Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–15.
ISSN-0037-8712
doi:10.5269/bspm.65679

Some characterizations of translation surface generated by a non-null curve and a curve on a spacelike surface in Minkowski 3-space *

Akhilesh Yadav and Ajay Kumar Yadav†

ABSTRACT: In this paper, we study translation surfaces generated by a non-null curve in $\mathbb{E}^3_1$ and a curve lying on a spacelike surface in Minkowski 3-space and obtain necessary and sufficient conditions for such surfaces to be flat or minimal. Further, we obtain normal curvature, geodesic curvature and geodesic torsion of the generating curves and find necessary and sufficient conditions for these curves to be geodesic, asymptotic line and line of curvature. Finally, we study some special cases of such translation surfaces according to the angle between the normal of the translation surface and the respective normals of the generating curves.

Key Words: Translation surface, Gauss curvature, mean curvature, geodesic curves, asymptotic curves, line of curvature.

Contents

1. Introduction

A surface of revolution is formed by translating a curve in such a way that a point on the curve moves along a circle. Translation surfaces are the generalization of the surfaces of revolution which are formed by as translating a curve along any other arbitrary curve which has a intersection with the first one. The parametrization of generalized type of a translation surface in 3-dimensional Euclidean space is given by

$$X(u,v) = \alpha(u) + \beta(v),$$

where α and β are space curves, called generating curves. Translation surface which is known as double curve in differential geometry are base for roofing structures. The construction and design of free form glass roofing structures are generally created with the help of curved (formed) glass panes or planar triangular glass facets.

Translation surfaces has been studied in Euclidean space as well as semi-Euclidean space by many authors. For instance, In [4], Liu obtained some characterizations about the translation surfaces with constant mean curvature or constant Gauss curvature in 3-dimensional Euclidean space $\mathbb{E}^3$ and 3-dimensional Minkowski space $\mathbb{E}^3_1$. In [3], Çetin and Tunçer studied surfaces parallel to translation surfaces in Euclidean 3-space. In [1], Ali et al., gave some results on some special points of the translation surfaces in $\mathbb{E}^3$. Since the translation surfaces are surfaces produced by two space curves, some basic calculations of the surface can be stated in terms of Frenet vectors and curvatures of the space curves. In [2], Çetin and Önder investigated translation surfaces according to Frenet frames in Minkowski 3-space and studied some properties of these surfaces. Furthermore, they calculated first fundamental form, second fundamental form, Gaussian curvature and mean curvature of the translation surface. Finally, they gave the

* The project is supported by Council of Scientific & Industrial Research, Government of India [09/013(0953)/2020-EMR-I].

† Corresponding author

Submitted November 02, 2022. Published December 29, 2024
2010 *Mathematics Subject Classification:* 53A04, 53A05.

conditions for the generator curves of the translation surface being a geodesic, an asymptotic line and a principal line. In [9], we studied the translation surfaces generated by spherical indicatrices of timelike curves in Minkowski 3-space.

Motivated by these studies, we study translation surfaces generated by a non-null space curve and a curve lying on a spacelike surface in Minkowski 3-space according to Frenet frame of the first curve and Darboux frame of the second one. We calculate normal curvature, geodesic curvature and geodesic torsion of the generating curves and obtain necessary and sufficient conditions for these curves to be geodesic, asymptotic and line of curvature. Finally, we study some special cases of such translation surfaces according to the angle between the normal of the surface and the normals of the generating curves. In first subsection of the section 3, we deal with a translation surface M generated by an arbitrary spacelike space curve α and an arbitrary curve β on a spacelike surface σ while in the second subsection we deal with a translation surface M generated by an arbitrary timelike space curve α and an arbitrary curve β on the spacelike surface σ.

2. Preliminaries

The Minkowski 3-space denoted by $\mathbb{E}^3_1$ is a three dimensional real vector space $\mathbb{R}^3$ endowed with the metric tensor $\langle .,. \rangle = -dx^2 + dy^2 + dz^2$. The (Lorentzian) scalar and cross product are defined by

$$\begin{cases} \langle x, y \rangle = -x_1y_1 + x_2y_2 + x_3y_3, \\ x \times y = (x_2y_3 - x_3y_2, x_1y_3 - x_3y_1, x_2y_1 - x_1y_2), \end{cases} \tag{2.1}$$

where $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$ belong to $\mathbb{E}^3_1$. This space is also known as Lorentz-Minkowski space. A vector $x \in \mathbb{E}^3_1$ is said to be spacelike when $\langle x, x \rangle > 0$ or $x = 0$, timelike when $\langle x, x \rangle < 0$ and lightlike(null) when $\langle x, x \rangle = 0$. A curve in $\mathbb{E}^3_1$ is called spacelike, timelike or lightlike when the velocity vector of the curve is spacelike, timelike or lightlike, respectively.

Let $\gamma = \gamma(s) : I \to \mathbb{E}^3_1$ be a regular curve. The curve γ is said to be a unit speed (or parameterized by the arc-length parameter s) if $\langle \gamma'(s), \gamma'(s) \rangle = \pm 1$ for any $s \in I$. Let $\{t(s), n(s), b(s)\}$ be the moving Frenet frame of γ.

For the timelike curve γ the Frenet frame satisfy following condition:

$$\begin{cases} \langle t, t \rangle = -\langle n, n \rangle = -\langle b, b \rangle = -1, \\ \langle t, n \rangle = \langle t, b \rangle = \langle b, n \rangle = 0, \\ t \times n = b, \; n \times b = -t, \; b \times t = n, \\ det(t, n, b) = 1, \end{cases} \tag{2.2}$$

and the Frenet-Serret equations are given by

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}, \tag{2.3}$$

where the $'$ denotes the derivative with respect to s. κ and τ are the curvature and torsion of the curve, respectively.

For a spacelike curve γ the Frenet frame satisfies following condition:

$$\begin{cases} t \times n = b, \; n \times b = -\epsilon t, \; b \times t = -n, \\ det(t, n, b) = -\epsilon, \end{cases} \tag{2.4}$$

where $\epsilon = \pm 1$.

The Frenet-Serret equations are given by

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\epsilon\kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix},$$

where $\langle t,t\rangle = 1$, $\langle n,n\rangle = \epsilon$, $\langle b,b\rangle = -\epsilon$, $\langle t,b\rangle = \langle t,n\rangle = \langle n,b\rangle = 0$. When $\epsilon = 1$, $\gamma(s)$ is a spacelike curve with spacelike principal normal n and timelike binormal b while if ϵ = -1 then γ is a spacelike curve with timelike principal normal n and spacelike binormal b.

Definition 2.1 *[8] Let v and w be two spacelike vectors in $\mathbb{E}_1^3$. Then there is a unique non-negative real number $\theta \geq 0$ such that $\langle v,w\rangle = \|v\|\|w\|\cos\theta$.*

Definition 2.2 *[8] Let v be a spacelike vector and w be a timelike vector in $\mathbb{E}_1^3$. Then there is a unique non-negative real number $\theta \geq 0$, such that $\langle v,w\rangle = \|v\|\|w\|\sinh\theta$.*

Definition 2.3 *[6] Let v be a timelike vector and w be a timelike vector in same time cone of $\mathbb{E}_1^3$, i.e. $\langle v,w\rangle < 0$. Then there is a unique non-negative real number $\theta \geq 0$, such that $\langle v,w\rangle = -\|v\|\|w\|\cosh\theta$.*

A surface in $\mathbb{E}_1^3$ is said to be a spacelike, timelike or lightlike if the metric on the surface is positive definite, indefinite or degenerate, respectively. Type of the surface can also be expressed in terms of the causal character of the normal on the surface by the following lemma.

