad \text{as } n\to\infty \text{ for all } t\in J$$

Next, we show that the function $r(t)$ is a solution of BVPHDEF (2.3) defined on J. Now, since $r(t,\epsilon_n)$ is a solution of BVPHDEF (5.2), we have

$$\begin{aligned}r(t,\epsilon_n)=f(t,r(t,\varepsilon_n))&\left[\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha)\left(g(s,r(s,\varepsilon_n))+\varepsilon_n\right)ds+\frac{c+\epsilon}{a+b}E_\alpha(\lambda t^\alpha)\right.\\ &\left.-\frac{b}{a+b}E_\alpha(\lambda t^\alpha)\int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(T-s)^\alpha)\left(g(s,r(s,\varepsilon_n))+\varepsilon_n\right)ds\right]\end{aligned}$$

for all $t\in J$. Taking the limit as $n\longrightarrow\infty$in the above equation yields

$$\begin{aligned}r(t)=f(t,r(t))&\left[\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha)\,g(s,r(s))+ds+\frac{c+\epsilon}{a+b}E_\alpha(\lambda t^\alpha)\right.\\ &\left.-\frac{b}{a+b}E_\alpha(\lambda t^\alpha)\int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(T-s)^\alpha)\,g(s,r(s))\,ds\right]\end{aligned}$$

for all $t\in J$. Thus, the function r is a solution of BVPHDEF (2.3) on J. Finally, from inequality (5.3) it follows that $u(t)\le r(t)$ for all $t\in J$. Hence, BVPHDEF (2.3) has a maximal solution on J. This completes the proof. □

6. Comparison theorems

The main problem of the differential inequalities is to estimate a bound for the solution set for the differential inequality related to BVPHDEF (2.3). In this section, we prove that the maximal and minimal solutions serve as bounds for the solutions of the related differential inequality to BVPHDEF (2.3) on $J=[0,T]$.

Theorem 6.1 *Assume that hypotheses $(A_0)-(A_2)$ and $(B_1)-(B_3)$ and condition (3.3) hold and a,b,c are real constants with $a+b\neq 0$ and $\frac{b}{a+b}\le 0$. Assume that there exists a real number $M>0$ such that*

$$g(t,x_1)-g(t,x_2)\le\frac{M}{1+t^\alpha}\left(\frac{x_1}{f(t,x_1)}-\frac{x_2}{f(t,x_2)}\right) \qquad a.e.\ t\in J$$

for all $x_1,x_2\in\mathbb{R}$ with $x_1\ge x_2$, where $M\le\Gamma(1+\alpha)$. Moreover, if there exists a function $u\in C(J,\mathbb{R})$ such that

$$\begin{cases} D^\alpha\left(\frac{u(t)}{f(t,x(t))}\right)-\lambda\left(\frac{u(t)}{f(t,u(t))}\right)\le g(t,u(t)) & a.e.\ t\in J=[0,T],\\ a\frac{u(0)}{f(0,u(0))}+b\frac{u(T)}{f(T,u(T))}\le c,\end{cases}\tag{6.1}$$

then

$$u(t)\le r(t) \qquad a.e.\ t\in J \tag{6.2}$$

where r is a maximal solution of BVPHDEF (2.3) on J.

Proof: Let ϵ be arbitrarily small.Accoeding to the theorem (5.2) $r(t,\epsilon)$ is a maximal solution of BVPHDEF (5.2) so that the limit

$$r(t) = \lim_{\epsilon \to 0} r(t,\epsilon) \tag{6.3}$$

is uniform on J and the function r is a maximal solution of BVPHDEF (2.3) on J. Thus, we obtain

$$\begin{cases} D^{\alpha}\left(\frac{r(t,\epsilon)}{f(t,r(t,\epsilon))}\right) - \lambda\left(\frac{r(t,\epsilon)}{f(t,r(t,\epsilon))}\right) = g(t,r(t,\epsilon)) + \epsilon & \text{a.e. } t \in J = [0,T], \\ a\frac{r(0,\epsilon)}{f(0,r(0,\epsilon))} + b\frac{r(T,\epsilon))}{f(T,r(T,\epsilon))} = c + \epsilon, \end{cases} \tag{6.4}$$

From the above equality it follows that

$$\begin{cases} D^{\alpha}\left(\frac{x(t)}{f(t,x(t))}\right) - \lambda\left(\frac{x(t)}{f(t,x(t))}\right) > g(t,x(t)) & \text{a.e. } t \in J = [0,T], \\ a\frac{x(0)}{f(0,x(0))} + b\frac{x(T)}{f(T,x(T))} = c + \epsilon, \end{cases} \tag{6.5}$$

From inequalities (6.1) and (6.5) we have

$$a\frac{x(0)}{f(0,x(0))} + b\frac{x(T)}{f(T,x(T))} \le c < c + \epsilon = a\frac{r(0,\epsilon)}{f(0,r(0,\epsilon))} + b\frac{r(T,\epsilon))}{f(T,r(T,\epsilon))}$$

According to the theorem (4.3) we infer that $u(t) < r(t,\epsilon)$ for all $t \in J$. Then , by the limit (6.3) inequality (6.2) holds on J.This completes the proof. □