Lemma 2.1 *[5] A surface in Minkowski 3-space is spacelike, timelike or lightlike if and only if, at every point of the surface there exists a normal that is timelike, spacelike or lightlike, respectively.*

Let M be a smooth spacelike surface in $\mathbb{E}_1^3$ and $\gamma : I \to M \subset \mathbb{E}_1^3$ be a unit speed spacelike curve on the surface. Then the Darboux frame $\{T, B = N \times T, N\}$ along the curve is well-defined and positively oriented along the curve, where T is the tangent vector field of γ, N is the unit normal of M and B is intrinsic normal of γ. The Darboux equations are given by

$$T^{'} = \kappa_g B + \kappa_n N,\ B^{'} = -\kappa_g T + \tau_g N,\ N^{'} = \kappa_n T + \tau_g B, \tag{2.5}$$

where κ_g, κ_n and τ_g are the geodesic curvature, normal curvature and geodesic torsion, respectively, and $\langle T,T\rangle = \langle B,B\rangle = 1$ and $\langle N,N\rangle = -1$, $\langle n,n\rangle = 1$.

Let $M : X = X(u,v) \in \mathbb{E}_1^3$ be a regular surface. The unit normal vector field of the surface M is determined by

$$N = \frac{X_u \times X_v}{\|X_u \times X_v\|}, \tag{2.6}$$

where X_u and X_v are derivatives of X with respect to u and v, respectively. The coefficients of the first fundamental form and second fundamental form are given by

$$E = \langle X_u, X_u\rangle,\ F = \langle X_u, X_v\rangle,\ G = \langle X_v, X_v\rangle$$

and

$$l = \langle X_{uu}, N\rangle,\ m = \langle X_{uv}, N\rangle,\ n = \langle X_{vv}, N\rangle.$$

Gaussian and mean curvatures of the surface M are expressed as follows [7]

$$K = \langle N,N\rangle \frac{ln - m^2}{EG - F^2} \tag{2.7}$$

and

$$H = \frac{1}{2}\frac{En + Gl - 2Fm}{|EG - F^2|}, \tag{2.8}$$

respectively.

Definition 2.4 *A surface in $\mathbb{E}_1^3$ is called flat when the Gaussian curvature vanishes and it is called minimal when the mean curvature vanishes.*

3. Translation surface generated by a space curve in $\mathbb{E}_1^3$ and a curve lying on a spacelike surface in Minkowski 3-space $\mathbb{E}_1^3$

Let $\alpha : I \subset R \to \mathbb{E}_1^3$ be a space curve with arc-length parameter u and $\beta : J \subset R \to \sigma \subset \mathbb{E}_1^3$ be a curve with arc-length parameter v on a spacelike surface σ in $\mathbb{E}_1^3$. Since σ is spacelike, the curve β is also spacelike. Let $\{t, n, b, \kappa_\alpha, \tau_\alpha\}$ be the Frenet apparatus of the curve α and let $\{T, B, N\}$ be the Darboux frame of β with κ_n, κ_g, τ_g be the normal curvature, geodesic curvature and geodesic torsion of β on σ respectively. In this section, we examine the translation surface generated by the curves α and β and find out some characterizations of the surface as well as of the generating curves of the surface.

3.1. Translation surface generated by a spacelike curve and a curve lying on a spacelike surface

The translation surface generated by a spacelike curve α and a spacelike curve β lying on a spacelike surface σ is given by

$$M : X(u, v) = \alpha(u) + \beta(v). \tag{3.1}$$

Differentiating the above equation (3.1) with respect to u and v, we obtain $X_u = t$ and $X_v = T$.

The unit normal vector $\bar{N}$ of the surface M is given by

$$\bar{N}(u, v) = \frac{X_u \times X_v}{\|X_u \times X_v\|}, \tag{3.2}$$

where $X_u \times X_v = t \times T$ and $\|X_u \times X_v\| = \sqrt{-\epsilon(EG - F^2)}$, where $\epsilon = \langle \bar{N}, \bar{N} \rangle$.

By Definition 2.1, we have $\langle t, T \rangle = \cos\theta$ and the coefficients of first fundamental form are obtained as follows

$$E = 1,\ F = \cos\theta,\ G = 1, \tag{3.3}$$

where θ is the smooth angle function between t and T. $EG - F^2 = 1 - \cos^2\theta = \sin^2\theta > 0$ shows that the surface M is spacelike and hence the unit normal $\bar{N}$ is timelike, i.e. $\langle \bar{N}, \bar{N} \rangle = -1$.

Thus, we obtain the timelike unit normal,

$$\bar{N}(u, v) = \frac{t \times T}{\sin\theta}, \tag{3.4}$$

so that $\langle \bar{N}, t \rangle = \langle \bar{N}, T \rangle = 0$ and necessarily θ is non-zero otherwise there does not exist such surface.

Now, we have two cases according to the causality of the principal normal n of the curve α,

Case (i): when n is spacelike, then $\langle t, t \rangle = 1, \langle n, n \rangle = 1, \langle b, b \rangle = -1$. Suppose the angle between n and $\bar{N}$ is ϕ and the angle between N and $\bar{N}$ is ψ then $\bar{N}$ can be expressed as follows [10]

$$\bar{N}(u, v) = \sinh\phi\, n + \cosh\phi\, b,$$
$$\bar{N}(u, v) = \cosh\psi\, N + \sinh\psi\, B.$$

The coefficients of second fundamental form of the surface M are given by

$$\begin{cases} l = \kappa_\alpha \sinh\phi, \\ m = 0, \\ n = \kappa_g \sinh\psi - \kappa_n \cosh\psi. \end{cases} \tag{3.5}$$

Now, using the equations (2.7), (2.8) and above calculations, we obtain the following results.

Theorem 3.1 *The Gaussian curvature and the mean curvature of the translation surface M is given as follows*

$$K = -\frac{(\kappa_\alpha \sinh\phi)(\kappa_g \sinh\psi - \kappa_n \cosh\psi)}{\sin^2\theta},$$

$$H = \frac{(\kappa_\alpha \sinh\phi) + (\kappa_g \sinh\psi - \kappa_n \cosh\psi)}{2\sin^2\theta}.$$

Corollary 3.1 *The timelike translation surface M is flat if and only if*

$$\kappa_\alpha \sinh\phi = 0 \text{ or } \kappa_g \sinh\psi - \kappa_n \cosh\psi = 0.$$

Proof: By putting $K = 0$ in the Theorem 3.1, we get the desired result. □

Corollary 3.2 *The translation surface M is minimal if and only if*

$$\kappa_\alpha \sinh\phi = -\kappa_g \sinh\psi + \kappa_n \cosh\psi.$$

Proof: By putting $H = 0$ in the Theorem 3.1, we get the required result. □

Theorem 3.2 *The normal curvature, geodesic curvature and geodesic torsion of the curve α lying on the translation surface M are found as follows*

$$\begin{cases} \kappa_n^\alpha = \kappa_\alpha \sinh\phi, \\ \kappa_g^\alpha = -\kappa_\alpha \cosh\phi, \\ \tau_g^\alpha = -\tau_\alpha - \phi_u, \end{cases}$$

where κ_α and τ_α are curvature and torsion of the curve α, respectively.

Proof: We have $\kappa_n^\alpha = \langle \alpha'', \bar{N}\rangle$ and since $\alpha' = t$, $\alpha'' = \kappa_\alpha n$ and $\bar{N} = \sinh\phi\ n + \cosh\phi\ b$, we get $\kappa_n^\alpha = \langle \kappa_\alpha n, \sinh\phi\ n + \cosh\phi\ b\rangle = \kappa_\alpha \sinh\phi$.

Also, $\kappa_g^\alpha = \langle \alpha'', \bar{N} \times t\rangle$, where $\bar{N} \times t = \sinh\phi\ (n \times t) + \cosh\phi\ (b \times t) = -\sinh\phi\, b - \cosh\phi\, n$. So, we get $\kappa_g^\alpha = \langle \kappa_\alpha n, -\sinh\phi\, b - \cosh\phi\, n\rangle = -\kappa_\alpha \cosh\phi$.