Theorem 6.2 *Assume that hypotheses $(A_0) - (A_2)$ and $(B_1) - (B_3)$ and condition (3.3) hold and a,b,c are real constants with $a+b \neq 0$ and $\frac{b}{a+b} \le 0$. Assume that there exists a real number $M > 0$ such that*

$$g(t,x_1) - g(t,x_2) \le \frac{M}{1+t^{\alpha}}\left(\frac{x_1}{f(t,x_1)} - \frac{x_2}{f(t,x_2)}\right) \qquad a.e.\ t \in J$$

for all $x_1, x_2 \in \mathbb{R}$ with $x_1 \ge x_2$, where $M \le \Gamma(1+\alpha)$. Moreover, if there exists a function $\eta \in C(J,\mathbb{R})$ such that

$$\begin{cases} D^{\alpha}\left(\frac{\eta(t)}{f(t,\eta(t))}\right) - \lambda\left(\frac{\eta(t)}{f(t,\eta(t))}\right) \ge g(t,\eta(t)) & a.e.\ t \in J = [0,T], \\ a\frac{\eta(0)}{f(0,\eta(0))} + b\frac{\eta(T)}{f(T,\eta(T))} > c, \end{cases}$$

then

$$\nu(t) \le \eta(t)$$

for all $t \in J$, where ν is a minimal solution of BNVPHDEF (2.3) on J.

Notice that Theorem (6.1) is used to prove the boundness and uniqueness of the solutions for BVPHDEF (2.3) on J.An outcome in this direction is as following.

Theorem 6.3 *Suppose that hypotheses $(A_0) - (A_2)$ and $(B_1) - (B_3)$ and condition (3.3) hold and a,b,c are real constants with $a+b \neq 0$ and $\frac{b}{a+b} \le 0$. Assume that there exists a real number $M > 0$ such that*

$$g(t,x_1) - g(t,x_2) \le \frac{M}{1+t^{\alpha}}\left(\frac{x_1}{f(t,x_1)} - \frac{x_2}{f(t,x_2)}\right) \qquad a.e.\ t \in J$$

for all $x_1, x_2 \in \mathbb{R}$ with $x_1 \ge x_2$, where $M \le \Gamma(1+\alpha)$. If an identically zero function is the only solution of the differential eqution

$$D^{\alpha}m(t) - \lambda m(t) = \frac{M}{1+t^{\alpha}}m(t) \qquad a.e.\ t \in J, \qquad am(0) + bm(T) = 0 \tag{6.6}$$

then BVPHDEF (2.3) has a unique solution on J.

Proof: According to the theorem (3.2) , BNPHDEF (2.3) has a solution on J.Assume that there exist two solutions υ_1 and υ_2 on J such that $\upsilon_1 > \upsilon_2$. Define a function $m : J \to \mathbb{R}$ by

$$m(t) = \frac{\upsilon_1(t)}{f(t, \upsilon_1(t))} - \frac{\upsilon_2(t)}{f(t, \upsilon_2(t))}$$

From hypothesis (A_0), we infer that $m(t) > 0$ thus we have

$$\begin{aligned}
D^\alpha m(t) - \lambda m(t) &= D^\alpha \left(\frac{\upsilon_1(t)}{f(t, \upsilon_1(t))} - \frac{\upsilon_2(t)}{f(t, \upsilon_2(t))} \right) - \lambda \left(\frac{\upsilon_1(t)}{f(t, \upsilon_1(t))} - \frac{\upsilon_2(t)}{f(t, \upsilon_2(t))} \right) \\
&= D^\alpha \left(\frac{\upsilon_1(t)}{f(t, \upsilon_1(t))} \right) - \lambda \frac{\upsilon_1(t)}{f(t, \upsilon_1(t))} - \left(D^\alpha \left(\frac{\upsilon_2(t)}{f(t, \upsilon_2(t))} \right) - \lambda \frac{\upsilon_2(t)}{f(t, \upsilon_2(t))} \right) \\
&= g(t, \upsilon_1(t)) - g(t, \upsilon_2(t)) \\
&\leq \frac{M}{1 + t^\alpha} \left(\frac{\upsilon_1}{f(t, \upsilon_1)} - \frac{\upsilon_2}{f(t, \upsilon_2)} \right) \\
&= \frac{M}{1 + t^\alpha} m(t)
\end{aligned}$$

Since υ_1 and υ_2 are solutions of BVPHDEF (2.3) , we have

$$a \frac{\upsilon_1(0)}{f(0, \upsilon_1(0))} + b \frac{\upsilon_1(T)}{f(T, \upsilon_1(T))} = c = a \frac{\upsilon_2(0)}{f(0, \upsilon_2(0))} + b \frac{\upsilon_2(T)}{f(T, \upsilon_2(T))}$$

thus

$$a \left(\frac{\upsilon_1(0)}{f(0, \upsilon_1(0))} - \frac{\upsilon_2(0)}{f(0, \upsilon_2(0))} \right) = b \left(\frac{\upsilon_2(T)}{f(T, \upsilon_2(T))} - \frac{\upsilon_1(T)}{f(T, \upsilon_1(T))} \right)$$

i.e

$$am(0) = -bm(T)$$

then

$$am(0) + bm(T) = 0$$

Now, By applying theorem (6.1) for $f(t, x) = 1$ and $c = 0$ we obtain $m(t) \leq 0$ for all $t \in J$, Where an identically zero function is the only solution of the differential equation (6.6)
$m(t) \leq 0$ is a contradiction with $m(t) > 0$. Then we obtain $\upsilon_1 = \upsilon_2$. This completes the proof. □

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