Finally, $\tau_g^\alpha = \langle \bar{N}_u, \bar{N} \times t\rangle$, where $\bar{N}_u = \frac{\partial \bar{N}}{\partial u} = \phi_u \cosh\phi\ n + \phi_u \sinh\phi\ b + \sinh\phi\ n_u + \cosh\phi\ b_u = -\kappa_\alpha \sinh\phi\ t + (\phi_u + \tau_\alpha)\cosh\phi\ n + (\phi_u + \tau_\alpha)\sinh\phi\ b$. Hence $\tau_g^\alpha = \langle \bar{N}_u, -\sinh\phi\ b - \cosh\phi\ n\rangle = -(\phi_u + \tau_\alpha)\cosh^2\phi + (\phi_u + \tau_\alpha)\sinh^2\phi = (-\phi_u - \tau_\alpha)(\cosh^2\phi - \sinh^2\phi) = -\phi_u - \tau_\alpha$. □

Case (ii): When n is timelike, then $\langle t, t\rangle = 1, \langle n, n\rangle = -1, \langle b, b\rangle = 1$. Suppose the angle between n and $\bar{N}$ is ϕ and the angle between N and $\bar{N}$ is ψ then $\bar{N}$ can be expressed as follows [10]

$$\bar{N}(u, v) = \cosh\phi\ n + \sinh\phi\ b,$$
$$\bar{N}(u, v) = \cosh\psi\ N + \sinh\psi\ B.$$

The coefficients of second fundamental form of the surface M are given by

$$\begin{cases} l = -\kappa_\alpha \cosh\phi, \\ m = 0, \\ n = \kappa_g \sinh\psi - \kappa_n \cosh\psi. \end{cases} \tag{3.6}$$

Now, using the equations (2.7), (2.8) and above calculations, we obtain the following results.

Theorem 3.3 *The Gaussian curvature and the mean curvature of the spacelike translation surface M are given as follows, respectively*

$$K = \frac{(\kappa_\alpha \cosh\phi)(\kappa_g \sinh\psi - \kappa_n \cosh\psi)}{\sin^2\theta},$$

$$H = \frac{-(\kappa_\alpha \cosh\phi) + (\kappa_g \sinh\psi - \kappa_n \cosh\psi)}{2\sin^2\theta}.$$

Corollary 3.3 *The spacelike translation surface M is flat if and only if*

$$\kappa_\alpha = 0 \text{ or } \kappa_g \sinh\psi - \kappa_n \cosh\psi = 0.$$

Proof: By putting $K = 0$ in the Theorem 3.3, we get the desired result. □

Corollary 3.4 *The spacelike translation surface M is minimal if and only if*

$$\kappa_\alpha \cosh\phi = \kappa_g \sinh\psi - \kappa_n \cosh\psi.$$

Proof: By putting $H = 0$ in the Theorem 3.3, we get the required result. □

Theorem 3.4 *The normal curvature, geodesic curvature and geodesic torsion of the curve α lying on the translation surface M are found as follows*

$$\begin{cases} \kappa_n^\alpha = -\kappa_\alpha \cosh\phi, \\ \kappa_g^\alpha = \kappa_\alpha \sinh\phi, \\ \tau_g^\alpha = -\tau_\alpha - \phi_u, \end{cases}$$

where κ_α and τ_α are curvature and torsion of the curve α, respectively.

Proof: We have that $\kappa_n^\alpha = \langle \alpha'', \bar{N}\rangle$ and since $\alpha' = t$, $\alpha'' = \kappa_\alpha n$ and $\bar{N} = \cosh\phi\; n + \sinh\phi\; b$, we get $\kappa_n^\alpha = \langle \kappa_\alpha n, \cosh\phi\; n + \sinh\phi\; b\rangle = -\kappa_\alpha \cosh\phi$.

Also, $\kappa_g^\alpha = \langle \alpha'', \bar{N}\times t\rangle$, where $\bar{N}\times t = \cosh\phi\;(n\times t) + \sinh\phi\;(b\times t) = -\cosh\phi\; b - \sinh\phi\; n$. So, we get $\kappa_g^\alpha = \langle \kappa_\alpha n, -\cosh\phi\; b - \sinh\phi\; n\rangle = \kappa_\alpha \sinh\phi$.

Finally, $\tau_g^\alpha = \langle \bar{N}_u, \bar{N}\times t\rangle$, where $\bar{N}_u = \frac{\partial \bar{N}}{\partial u} = \phi_u \cosh\phi\; b + \phi_u \sinh\phi\; n + \cosh\phi\; n_u + \sinh\phi\; b_u = \kappa_\alpha \cosh\phi\; t + (\phi_u + \tau_\alpha)\sinh\phi\; n + (\phi_u + \tau_\alpha)\cosh\phi\; b$. Hence $\tau_g^\alpha = \langle \bar{N}_u, -\cosh\phi\; b - \sinh\phi\; n\rangle = (\phi_u + \tau_\alpha)\sinh^2\phi - (\phi_u + \tau_\alpha)\cosh^2\phi = (-\phi_u - \tau_\alpha)(\cosh^2\phi - \sinh^2\phi) = -\phi_u - \tau_\alpha$. □

Theorem 3.5 *The normal curvature, geodesic curvature and geodesic torsion of the curve β lying on the translation surface M (in both cases) are found as follows*

$$\begin{cases} \kappa_n^\beta = \kappa_g \sinh\psi - \kappa_n \cosh\psi, \\ \kappa_g^\beta = \kappa_g \cosh\psi - \kappa_n \sinh\psi, \\ \tau_g^\beta = \tau_g + \psi_v, \end{cases}$$

where κ_n, κ_g and τ_g are the normal curvature, geodesic curvature and geodesic torsion of the curve β lying on the surface σ, respectively.

Proof: We have $\kappa_n^\beta = \langle \beta'', \bar{N}\rangle$, and since $\beta' = T$, $\beta'' = \kappa_g B + \kappa_n N$ and $\bar{N} = \cosh\psi\; N + \sinh\psi\; B$, we get $\kappa_n^\beta = \langle \kappa_g B + \kappa_n N, \cosh\psi\; N + \sinh\psi\; B\rangle = \kappa_g \sinh\psi - \kappa_n \cosh\psi$.

Also, $\kappa_g^\beta = \langle \beta'', \bar{N}\times T\rangle$, where $\bar{N}\times T = \cosh\psi\;(N\times T) + \sinh\psi\;(B\times T) = \cosh\psi\; B + \sinh\psi\; N$. So, we get $\kappa_g^\beta = \langle \kappa_g B + \kappa_n N, \cosh\psi\; B + \sinh\psi\; N\rangle = \kappa_g \cosh\psi - \kappa_n \sinh\psi$.

Finally, $\tau_g^\beta = \langle \bar{N}_v, \bar{N}\times T\rangle$, where $\bar{N}_v = \frac{\partial \bar{N}}{\partial v} = \psi_v \cosh\psi\; B + \psi_v \sinh\psi\; N + \cosh\psi\; N_v + \sinh\psi\; B_v = (\kappa_n \cosh\psi - \kappa_g \sinh\psi)T + (\psi_v + \tau_g)\cosh\psi\; B + (\psi_v + \tau_g)\sinh\psi\; N$. Hence, $\tau_g^\beta = \langle \bar{N}_u, \cosh\psi\; B + \sinh\psi\; N\rangle = (\psi_v + \tau_g)\cosh^2\psi - (\psi_v + \tau_g)\sinh^2\psi = \psi_v + \tau_g$. □

Corollary 3.5 *The relation between Gaussian curvature and mean curvature of the translation surface M and normal curvatures of the generating curves are given as follows*

$$K = -\frac{\kappa_n^\alpha \kappa_n^\beta}{\sin^2\theta},$$

$$H = \frac{\kappa_n^\alpha + \kappa_n^\beta}{2\sin^2\theta}.$$

Proof: Using Theorem 3.1, 3.2, 3.3, we get the desired result. □

Theorem 3.6 *The curve α with spacelike principal normal is an asymptotic curve on M if and only if either α is a straight line or the binormal b of the curve α is parallel to the unit normal $\bar{N}$ of the surface M.*

Proof: From Theorem 3.2, we have $\kappa_n^\alpha = \kappa_\alpha \sinh\phi$. So α is an asymptotic curve on M if and only if $\kappa_\alpha \sinh\phi = 0$ if and only if either $\kappa_\alpha = 0$ or $\sinh\phi = 0$ if and only if either α is a straight line or the binormal b of the curve α is parallel to the unit normal $\bar{N}$ of the surface M. □

Theorem 3.7 *The curve α with spacelike principal normal is a geodesic curve on M if and only if α is a straight line.*

Proof: From Theorem 3.2, we have $\kappa_g^\alpha = -\kappa_\alpha \cosh\phi$. We know that a curve on a surface is a geodesic curve if and only if the geodesic curvature is zero, so α is a geodesic curve on M if and only if $\kappa_\alpha \cosh\phi = 0$ if and only if $\kappa_\alpha = 0$, since $\cosh\phi$ is never zero. Hence the curve α with spacelike principal normal is a geodesic curve on M if and only if α is a straight line. □

Theorem 3.8 *The curve α with timelike principal normal is an asymptotic curve on M if and only if α is a straight line.*

Proof: From Theorem 3.4, we have $\kappa_n^\alpha = -\kappa_\alpha \cosh\phi$. We know that a curve on a surface is an asymptotic curve if and only if the normal curvature is zero, so α is an asymptotic curve on M if and only if $\kappa_\alpha \cosh\phi = 0$ if and only if $\kappa_\alpha = 0$, since $\cosh\phi$ is never zero. Hence the curve α with timelike principal normal is an asymptotic curve on M if and only if α is a straight line. □

Theorem 3.9 *The curve α with timelike principal normal is a geodesic curve on M if and only if either α is a straight line or the principal normal n of the curve α is parallel to the normal $\bar{N}$ of the surface M.*

Proof: From Theorem 3.4, we have $\kappa_g^\alpha = \kappa_\alpha \sinh\phi$. So α is a geodesic curve on M if and only if $\kappa_\alpha \sinh\phi = 0$ if and only if either $\kappa_\alpha = 0$ or $\sinh\phi = 0$ if and only if either α is a straight line or the principal normal n and the normal $\bar{N}$ to the surface M are parallel together. □

Theorem 3.10 *The curve β is an asymptotic curve on M if and only if either β is a straight line or the angle between the normal N to the surface σ and the normal $\bar{N}$ to the surface M is given as follows*

$$\tanh\psi = \frac{\kappa_n}{\kappa_g}.$$

Proof: From Theorem 3.5, we have $\kappa_n^\beta = \kappa_g \sinh\psi - \kappa_n \cosh\psi$. So β is an asymptotic curve on M if and only if $\kappa_g \sinh\psi - \kappa_n \cosh\psi = 0$ if and only if $\tanh\psi = \frac{\kappa_n}{\kappa_g}$, when $\kappa_g \neq 0$. In case $\kappa_g = 0$, we get $\kappa_n \cosh\psi = 0$, which implies $\kappa_n = 0$, then the curve β is a straight line. □

Theorem 3.11 *The curve β is a geodesic curve on M if and only if either β is a straight line or the angle between the normal N to the surface σ and the normal $\bar{N}$ to the surface M is given as follows*

$$\tanh\psi = \frac{\kappa_g}{\kappa_n}.$$

Proof: From Theorem 3.5, we have $\kappa_g^\beta = \kappa_g \cosh\psi - \kappa_n \sinh\psi$. So β is a geodesic curve on M if and only if $\kappa_g \cosh\psi - \kappa_n \sinh\psi = 0$ if and only if $\tanh\psi = \frac{\kappa_g}{\kappa_n}$, when $\kappa_n \neq 0$. In case $\kappa_n = 0$, we get $\kappa_g \cosh\psi = 0$, which implies $\kappa_g = 0$, then the curve β is a straight line. □

Theorem 3.12 *The curve α with timelike/spacelike principal normal is a line of curvature on M if and only if the angle between the normal $\bar{N}$ to the surface M and the principal normal n of α is given as $\phi = -\int \tau_\alpha du + c(v)$.*

Proof: From Theorem 3.2 and 3.4, we have $\tau_g^\alpha = -\tau_\alpha - \phi_u$. We know that a curve on a surface is a line of curvature if and only if geodesic torsion is zero, so α is a line of curvature on M if and only if $\tau_\alpha + \phi_u = 0$ if and only if $\phi_u = -\tau_\alpha$, integrating both side with respect to u we get $\phi = -\int \tau_\alpha du + c(v)$, where $c(v)$ is some function of v. □

Theorem 3.13 *The curve β is a line of curvature on M if and only if the angle between the normal $\bar{N}$ to the surface M and the normal N to the surface σ along β is given as $\psi = -\int \tau_g dv + c(u)$.*

Proof: Using Theorem 3.3, we can prove it in similar way to the above theorem. □

Theorem 3.14 *If the curve α with spacelike principal normal is an asymptotic curve on the translation surface M then α is a planar curve.*

Proof: We have, $\bar{N} = \frac{t \times T}{\sin\theta}$ also $\bar{N} = \sinh\phi\, n + \cosh\phi\, b$, which implies $\sinh\phi = \langle \bar{N}, n\rangle = \frac{1}{\sin\theta}\langle t \times T, n\rangle = \frac{1}{\sin\theta}\langle n \times t, T\rangle = -\frac{1}{\sin\theta}\langle b, T\rangle$. Differentiating both sides with respect to u, we get

$$\begin{aligned}
\cosh\phi\ \phi_u &= \frac{\cot\theta}{\sin\theta}\theta_u\langle b, T\rangle - \frac{1}{\sin\theta}\langle b', T\rangle \\
&= \frac{\cot\theta}{\sin\theta}\theta_u\langle b, T\rangle - \frac{\tau_\alpha}{\sin\theta}\langle n, T\rangle \\
&= -\theta_u \cot\theta \sinh\phi - \tau_\alpha \cosh\phi. \qquad (3.7)
\end{aligned}$$

Thus, if α is an asymptotic curve then we have $\sinh\phi = 0$, which implies that $\tau_\alpha = -\phi_u = 0$, and hence α is a planar curve.

□

Theorem 3.15 *If the curve α with timelike principal normal is a geodesic curve on the translation surface M then either θ is a function of v only or $\theta = 90°$. Conversely, if either θ is a function of v only or $\theta = 90°$ then the curve α with timelike principal normal is a geodesic curve or a line of curvature on the translation surface M.*

Proof: We have, $\bar{N} = \frac{t \times T}{\sin\theta}$ also $\bar{N} = \cosh\phi\, n + \sinh\phi\, b$, which implies $\cosh\phi = \langle \bar{N}, n\rangle = \frac{1}{\sin\theta}\langle t \times T, n\rangle = \frac{1}{\sin\theta}\langle n \times t, T\rangle = -\frac{1}{\sin\theta}\langle b, T\rangle$. Differentiating both sides with respect to u, we obtain

$$\begin{aligned}
\sinh\phi\ \phi_u &= -\frac{\cot\theta}{\sin\theta}\theta_u\langle b, T\rangle + \frac{1}{\sin\theta}\langle b', T\rangle \\
&= -\frac{\cot\theta}{\sin\theta}\theta_u\langle b, T\rangle + \frac{\tau_\alpha}{\sin\theta}\langle n, T\rangle \\
&= -\theta_u \cot\theta \cosh\phi - \tau_\alpha \sinh\phi. \qquad (3.8)
\end{aligned}$$

Thus, if α is a geodesic curve then we have $\sinh\phi = 0$, which implies $\theta_u \cot\theta \cosh\phi = 0 \implies \theta_u \cot\theta = 0 \implies \theta_u = 0$ or $\cot\theta = 0$, hence α is a geodesic curve if either θ is a function of v only or $\theta = 90°$. Now, assuming either θ is a function of v only or $\theta = 90°$, (3.8) gives $\sinh\phi\ (\tau_\alpha + \phi_u) = 0$, then using Theorem 3.4 we get that either $\tau_g^\alpha = 0$ or $\sinh\phi = 0$ which implies that either α is line of curvature or a geodesic curve.

□

Theorem 3.16 *If the normal $\bar{N}$ to the translation surface M is parallel to the timelike principal normal n of the generating curve α, then α is geodesic curve on M and β is a straight line, hence M is a cylindrical surface with the mean curvature $H = -\frac{\kappa_\alpha}{2}$.*

Proof: The unit normal of the translation surface M is given by

$$\bar{N} = \cosh\phi\; n + \sinh\phi\; b = \frac{t \times T}{\sin\theta}.$$

Since $\bar{N}$ is parallel to n, $\phi = 0$ so $\kappa_g^\alpha = \kappa_\alpha \sinh\phi = 0$, which implies that α is geodesic curve. Also we have $\langle T, t\rangle = \cos\theta$, hence we can write

$$T = \cos\theta\; t + \sin\theta\; b. \tag{3.9}$$

Now, $\kappa_n^\beta = \langle T', \bar{N}\rangle = \langle \frac{\partial(\cos\theta\; t + \sin\theta\; b)}{\partial v}, \cosh\phi\; n + \sinh\phi\; b\rangle = \langle -\theta_v \sin\theta\; t + \theta_v \cos\theta\; b, n\rangle = 0$. Hence β is an asymptotic curve on the surface M. Similarly, $\kappa_g^\beta = \langle T', B\rangle = \langle \frac{\partial(\cos\theta\; t+\sin\theta\; b)}{\partial v}, \bar{N} \times T\rangle = \langle -\theta_v \sin\theta\; t + \theta_v \cos\theta\; b, n \times T\rangle = \langle -\theta_v \sin\theta\; t + \theta_v \cos\theta\; b, \cos\theta\; b - \sin\theta\; t\rangle = \theta_v$. Now using Corollary 3.5, and $\kappa_n^\alpha = -\kappa_\alpha \cosh\phi = -\kappa_\alpha$, we get $K = 0$ and $H = \frac{\kappa_n^\alpha}{2\sin^2\theta} = -\frac{\kappa_\alpha}{2\sin^2\theta}$.

Also, since $\bar{N} = n$ we get $\langle \bar{N}, T\rangle = 0 \implies \langle n, T\rangle = 0$. If we differentiate this with respect to u and use Frenet equations of α, we get

$$\begin{aligned} \langle n', T\rangle &= 0, \\ \kappa_\alpha \langle t, T\rangle + \tau_\alpha \langle b, T\rangle &= 0, \\ \kappa_\alpha \cos\theta + \tau_\alpha \sin\theta &= 0, \end{aligned} \tag{3.10}$$

which implies that θ is a function of u only i.e. $\theta_v = 0$.

Now, since α is a geodesic curve, by Theorem 3.15, we get that $\theta_u = 0$ or $\theta = 90°$. Hence, we get that θ is a constant function which is equal to $90°$ and $\kappa_g^\beta = -\theta_v \implies \kappa_g^\beta = 0$. So β is a straight line which implies that M is a cylindrical surface. Finally, $\sin\theta = \sin 90° = 1$, so mean curvature is obtained as $H = -\frac{\kappa_\alpha}{2}$. □

Theorem 3.17 *If the normal $\bar{N}$ to the translation surface M is parallel to the timelike binormal b of the generating curve α, then α is a planar curve and M is a plane.*

Proof: If $\bar{N}$ is parallel to b then $\sinh\phi = 0$ and hence $\phi = 0$. Putting this into (3.7), we obtain that $\tau_\alpha = 0$, which implies that α is planar curve and hence b is a fixed vector consequently $\bar{N}$ is also a fixed vector which implies M is a plane. □

Using Theorem 3.6 and 3.17 we find the following result,

Theorem 3.18 *The curve α with spacelike principal normal is an asymptotic curve on M if and only if either α is a straight line or a planar curve.*

Note: Since $\bar{N} = \cosh\psi\; N + \sinh\psi\; B$, $\bar{N}$ is never parallel to B as $\cosh\psi \neq 0$.

Theorem 3.19 *If the normal $\bar{N}$ to the translation surface M is parallel to the normal N of the surface σ along the curve β, then the generating curve α is a straight line as a result M is a cylindrical surface with mean curvature $H = \frac{\tau_g^2 + \kappa_n^2}{2\kappa_n}$ when β is not an asymptotic curve on σ, otherwise M is a plane.*

Proof: The unit normal of the translation surface M is given by,

$$\bar{N} = \cosh\psi\; N + \sinh\psi\; B = \frac{t \times T}{\sin\theta}.$$

Thus, if $\bar{N}$ is parallel to N then $\langle \bar{N}, t\rangle = \langle N, t\rangle = 0$. We have $\langle t, T\rangle = \cos\theta$, hence we can write

$$t = \cos\theta\; T + \sin\theta\; B. \tag{3.11}$$

Now, $\kappa_n^\alpha = \langle t', \bar{N}\rangle = \langle \frac{\partial(\cos\theta\ T+\sin\theta\ B)}{\partial u}, \cosh\psi\ N + \sinh\psi\ B\rangle = \langle -\theta_u \sin\theta\ T + \theta_u\cos\theta\ B, N\rangle = 0$. Hence, α is an asymptotic curve on the surface M. Similarly, $\kappa_g^\alpha = \langle t', \bar{N}\times t\rangle = \langle t', N\times(\cos\theta\ T + \sin\theta\ B)\rangle = \langle -\theta_u\sin\theta\ \ T + \theta_u\cos\theta\ \ B, \cos\theta\ \ B - \sin\theta\ \ T\rangle = \theta_u$. Now, using Corollary 3.5 and $\kappa_n^\beta = \kappa_g\sinh\psi - \kappa_n\cosh\psi = -\kappa_n$, we get $K = 0$ and $H = \frac{\kappa_n^\beta}{2\sin^2\theta} = -\frac{\kappa_n}{2\sin^2\theta}$.

Now, since $\bar{N} = N$, we get $\langle \bar{N}, t\rangle = 0 = \langle N, t\rangle$. If we differentiate this with respect to v and use Darboux equations of β lying on σ we get

$$\begin{aligned}\langle N', t\rangle &= 0,\\ \kappa_n\langle T, t\rangle + \tau_g\langle B, t\rangle &= 0,\\ \kappa_n\cos\theta + \tau_g\sin\theta &= 0,\end{aligned} \tag{3.12}$$

which implies that θ is a function of v only i.e. $\theta_u = 0$. So $\kappa_g^\alpha = 0$ also. Hence α is a straight line. Now, if $\kappa_n = 0$ then the mean curvature $H = \frac{\kappa_n^\beta}{2\sin^2\theta} = -\frac{\kappa_n}{2\sin^2\theta} = 0$, which along with $K = 0$ implies that M is a plane. Also, when $\kappa_n \neq 0$, (3.12) implies that $\cot\theta = -\frac{\tau_g}{\kappa_n} \implies \sin\theta = -\frac{\kappa_n}{\sqrt{\tau_g^2+\kappa_n^2}}$ which gives $H = \frac{\tau_g^2+\kappa_n^2}{2\kappa_n}$. Hence, M is a cylindrical surface with mean curvature $H = \frac{\tau_g^2+\kappa_n^2}{2\kappa_n}$.

□

Example 3.1 Let α be a spacelike curve in $\mathbb{E}_1^3$ given by $\alpha(s) = (\sqrt{2}\cosh s, -\sinh s, \cosh s)$ and $\beta(t) = (\cosh t, 0, \sinh t)$ be an spacelike curve lying on the surface $\sigma(u,v) = (\cosh u\cosh v, \sinh u\cosh v, \sinh v)$ which is a spacelike surface, where α and β are curves given by the arc-length parameters s and t, respectively. The translation surface generated by the curves α and β is given by $M_1 : X(s,t) = \alpha(s) + \beta(t) = (\sqrt{2}\cosh s + \cosh t, -\sinh s, \cosh s + \sinh t)$. See figure 1.

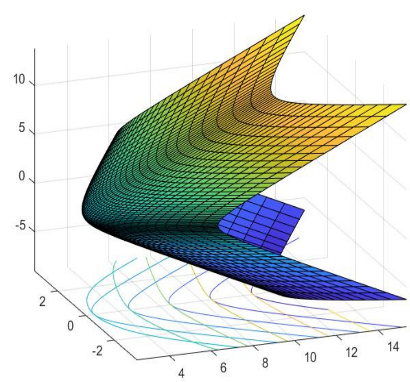

Figure 1: Translation surface generated by a spacelike curve in $\mathbb{E}_1^3$ and a curve lying on a spacelike surface.

3.2. Translation surface generated by a timelike curve and a curve lying on a spacelike surface

The translation surface generated by a timelike curve α and a spacelike curve β lying on a spacelike surface σ is given by

$$M : X(u,v) = \alpha(u) + \beta(v). \tag{3.13}$$

Differentiating the above equation (3.13) with respect to u and v, we obtain $X_u = t$ and $X_v = T$.

The unit normal vector $\bar{N}$ of the surface M is given by

$$\bar{N}(u,v) = \frac{X_u\times X_v}{\|X_u\times X_v\|}, \tag{3.14}$$

where $X_u \times X_v = t \times T$ and $\|X_u \times X_v\| = \sqrt{-\epsilon(EG - F^2)}$, where $\epsilon = \langle \bar{N}, \bar{N} \rangle$.

By Definition 2.2, we have $\langle t, T \rangle = \sinh\theta$ and the coefficients of first fundamental form are obtained as follows

$$E = -1, \; F = \sinh\theta, \; G = 1, \tag{3.15}$$

where θ is the smooth hyperbolic angle function between t and T. $EG - F^2 = -1 - \sinh^2\theta = -\cosh^2\theta < 0$ shows that the surface M is timelike and hence the unit normal $\bar{N}$ is spacelike, i.e. $\langle \bar{N}, \bar{N} \rangle = 1$.

Thus, we obtain the spacelike unit normal,

$$\bar{N}(u, v) = \frac{t \times T}{\cosh\theta}, \tag{3.16}$$

so that $\langle \bar{N}, t \rangle = \langle \bar{N}, T \rangle = 0$.

Now, suppose the angle between n and $\bar{N}$ is ϕ and the angle between N and $\bar{N}$ is ψ then $\bar{N}$ can be expressed as follows [10]

$$\bar{N}(u, v) = \cos\phi \, n + \sin\phi \, b,$$
$$\bar{N}(u, v) = \sinh\psi N + \cosh\psi B.$$

The coefficients of second fundamental form of the surface M are given by

$$\begin{cases} l = \kappa_\alpha \cos\phi, \\ m = 0, \\ n = \kappa_g \cosh\psi - \kappa_n \sinh\psi. \end{cases} \tag{3.17}$$

Now, using the equations (2.7) and (2.8) and above calculations, we obtain the following results.

Theorem 3.20 *The Gaussian curvature and the mean curvature of the timelike translation surface M are given as follows, respectively*

$$K = -\frac{(\kappa_\alpha \cos\phi)(\kappa_g \cosh\psi - \kappa_n \sinh\psi)}{\cosh^2\theta},$$

$$H = \frac{(\kappa_\alpha \cos\phi) + (\kappa_g \cosh\psi - \kappa_n \sinh\psi)}{2\cosh^2\theta}.$$

Corollary 3.6 *The timelike translation surface M is flat if and only if*

$$\kappa_\alpha \cos\phi = 0 \text{ or } \kappa_g \cosh\psi - \kappa_n \sinh\psi = 0.$$

Proof: By putting $K = 0$ in the Theorem 3.20, we get the desired result. □

Corollary 3.7 *The timelike translation surface M is minimal if and only if*

$$\kappa_\alpha \cos\phi = -\kappa_g \cosh\psi + \kappa_n \sinh\psi.$$

Proof: By putting $H = 0$ in the Theorem 3.20, we get the required result. □

Theorem 3.21 *The normal curvature, geodesic curvature and geodesic torsion of the curve α lying on the translation surface M are found as follows*

$$\begin{cases} \kappa_n^\alpha = \kappa_\alpha \cos\phi, \\ \kappa_g^\alpha = \kappa_\alpha \sin\phi, \\ \tau_g^\alpha = -\tau_\alpha - \phi_u, \end{cases}$$

where κ_α and τ_α are curvature and torsion of the curve α, respectively.

Proof: We know that $\kappa_n^\alpha = \langle \alpha'', \bar{N} \rangle$. Since $\alpha' = t$, $\alpha'' = \kappa_\alpha n$ and using $\bar{N} = \cos\phi\, n + \sin\phi\, b$, we get $\kappa_n^\alpha = \langle \kappa_\alpha n, \cos\phi\, n + \sin\phi\, b \rangle = \kappa_\alpha \cos\phi$.

Also, $\kappa_g^\alpha = \langle \alpha'', \bar{N} \times t \rangle$, where $\bar{N} \times t = \cos\phi\, (n \times t) + \sin\phi\, (b \times t) = -\cos\phi\, b + \sin\phi\, n$. So we get $\kappa_g^\alpha = \langle \kappa_\alpha n, -\cos\phi\, b + \sin\phi\, n \rangle = \kappa_\alpha \sin\phi$.

Finally, $\tau_g^\alpha = \langle \bar{N}_u, \bar{N} \times t \rangle$, where $\bar{N}_u = \frac{\partial \bar{N}}{\partial u} = \phi_u \cos\phi\; b - \phi_u \sin\phi\; n + \cos\phi\; n_u + \sin\phi\; b_u = \kappa_\alpha \cos\phi\; t - (\phi_u + \tau_\alpha)\sin\phi\; n + (\phi_u + \tau_\alpha)\cos\phi\; b$. Hence, $\tau_g^\alpha = \langle \bar{N}_u, -\cos\phi\; b + \sin\phi\; n \rangle = -(\phi_u + \tau_\alpha)\sin^2\phi - (\phi_u + \tau_\alpha)\cos^2\phi = -\phi_u - \tau_\alpha$. □

Theorem 3.22 *The normal curvature, geodesic curvature and geodesic torsion of the curve β lying on the translation surface M are found as follows*

$$\begin{cases} \kappa_n^\beta = \kappa_g \cosh\psi - \kappa_n \sinh\psi, \\ \kappa_g^\beta = \kappa_g \sinh\psi - \kappa_n \cosh\psi, \\ \tau_g^\beta = -\tau_g - \psi_v, \end{cases}$$

where κ_n, κ_g and τ_g are the normal curvature, geodesic curvature and geodesic torsion of the curve β lying on the surface σ, respectively.

Proof: We know that $\kappa_n^\beta = \langle \beta'', \bar{N} \rangle$. Since $\beta' = T$, $\beta'' = \kappa_g B + \kappa_n N$ and using $\bar{N} = \sinh\psi\; N + \cosh\psi\; B$, we get $\kappa_n^\beta = \langle \kappa_g B + \kappa_n N, \sinh\psi\; N + \cosh\psi\; B \rangle = \kappa_g \cosh\psi - \kappa_n \sinh\psi$.

Also, $\kappa_g^\beta = \langle \beta'', \bar{N} \times T \rangle$, where $\bar{N} \times T = \sinh\psi\; (N \times T) + \cosh\psi\; (B \times T) = \sinh\psi\; B + \cosh\psi\; N$. So we get $\kappa_g^\beta = \langle \kappa_g B + \kappa_n N, \sinh\psi\; B + \cosh\psi\; N \rangle = \kappa_g \sinh\psi - \kappa_n \cosh\psi$.

Finally, using the Darboux equations we get $\tau_g^\beta = \langle \bar{N}_v, \bar{N} \times T \rangle$, where $\bar{N}_v = \frac{\partial \bar{N}}{\partial v} = \psi_v \sinh\psi\; B + \psi_v \cosh\psi\; N + \sinh\psi\; N_v + \cosh\psi\; B_v = (\kappa_n \sinh\psi - \kappa_g \cosh\psi) T + (\psi_v + \tau_g)\sinh\psi\; B + (\psi_v + \tau_n)\cosh\psi\; N$. Hence, $\tau_g^\beta = \langle \bar{N}_u, \sinh\psi\; B + \cosh\psi\; N \rangle = -(\psi_v + \tau_g)\cosh^2\psi + (\psi_v + \tau_g)\sinh^2\psi = -\psi_v - \tau_g$. □

Theorem 3.23 *The timelike curve α is a geodesic curve on M if and only if either α is a straight line or the principal normal n and the normal $\bar{N}$ to the surface M are parallel.*

Proof: From Theorem 3.21, we have $\kappa_g^\alpha = \kappa_\alpha \sin\phi$. Thus α is a geodesic curve on M if and only if $\kappa_\alpha \sin\phi = 0$ if and only if either $\kappa_\alpha = 0$ or $\sin\phi = 0$ if and only if either α is a straight line or the principal normal n and the normal $\bar{N}$ to the surface M are parallel. □

Theorem 3.24 *The timelike curve α is an asymptotic curve on M if and only if either α is a straight line or the binormal b and the normal $\bar{N}$ to the surface M are parallel.*

Proof: From Theorem 3.21, we have $\kappa_n^\alpha = \kappa_\alpha \cos\phi$. Thus α is an asymptotic curve on M if and only if $\kappa_\alpha \cos\phi = 0$ if and only if either $\kappa_\alpha = 0$ or $\cos\phi = 0$ if and only if either α is a straight line or the binormal b and the normal $\bar{N}$ to the surface M are parallel. □

Theorem 3.25 *The curve β is an asymptotic curve on M if and only if either β is a straight line or the angle between the normal N to the surface σ and the normal $\bar{N}$ to the surface M is given as follows*

$$\tanh\psi = \frac{\kappa_g}{\kappa_n}.$$

Proof: From Theorem 3.22, we have $\kappa_n^\beta = \kappa_g \cosh\psi - \kappa_n \sinh\psi$. So β is an asymptotic curve on M if and only if $\kappa_g \cosh\psi - \kappa_n \sinh\psi = 0$ if and only if $\tanh\psi = \frac{\kappa_g}{\kappa_n}$, when $\kappa_n \neq 0$. In case $\kappa_n = 0$, we get that $\kappa_g \cosh\psi = 0$ which implies $\kappa_g = 0$, then it follows that the curve β is a straight line. □

Theorem 3.26 *The curve β is a geodesic curve on M if and only if either β is a straight line or the angle between the normal N to the surface σ and the normal $\bar{N}$ to the surface M is given as follow*

$$\tanh\psi = \frac{\kappa_n}{\kappa_g}.$$

Proof: From Theorem 3.22, we have $\kappa_g^\beta = \kappa_g \sinh\psi - \kappa_n \cosh\psi$. So β is a geodesic curve on M if and only if $\kappa_g \sinh\psi - \kappa_n \cosh\psi = 0$ if and only if $\tanh\psi = \frac{\kappa_n}{\kappa_g}$, when $\kappa_g \neq 0$. In case $\kappa_g = 0$ we get that $\kappa_n \cosh\psi = 0$ which implies $\kappa_n = 0$ further which implies that the curve β is a straight line. □

Theorem 3.27 *If the timelike curve α is an asymptotic curve of the translation surface M then α is a planar curve.*

Proof: We have, $\bar{N} = \frac{t\times T}{\cosh\theta}$ and $\bar{N} = \cos\phi\ n + \sin\phi\ b$, thus $\cos\phi = \langle \bar{N}, n\rangle = \frac{1}{\cosh\theta}\langle t\times T, n\rangle = \frac{1}{\cosh\theta}\langle n\times t, T\rangle = -\frac{1}{\cosh\theta}\langle b, T\rangle$. Differentiating it with respect to u,

$$\begin{aligned} \sin\phi\ \phi_u &= -\frac{\tanh\theta}{\cosh\theta}\theta_u\langle b,T\rangle + \frac{1}{\cosh\theta}\langle b', T\rangle \\ &= -\frac{\tanh\theta}{\cosh\theta}\theta_u\langle b,T\rangle + \frac{\tau_\alpha}{\cosh\theta}\langle n, T\rangle \\ &= \theta_u \tanh\theta\cos\phi + \tau_\alpha \sin\phi. \end{aligned} \tag{3.18}$$

Thus, if α is an asymptotic curve then we have $\cos\phi = 0$ or $\kappa_\alpha = 0$, which implies that $\tau_\alpha = \phi_u = 0$, hence α is a planar curve. □

Theorem 3.28 *If the normal $\bar{N}$ to the translation surface M is parallel to the principal normal n of the generating curve α, then α is a geodesic curve on M and β is a straight line, hence M is a cylindrical surface with the mean curvature $H = \frac{\kappa_\alpha}{2\cosh^2\theta}$.*

Proof: The unit normal of translation surface M is given as

$$\bar{N} = \cos\phi\ n + \sin\phi\ b = \frac{t\times T}{\cosh\theta}.$$

Thus, if $\bar{N}$ is parallel to n then $\phi = 0$ so $\kappa_g^\alpha = \kappa_\alpha \sin\phi = 0$, which implies that α is a geodesic curve. We have $\langle T, t\rangle = \sinh\theta$, hence we can write

$$T = -\sinh\theta\ t + \cosh\theta\ b. \tag{3.19}$$

Now, $\kappa_n^\beta = \langle T', \bar{N}\rangle = \langle \frac{\partial(-\sinh\theta\ t + \cosh\theta\ b)}{\partial v}, \cos\phi\ n + \sin\phi\ b\rangle = \langle -\theta_v\cosh\theta\ t + \theta_v \sinh\theta\ b, n\rangle = 0$. Hence β is an asymptotic curve on the surface M. Similarly, $\kappa_g^\beta = \langle T', B\rangle = \langle \frac{\partial(-\sinh\theta\ t + \cosh\theta\ b)}{\partial v}, \bar{N}\times T\rangle = \langle -\theta_v\cosh\theta\ t + \theta_v\sinh\theta\ b, n\times T\rangle = \langle -\theta_v\cosh\theta\ t + \theta_v\sinh\theta\ b, \sinh\theta\ b - \cosh\theta\ t\rangle = -\theta_v$. Now, using Corollary 3.5 and $\kappa_n^\alpha = \kappa_\alpha\cos\phi = \kappa_\alpha$, $\kappa_n^\beta = 0$, we get $K = 0$ and $H = \frac{\kappa_n^\alpha}{2\cosh^2\theta} = \frac{\kappa_\alpha}{2\cosh^2\theta}$.

Also, since $\bar{N} = n$, we get $\langle \bar{N}, T\rangle = 0 \implies \langle n, T\rangle = 0$. If we differentiate this with respect to u and use Frenet equations of α, we get

$$\begin{aligned} \langle n', T\rangle &= 0, \\ \kappa_\alpha\langle t, T\rangle + \tau_\alpha\langle b, T\rangle &= 0, \\ \kappa_\alpha \sinh\theta + \tau_\alpha\cosh\theta &= 0, \end{aligned} \tag{3.20}$$

which implies that θ is a function of u only i.e. $\theta_v = 0$ and $\kappa_g^\beta = -\theta_v = 0$. So β is a straight line and hence M is a cylindrical surface with mean curvature $H = \frac{\kappa_\alpha}{2\cosh^2\theta}$. □

Theorem 3.29 *If the normal $\bar{N}$ to the translation surface M is parallel to the binormal b of the generating curve α then α is a planar curve and M is a plane.*

Proof: When $\bar{N}$ is parallel to b, $\cos\phi = 0$ and $\phi = 90°$, putting this into (3.18) we obtain that $\tau_\alpha = 0$, which implies that α is planar curve and hence b is a fixed vector consequently $\bar{N}$ is also a fixed vector which implies M is a plane. □

Using Theorem 3.24 and 3.29, we find the following result.

Theorem 3.30 *The timelike curve α is an asymptotic curve on M if and only if either α is a straight line or a planar curve.*

Theorem 3.31 *If the normal $\bar{N}$ to the translation surface M is parallel to the intrinsic normal B of the curve β on the surface σ then the generating curve α is a straight line as a result M is a cylindrical surface with mean curvature $H = \frac{\kappa_g^2-\tau_g^2}{2\kappa_g}$ when β is not a geodesic curve on σ, otherwise M is a plane.*

Proof: The unit normal of translation surface M is given by

$$\bar{N} = \sinh\psi\ N + \cosh\psi\ B = \frac{t \times T}{\cosh\theta}.$$

If $\bar{N}$ is parallel to B then$\langle \bar{N}, t\rangle = \langle B, t\rangle = 0$. We have $\langle t, T\rangle = \sinh\theta$, hence we can write

$$t = \sinh\theta\ T + \cosh\theta\ N. \tag{3.21}$$

Now, $\kappa_n^\alpha = \langle t', \bar{N}\rangle = \langle \frac{\partial(\sinh\theta\ T+\cosh\theta\ B)}{\partial u}, \sinh\psi\ N + \cosh\psi\ B\rangle = \langle \theta_u \cosh\theta\ T + \theta_u \sinh\theta\ N, B\rangle = 0$. Hence α is an asymptotic curve on the surface M. Similarly $\kappa_g^\alpha = \langle t', \bar{N} \times t\rangle = \langle t', B \times (\sinh\theta\ T + \cosh\theta\ N)\rangle = \langle \theta_u \cosh\theta\ T + \theta_u \sinh\theta\ N, \sinh\theta\ N + \cosh\theta\ T\rangle = \theta_u$. Now, using Corollary 3.5 and $\kappa_n^\beta = \kappa_g \cosh\psi - \kappa_n \sinh\psi = \kappa_g$, we get $K = 0$ and $H = \frac{\kappa_n^\beta}{2\cosh^2\theta} = \frac{\kappa_g}{2\cosh^2\theta}$.

Also, since $\bar{N} = B$, we get $\langle \bar{N}, t\rangle = 0 = \langle B, t\rangle$. If we differentiate this with respect to v and use Darboux equations of β lying on σ, we get

$$\begin{aligned} \langle B', t\rangle &= 0, \\ -\kappa_g\langle T, t\rangle + \tau_g\langle N, t\rangle &= 0, \\ -\kappa_g \sinh\theta + \tau_g \cosh\theta &= 0, \end{aligned} \tag{3.22}$$

which implies that θ is a function of v only i.e. $\theta_u = 0 \implies \kappa_g^\alpha = 0$. Hence α is a straight line. Now, if $\kappa_g = 0$ then the mean curvature $H = \frac{\kappa_g}{2\cosh^2\theta} = 0$, which along with $K = 0$ implies that M is a plane. Also when $\kappa_g \neq 0$, from (3.22), we get that $\tanh\theta = \frac{\tau_g}{\kappa_g} \implies \cosh^2\theta = \frac{\kappa_g^2}{\kappa_g^2-\tau_g^2}$, using this we get $H = \frac{\kappa_g^2-\tau_g^2}{2\kappa_g}$. Hence, M is a cylindrical surface with mean curvature $H = \frac{\kappa_g^2-\tau_g^2}{2\kappa_g}$.

□

Example 3.2 Let α be a timelike curve in $\mathbb{E}_1^3$ given by $\alpha(s) = (\sqrt{2}\sinh s, \sqrt{2}\cosh s, s)$ and $\beta(t) = (\cosh t, 0, \sinh t)$ be an spacelike curve lying on the surface $\sigma(u, v) = (\cosh u \cosh v, \sinh u \cosh v, \sinh v)$ which is a spacelike surface, where α and β are curves given by the arc-length parameters s and t, respectively. The translation surface generated by the curves α and β is given by $M_2 : X(s, t) = \alpha(s) + \beta(t) = (\sqrt{2}\sinh s + \cosh t, \sqrt{2}\cosh s, s + \sinh t)$. See figure 2.

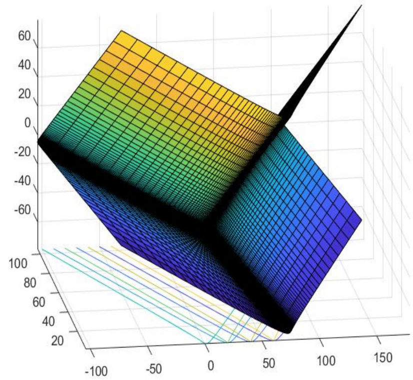

Figure 2: Translation surface generated by a timelike curve and a curve lying on a spacelike surface.

Acknowledgments

We thank the referee and editor for their valuable suggestions.

References

1. Ali, A. T., Abdel Aziz, H. S., Sorour Adel, H., *On curvatures and points of the translation surfaces in Euclidean 3-space*, J. Egyptian Math. Soc. 23, 167-172, (2015).
2. Çetin, M., Kocayiğit, H., Önder, M., *Translation surfaces according to Frenet frame in Minkowski* 3*-space*, Int. J. Phys. Sci., Vol. 7(47), 6135-6143, (2012).
3. Çetin, M., Tunçer, Y., *Parallel surfaces to translation surfaces in Euclidean* 3*-space*, Commun Fac. Sci. Univ. Ank. Series A1 Math. Stat., 64(2), 47-54, (2015).
4. Liu, H., *Translation surfaces with constant mean curvature in 3-dimensional spaces*, J. Geometry, 64, 141-149, (1999).
5. Kuhnel, W., *Differential Geometry: Curves - Surfaces - Manifolds*, Wiesdaden, Braunchweig, (1999).
6. O'Neill, B., *Semi-Riemannian Geometry, With Application to Relativity, Pure and Applied Mathematics*, 103, Academic Press, Inc. New York, (1983).
7. Osman, K., Ali, C., *The Frenet and Darboux instantaneous rotation vectors for curves and spacelike surfaces*, Math. Comput. Appl., 1(2), 77-86, (1996).
8. Ratcliffe, J. G., *Foundation of Hyperbolic Manifolds*, Second Edition, Graduate Text in Mathematics, 149, Springer, New York, (2006).
9. Yadav, A., Yadav, A. K., *Some characterizations of translation surface generated by spherical indicatrices of timelike curves in minkowski* 3*-space*, Int. Electron. J. Geom., 16(1), 48-61, (2023).
10. Zoubir, H., Baba-Hamed, Ch., Bekkar, M., *Translation surfaces in the three-dimensional Lorentz-Minkowski space satisfying* $\Delta r_i = \lambda_i r_i$, Int. J. Math. Anal. 4(17), 797-808, (2010).

Akhilesh Yadav,
Department of Mathematics,
Institute of Science, Banaras Hindu University, Varanasi
India.
E-mail address: akhilesha68@gmail.com

and

Ajay Kumar Yadav,
Department of Mathematics,
Institute of Science, Banaras Hindu University, Varanasi
India.
E-mail address: ajaykumar74088@gmail.com